J. Indones. Math. Soc.
Vol. 32, No. 4 (2026), pp. 1–14.

Eigenvalues and Eigenvectors of Latin squares over Maxmin-ω Systems

Nur Millatul Af Idah[1] and Muhammad Syifa'ul Mufid[1*]

[1]Department of Mathematics, Institut Teknologi Sepuluh Nopember, Indonesia

Abstract. Maxmin-ω systems are the generalization of max-plus and min-plus systems by introducing a threshold parameter $\omega \in (0, 1]$. For a set $S = \{s_1, s_2, \ldots, s_n\}$, possibly containing non-finite elements, $\bigoplus_\omega S$ returns the $\lceil \omega n \rceil$-th smallest element of S. Thus, the cases $\lceil \omega n \rceil = 1$ and $\lceil \omega n \rceil = n$ correspond to the min-plus and max-plus systems, respectively. Unlike in the max-plus and min-plus settings, existing results and algorithms for eigenproblems do not directly extend to maxmin-ω systems. In this paper, we solve the eigenproblem for maxmin-ω systems when the matrix is a Latin square, that is, when each row and each column consists of distinct elements. We prove that the eigenvalue is unique and is equal to the $\lceil \omega n \rceil$-th smallest entry of the matrix. We also characterize the corresponding eigenvectors in terms of the permutation cycles associated with the entries equal to the eigenvalue. These results show that the Latin square structure allows the eigenproblem in maxmin-ω systems to be resolved explicitly.

Key words and Phrases: maxmin-ω, max-plus, min-plus, Latin square, eigenproblem.

1. Introduction

Algebraic systems built on combinations of the minimum, maximum, and addition operations have been widely studied due to their rich theoretical properties and practical applications. Among these, the max-plus system, which relies solely on the maximum and addition operations, has become particularly significant. Over the past decades, max-plus algebra has been employed to model and solve diverse problems, including transportation networks [1], flow shop scheduling [2], urban bus routing [3], and ribosome dynamics [4]. Closely related to this is the min-plus algebra, differing only in the use of the minimum operation in place of the

*Corresponding author : syifaul.mufid@its.ac.id
2020 Mathematics Subject Classification: 05B15, 15A18, 15A80
Received: 14-10-2025, accepted: 03-06-2026.

maximum. Beyond these two, other algebraic frameworks—such as the min-max-plus system and its bipartite versions—have also been investigated [5].

Building on this foundation, recent research has introduced the maxmin-ω system [6], a generalization of max-plus and min-plus structures. This system incorporates a threshold parameter $\omega \in (0, 1]$, where $q = \lceil \omega n \rceil$ determines the number of inputs influencing the update of a node among n total nodes. When $q = 1$, the system behaves like min-plus system; when $q = n$, it coincides with max-plus system. The flexibility of this framework allows it to capture dynamics where neither extreme suffices, as explored in studies on solving linear equations over maxmin-ω [7]. However, many techniques from traditional max-plus and min-plus systems do not directly transfer to this setting, particularly for problems involving eigenvalues and eigenvectors.

This paper focuses on the eigenproblem for maxmin-ω systems when the underlying matrix is a Latin square—an $n \times n$ matrix in which each row and column contains all n distinct symbols exactly once. Latin squares, introduced by Euler, have long been central in combinatorics and related fields [8, 9]. Their structured nature allows explicit characterization of eigenvalues and eigenvectors in maxmin-ω systems, even though conventional max-plus or min-plus results cannot be applied directly.

In particular, we show that for any Latin square of order n, the eigenvalue under the maxmin-ω system is unique and equals the $\lceil \omega n \rceil$-th smallest element of the matrix. Furthermore, we develop an approach to determine both the trivial and non-trivial eigenvectors, leveraging the permutation cycles associated with the eigenvalue. This study not only extends the understanding of eigenproblems beyond classical max-plus settings in [8] but also opens pathways for applying maxmin-ω structures to broader classes of matrices and real-world systems.

Beyond their theoretical interest, these results may be useful in applications where the update rule depends on an intermediate order statistic. For instance, in an information-propagation network, the quantity $(L \otimes_\omega x)_i$ may represent the state of agent i after receiving the q-th ranked incoming signal. Similarly, in a distributed computation or asynchronous production system, it may describe the time at which a processor proceeds after the q-th required input has arrived. In such settings, Latin squares model balanced structures in which each node receives the same collection of weights or delays, but in a different order. The present results therefore provide a class of systems for which the long-term behaviour can be analyzed explicitly.

The remainder of the paper is organized as follows. Section 2 reviews max-plus, min-plus, and maxmin-ω algebra, along with the relevant concepts of Latin squares and eigenproblems. Section 3 presents the main results, including the derivation of eigenvalues and eigenvectors for Latin squares under maxmin-ω, with illustrative examples. Section 4 concludes the paper and suggests directions for future research.

2. Preliminaries

2.1. Max-Plus and Min-Plus Algebra.

The max-plus algebra plays a vital role in various mathematical fields and is intricately linked to the concept of a semiring, which is explained in the following definition.

Definition 2.1. [10] *Max-plus algebra is a semiring* $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ *where* $\mathbb{R}_\varepsilon$ *is the union of all real numbers and* $\varepsilon = -\infty$*, and the operations* $\oplus$ *and* $\otimes$ *on this structure are defined as*

$$a \oplus b = \max\{a, b\},$$
$$a \otimes b = a + b,$$

for every $a, b \in \mathbb{R}_\varepsilon$*. Furthermore, the notation* $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ *will be denoted by* $\mathbb{R}_{\max}$*.*

It is known that the neutral element and identity element of $\mathbb{R}_\varepsilon$ are ε and 0, respectively. The operations defined on matrices in $\mathbb{R}_\varepsilon^{m\times n}$ are extensions of the operations $\oplus$ and $\otimes$ on $\mathbb{R}_\varepsilon$, namely, for all $A, B \in \mathbb{R}_\varepsilon^{m\times n}$, with $C \in \mathbb{R}_\varepsilon^{n\times p}$ and $\alpha \in \mathbb{R}_\varepsilon$, the following holds:

$$[\alpha \otimes A](i,j) = \alpha + A(i,j),$$
$$[A \oplus B](i,j) = A(i,j) \oplus B(i,j),$$
$$[A \otimes C](i,j) = \bigoplus_{k=1}^{n} A(i,k) \otimes C(k,j).$$

Then, for all $A \in \mathbb{R}_\varepsilon^{n\times n}$ and $t \in \mathbb{N}$, $A^{\otimes t}$ denotes $A \otimes \cdots \otimes A$ (t times). For $t = 0$, we obtain $A^{\otimes 0}$, which is the max-plus identity matrix of dimension n, , i.e., a matrix with all main diagonal elements equal to 0 and $\varepsilon = -\infty$ for elements in other positions [11].

Another structure that is isomorphic to the max-plus algebra structure $\mathbb{R}_{\max}$ is the min-plus algebra $\mathbb{R}_{\min} = (\mathbb{R}_{\varepsilon'}, \oplus', \otimes)$, which will be explained in detail in the following definition.

Definition 2.2. [10] *The min-plus algebra is a semiring* $(\mathbb{R}_{\varepsilon'}, \oplus', \otimes)$ *where* $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\}$ *and*

$$c \oplus' d = \min\{c, d\},$$
$$c \otimes d = c + d,$$

for all $c, d \in \mathbb{R}_{\varepsilon'}$ *and* $\varepsilon' = +\infty$*.*

2.2. Maxmin-ω Systems.

The maxmin-omega system involves two operations, namely $\bigoplus_\omega$ and $\bigotimes_\omega$. Given a set of real numbers $A = \{a_1, a_2, \cdots, a_n\}$, the operation

$$\bigoplus_\omega A$$

in maxmin-ω yields the $\lceil \omega n \rceil$-th smallest element of A for $\omega \in (0, 1]$. Since the operation depends on ω only through the quantity $q = \lceil \omega n \rceil$, it is sufficient to consider representative values $\omega = q/n$ with $q \in \{1, 2, \ldots, n\}$. Indeed, for every

$i \in \{1, 2, \ldots, n\}$, all values of ω in the interval $((i-1)/n, i/n]$ yield the same value $q = i$. Throughout the paper, we write $[n] = \{1, 2, \ldots, n\}$.

The notation $\bigoplus_\omega A$ can also be expressed as a combination of operations $\oplus$ and $\oplus'$ as follows.

$$\bigoplus\nolimits_\omega A = \bigoplus_{D \in \mathcal{Q}(A,q)}{}' \left\{ \bigoplus_{i \in D} a_i \right\} \tag{1}$$

$$= \bigoplus_{D \in \mathcal{Q}(A,n+1-q)} \left\{ \bigoplus_{i \in D}{}' a_i \right\} \tag{2}$$

where $\mathcal{Q}(A, q)$ and $\mathcal{Q}(A, n+1-q)$ are respectively the collections of all subsets of A having q and $n+1-q$ elements [6]. The two equations in (1) and (2) yield the same result, with a different order of applying the maximum and minimum operations.

Furthermore, for each $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$ and $\alpha \in \mathbb{R}$, we define scalar multiplication by

$$[\alpha \otimes A](i, j) = \alpha + A(i, j),$$

while the maxmin-ω product is defined by

$$[A \otimes_\omega B](i, j) = \bigoplus\nolimits_\omega \{A(i, k) + B(k, j) \mid k = 1, 2, \cdots, n\}.$$

Then, the maxmin-omega system can be written as follows.

Definition 2.3. [6] *A maxmin-omega system of dimension n for $\omega \in (0, 1]$ and $q \in [n]$ is a system that evolves according to*

$$\boldsymbol{v}(k+1) = \mathcal{M}(\boldsymbol{v}(k)),\ k = 1, 2, \cdots$$

with $\boldsymbol{v}(k) = [\boldsymbol{v}_1(k)\ \ \boldsymbol{v}_2(k)\ \ \cdots\ \ \boldsymbol{v}_n(k)]^\top$, *and*

$$\mathcal{M}(\boldsymbol{v}(k)) = U \otimes_\omega \boldsymbol{v}(k), \tag{3}$$

for $\boldsymbol{v}(k) \in \mathbb{R}^n$, $U \in \mathbb{R}^{n \times n}$.

The maxmin-ω system (3) can be expressed as a maximum and minimum representation of $\mathcal{M}$ as follows:

$$\mathcal{M}(\mathbf{v}(k)) = C_1 \otimes \mathbf{v}(k) \oplus' C_2 \otimes \mathbf{v}(k) \oplus' \cdots \oplus' C_{\binom{n}{q}} \otimes \mathbf{v}(k) \tag{4}$$

$$\mathcal{M}(\mathbf{v}(k)) = D_1 \otimes' \mathbf{v}(k) \oplus D_2 \otimes \mathbf{v}(k) \oplus \cdots \oplus D_{\binom{n}{q-1}} \otimes \mathbf{v}(k) \tag{5}$$

For each maximum projection C_i, in every row we keep exactly q entries of U and replace all remaining entries by ε. Thus,

$$C_i(l, m) = \begin{cases} U(l, m), & \text{if } (l, m) \text{ is one of the selected } q \text{ entries in row } l, \\ \varepsilon, & \text{otherwise,} \end{cases}$$

Similarly, for each minimum projection D_j, in every row we keep exactly $n+1-q$ entries of U and replace all remaining entries by ε'. Thus,

$$D_j(l,m) = \begin{cases} U(l,m), & \text{if } (l,m) \text{ is one of the selected } n+1-q \\ & \text{entries in row } l, \\ \varepsilon', & \text{otherwise.} \end{cases}$$

It is stated in [6] that the maximum representation form C_i of this system is not unique, but there are as many as $\left(\binom{n}{q}!\right)^n$ such representations. Similarly, there are $\left(\binom{n}{q-1}!\right)^n$ minimum representation. Although the families of projections C_i, D_i are not unique, all such representations define the same operator $\mathcal{M}$. Hence, (3), (4), and (5) are always equivalent.

The following is an example of a maxmin-omega system, such that the result of (3) is the same as the maximum and minimum representations in (4) and (5).

Example 2.4. *Given the matrix*

$$U = \begin{bmatrix} 0 & -1 & -2 \\ 3 & 1 & 2 \\ 1 & 4 & 5 \end{bmatrix}$$

and the vector

$$\boldsymbol{v}(1) = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}.$$

By taking $\omega = 2/3$, we obtain $q = 2$, so by substituting U and $\boldsymbol{v}(1)$ into (3), we get

$$\boldsymbol{v}(2) = \mathcal{M}(\boldsymbol{v}(1)) = U \otimes_{2/3} \boldsymbol{v}(1) = \begin{bmatrix} 0 & -1 & -2 \\ 3 & 1 & 2 \\ 1 & 4 & 5 \end{bmatrix} \otimes_{2/3} \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$$
$$= \begin{bmatrix} \bigoplus_{2/3}\{1,3,4\} \\ \bigoplus_{2/3}\{4,5,8\} \\ \bigoplus_{2/3}\{2,8,11\} \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}.$$

Next, the maximum and minimum projections C_i and D_j for $i \in \{1,2,\binom{3}{2}\}$ and $j \in \{1,2,\binom{3}{1}\}$ are given as follows:

$$C_1 = \begin{bmatrix} 0 & -1 & -\infty \\ -\infty & 1 & 2 \\ 1 & -\infty & 5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & -\infty & -2 \\ 3 & 1 & -\infty \\ -\infty & 4 & 5 \end{bmatrix},$$
$$C_3 = \begin{bmatrix} -\infty & -1 & -2 \\ 3 & -\infty & 2 \\ 1 & 4 & -\infty \end{bmatrix}, D_1 = \begin{bmatrix} 0 & -1 & +\infty \\ +\infty & 1 & 2 \\ 1 & +\infty & 5 \end{bmatrix},$$
$$D_2 = \begin{bmatrix} 0 & +\infty & -2 \\ 3 & 1 & +\infty \\ +\infty & 4 & 5 \end{bmatrix}, \quad D_3 = \begin{bmatrix} +\infty & -1 & -2 \\ 3 & +\infty & 2 \\ 1 & 4 & +\infty \end{bmatrix}.$$

By substituting all maximum and minimum projections as well as $\mathbf{v}(1)$ *into* (4) *and* (5)*, the maximum and minimum representations of* $\mathcal{M}$ *are obtained as follows:*

$$\begin{aligned}\boldsymbol{v}(2) =&\mathcal{M}(\boldsymbol{v}(1)) = C_1 \otimes \boldsymbol{v}(1) \oplus' C_2 \otimes \boldsymbol{v}(1) \oplus' C_3 \otimes \boldsymbol{v}(1)\\ =&\left(\begin{bmatrix} 0 & -1 & -\infty \\ -\infty & 1 & 2 \\ 1 & -\infty & 5 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}\right) \oplus' \left(\begin{bmatrix} 0 & -\infty & -2 \\ 3 & 1 & -\infty \\ -\infty & 4 & 5 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}\right) \oplus'\\ &\left(\begin{bmatrix} -\infty & -1 & -2 \\ 3 & -\infty & 2 \\ 1 & 4 & -\infty \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 8 \\ 11 \end{bmatrix} \oplus' \begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix} \oplus' \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\boldsymbol{v}(2) =&\mathcal{M}(\boldsymbol{v}(1)) = D_1 \otimes' \boldsymbol{v}(1) \oplus D_2 \otimes' \boldsymbol{v}(1) \oplus D_3 \otimes' \boldsymbol{v}(1)\\ =&\left(\begin{bmatrix} 0 & -1 & +\infty \\ +\infty & 1 & 2 \\ 1 & +\infty & 5 \end{bmatrix} \otimes' \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}\right) \oplus \left(\begin{bmatrix} 0 & +\infty & -2 \\ 3 & 1 & +\infty \\ +\infty & 4 & 5 \end{bmatrix} \otimes' \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}\right) \oplus\\ &\left(\begin{bmatrix} +\infty & -1 & -2 \\ 3 & +\infty & 2 \\ 1 & 4 & +\infty \end{bmatrix} \otimes' \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \oplus \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} \oplus \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}\end{aligned}$$

It can be concluded that both results in (4) *and* (5) *are equal to* (3)*.*

2.3. **Latin Squares.**

A Latin square of order n is an $n \times n$ matrix whose entries consist of n distinct elements such that each element appears exactly once in every row and every column. This structure, first introduced by Euler, has played a central role in combinatorics due to its rich algebraic and combinatorial properties [9]. In particular, each row and column of a Latin square can be viewed as a permutation of the set $\{1, 2, \ldots, n\}$, ensuring that no symbol is repeated within a row or column.

A permutation is defined in algebra as a bijective mapping on a set W. The set of all permutations of W, denoted by S_W and called the **symmetry group** of the set W, for $W = \{1, 2, \cdots, n\}$ is denoted by S_n. In this study, $\{1, 2, \cdots, n\}$ will be denoted by $[n]$ in subsequent writing. These permutations can be written in cycle form, for example, if a mapping $\alpha : W \to W$, written as $(x_1 \;\; x_2 \;\; x_3 \;\; \cdots \;\; x_t)$ if $\alpha(x_1) = x_2$, $\alpha(x_2) = x_3$, $\cdots$, $\alpha(x_{t-1}) = x_t$, and $\alpha(x_t) = x_1$, which is then called a t-cycle or a cycle of length t, for $x_i \in W$ with $i = 1, 2, \ldots, t$. An identity permutation, where $\alpha(i) = i$ for all $i \in [n]$, is denoted by $\alpha = ()$.

Given a Latin square L with elements in $\{s_1, s_2, \ldots, s_n\}$, one can obtain a permutation $\sigma : [n] \to [n]$ associated with s_k where

$$\sigma_{s_k}(i) = j \text{ if } L(i, j) = s_k.$$

The following is an example of a Latin square with its corresponding permutations associated with each element.

Example 2.5. *Suppose we have a Latin square* L

$$L = \begin{bmatrix} -1 & -2 & -3 \\ -3 & -1 & -2 \\ -2 & -3 & -1 \end{bmatrix}.$$

It is straightforward to see that $\sigma_{-1} = (), \sigma_{-2} = (1\ 2\ 3)$*, and* $\sigma_{-3} = (1\ 3\ 2)$.

2.4. **Eigenproblem.**

The eigenproblem in linear algebra is how to find $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ of a matrix $T \in \mathbb{R}^{n\times n}$ as a solution to the equation $T \times \mathbf{v} = \lambda \times \mathbf{v}$, where λ and $\mathbf{v}$ are called the eigenvalue and eigenvector of T, respectively. The same applies in max-plus and min-plus algebra, namely for each $U \in \mathbb{R}_{\varepsilon}^{n\times n}$ and $W \in \mathbb{R}_{\varepsilon'}^{n\times n}$, with the equations $U \otimes \mathbf{v} = \lambda \otimes \mathbf{v}$ and $W \otimes' \mathbf{v} = \lambda \otimes \mathbf{v}$.

A matrix is called irreducible if its associated directed graph is strongly connected, that is, for every pair of distinct nodes i and j, there exists a directed path from i to j. A circuit is a directed path that starts and ends at the same node. In max-plus algebra, a critical circuit is a circuit with maximum average weight, where the average weight is defined as the total weight divided by the length of the circuit [11]. It is well known that every irreducible max-plus matrix has a unique eigenvalue, equal to the average weight of a critical circuit [12]. In the special case of a Latin square of order n, since each row and column contains the same n distinct elements, this eigenvalue is simply the maximum element of the set [8].

Given a Latin square L of order n and let $\lambda(L)$ be its eigenvalue, with $L_\lambda = -\lambda \otimes L$, then the i-th column for $i \in [n]$ of

$$L_\lambda^+ = L_\lambda \oplus L_\lambda^{\otimes 2} \oplus \cdots \oplus L_\lambda^{\otimes n}. \tag{6}$$

is an eigenvector of L if it satisfies $L_\lambda^+(i,i) = 0$ [8].

3. Eigenproblem of Latin Squares in Maxmin-ω Systems

3.1. **Eigenvalue and Trivial Eigenvector.**

The set of *Latin squares* of order n whose elements are members of $S = \{s_1, s_2, \cdots, s_n\}$ with $s_1 < s_2 < \cdots < s_n$ and for every $i \in [n]$, $s_i \in \mathbb{Z}$, is denoted by $\mathbb{L}_n(S)$. The eigenvalue and eigenvector of $L \in \mathbb{L}_n(S)$ associated with each $q \in [n]$ are the solutions to the following equation

$$L \otimes_\omega \mathbf{v}_q(L) = \lambda_q(L) \otimes \mathbf{v}_q(L). \tag{7}$$

Theorem 3.1. *Given a set* $S = \{s_1, \ldots, s_n\}$ *where* $s_1 < s_2 < \ldots < s_n$ *and* $\omega = q/n$ *for* $q \in [n]$*. If* $L \in \mathbb{L}_n(S)$ *then*

label=i. $[0\ 0\ \cdots\ 0]^\top \in \mathbb{R}^n$ *is the trivial eigenvector of* L*.*
lbbel=ii. The eigenvalue of L *is* $\bigoplus_\omega S = s_q$*.*

Proof. Take any $L \in \mathbb{L}_n(S)$ and a vector $[0\ 0\ \cdots\ 0]^\top \in \mathbb{R}^n$ such that when substituted into (7) we obtain

$$\left(L \otimes_\omega [0\ 0\ \cdots\ 0]^\top\right)_i = \bigoplus\nolimits_\omega \{L(i,k) + 0 | k = 1, 2, \cdots, n\}$$

$$= \bigoplus_{\omega} S \otimes 0$$
$$= s_q \otimes 0$$

for every $i \in [n]$, hence

$$L \otimes_\omega [0\ 0\ \cdots\ 0]^\top = s_q \otimes [0\ 0\ \cdots\ 0]^\top.$$

Since this expression satisfies (7), then s_q and $[0\ 0\ \cdots\ 0]^\top \in \mathbb{R}^n$ are the eigenvalue and eigenvector of L corresponding to q. □

3.2. **Non-Trivial Eigenvectors.**

From Theorem 3.1, the eigenvalue as well as the trivial eigenvector solution of $L \in \mathbb{L}_n(S)$ under maxmin-omega have been obtained. Therefore, we now proceed to discuss the non-trivial eigenvector. Before that, we present several supporting propositions to help derive the non-trivial eigenvector.

Proposition 3.2. *Let $S = \{s_1, \ldots, s_n\}$ be a set such that $s_1 < s_2 < \ldots < s_n$, and let $\omega = q/n$ for $q \in [n]$. If $L \in \mathbb{L}_n(S)$, then for every $\mathbf{x} \in \mathbb{R}^n$ it holds that*

$$(L \otimes_\omega \mathbf{x})_i - (L \otimes_\omega \mathbf{x})_j \leq s_n - s_1$$

Proof. We will prove this by contradiction. Assume there exists $b \in \mathbb{R}^n$ such that $L \otimes_\omega \mathbf{x} = b$, where there exist indices i, j satisfying $(L \otimes_\omega \mathbf{x})_i - (L \otimes_\omega \mathbf{x})_j > s_n - s_1$ for some $\omega = q/n$ with $q \in [n]$. First, expand the expressions of $(L \otimes_\omega \mathbf{x})_i$ and $(L \otimes_\omega \mathbf{x})_j$

$$(L \otimes_\omega \mathbf{x})_i = \bigoplus_{\omega} \{L(i,m) + x_m \mid m = 1, 2, \ldots, n\}$$
$$(L \otimes_\omega \mathbf{x})_j = \bigoplus_{\omega} \{L(j,t) + x_t \mid t = 1, 2, \ldots, n\}$$

Thus, subsequently, with $\mathcal{S}_1$ and $\mathcal{S}_2$ being the collections containing all subsets of $[n]$ with q and $n+1-q$ elements, respectively, it follows that

$$\begin{aligned}
(L \otimes_\omega \mathbf{x})_i - (L \otimes_\omega \mathbf{x})_j &= \bigoplus_{M \in \mathcal{S}_1}{}' \left\{ \bigoplus_{m \in M} L(i,m) + x_m \right\} - \bigoplus_{T \in \mathcal{S}_2} \left\{ \bigoplus_{t \in T}{}' L(j,t) + x_t \right\} \\
&= \bigoplus_{M \in \mathcal{S}_1}{}' \left\{ \bigoplus_{m \in M} L(i,m) + x_m \right\} + \bigoplus_{T \in \mathcal{S}_2}{}' \left\{ \bigoplus_{t \in T} -L(j,t) - x_t \right\} \\
&= \bigoplus_{(M,T) \in \mathcal{S}_1 \times \mathcal{S}_2}{}' \left\{ \bigoplus_{(m,t) \in S_1 \times S_2} L(i,m) - L(j,t) + x_m - x_t \right\} \\
&> s_n - s_1
\end{aligned}$$

Thus, for all $(M, T) \in \mathcal{S}_1 \times \mathcal{S}_2$,

$$\bigoplus_{(m,t) \in S_1 \times S_2} \{L(i,m) - L(j,t) + x_m - x_t\} > s_n - s_1$$

It is clear that for all m, t we have $L(i,m) - L(j,t) \leq s_n - s_1$. Assuming $x_1 \leq x_2 \leq \cdots \leq x_n$, then among the $\frac{n(n+1)}{2}$ possible differences $x_i - x_j$, at least $q(n+1-q)$ of them satisfy $x_i - x_j \leq 0$. Therefore, the set $\{L(i,m) - L(j,t) + x_m - x_t\}$ contains at least $q(n+1-q)$ elements, each less than or equal to $s_n - s_1$, so we must have:

$$\bigoplus_{(m,t)\in S_1\times S_2} \{L(i,m) - L(j,t) + x_m - x_t\} \leq s_n - s_1$$

This contradicts the previous inequality. Hence, we conclude that $(L \otimes_\omega \mathbf{x})_i - (L \otimes_\omega \mathbf{x})_j \leq s_n - s_1$ for all $\mathbf{x} \in \mathbb{R}^n$. □

Without loss of generality, the eigenvector of $L \in \mathbb{L}_n(S)$ satisfies the following assumptions.

Definition 3.3. *For every Latin square $L \in \mathbb{L}_n(S)$ with eigenvalue λ_q for $q \in [n]$, define $\mathbb{E}_q(L)$ as the set of vectors $\mathbf{x}$ satisfying:*

- $L \otimes_{q/n} \mathbf{x} = \lambda_q \otimes \mathbf{x}$,
- $x_i \geq 0$ *for all* $i \in [n]$,
- $x_j = 0$ *for some* $j \in [n]$.

Generally speaking, $\mathbb{E}_q(L)$ denotes the set of eigenvectors $\mathbf{x}$ of L corresponding to the eigenvalue λ_q, normalized so that all entries are nonnegative and at least one entry is equal to zero. The following proposition asserts the over-approximation for $\mathbb{E}_q(L)$. Hence, in general, the set $\{\mathbf{x} \in \mathbb{R}^n \mid 0 \leq x_i \leq s_n - s_1\}$ may contain vectors that do not satisfy the eigenvector equation.

Proposition 3.4. *Let $S = \{s_1, \ldots, s_n\}$ be a set where $s_1 < s_2 < \ldots < s_n$ and let $\omega = q/n$ for $q \in [n]$. If $L \in \mathbb{L}_n(S)$ and $\mathbf{x} \in \mathbb{E}_q(L)$, then $0 \leq x_i \leq s_n - s_1$. In other words, $\mathbb{E}_q(L) \subseteq \{\mathbf{x} \in \mathbb{R}^n \mid 0 \leq x_i \leq s_n - s_1\}$ for every $i \in [n]$.*

Proof. We will prove that $\mathbb{E}_q(L) \subseteq \{\mathbf{x} \in \mathbb{R}^n \mid 0 \leq x_i \leq s_n - s_1\}$ for every $i \in [n]$. Let $S = \{s_1, \ldots, s_n\}$ where $s_1 < s_2 < \ldots < s_n$, and let $\omega = q/n$ for $q \in [n]$. Take $L \in \mathbb{L}_n(S)$ and suppose $\mathbf{x} \in \mathbb{E}_q(L)$ so that

$$L \otimes_{q/n} \mathbf{x} = \lambda_q \otimes \mathbf{x},$$

$x_i \geq 0$ for all $i \in [n]$, and there exists some $t \in [n]$ such that $x_t = 0$. Then, by Proposition 3.2, we obtain

$$(L \otimes_\omega \mathbf{x})_i - (L \otimes_\omega \mathbf{x})_j \leq s_n - s_1$$

for every $i, j \in [n]$. Therefore, we have:

$$\begin{aligned}(\lambda_q \otimes \mathbf{x})_i - (\lambda_q \otimes \mathbf{x})_j &\leq s_n - s_1\\ (\lambda_q + x_i) - (\lambda_q + x_j) &\leq s_n - s_1\\ x_i - x_j &\leq s_n - s_1.\end{aligned}$$

Now take $j = t$ such that $x_t = 0$, then for every $i \in [n]$ we have

$$x_i \leq s_n - s_1$$

and since $x_i \geq 0$, it follows that $0 \leq x_i \leq s_n - s_1$. Therefore, we conclude that $\mathbb{E}_q(L) \subseteq \{\mathbf{x} \in \mathbb{R}^n \mid 0 \leq x_i \leq s_n - s_1\}$ for all $i \in [n]$. □

Proposition 3.5. *Let $S = \{s_1, \ldots, s_n\}$ be a set such that $s_1 < s_2 < \ldots < s_n$, and let $\omega = q/n$ for $q \in [n]$. If $L \in \mathbb{L}_n(S)$ and $\mathbf{x} \in \mathbb{E}_q(L)$, then for every $d \in \mathbb{R}$ it holds that $\lambda_q(d \otimes L) = d \otimes \lambda_q(L)$ and $\mathbf{x} \in \mathbb{E}_q(d \otimes L)$.*

Proof. Let $S = \{s_1, \ldots, s_n\}$ be a set such that $s_1 < s_2 < \ldots < s_n$, and let $\omega = q/n$ for $q \in [n]$. Take any $L \in \mathbb{L}_n(S)$ and $\mathbf{x} \in \mathbb{E}_q(L)$. We will prove that for every $d \in \mathbb{R}$,

$$(d \otimes L) \otimes_{q/n} \mathbf{x} = (d \otimes \lambda_q(L)) \otimes \mathbf{x},$$

or equivalently, $\mathbf{x} \in \mathbb{E}_q(d \otimes L)$. First, since $\mathbf{x} \in \mathbb{E}_q(L)$, it satisfies

$$L \otimes_{q/n} \mathbf{x} = \lambda_q(L) \otimes \mathbf{x}.$$

Now observe that

$$\begin{aligned}(d \otimes L) \otimes_{q/n} \mathbf{x} &= d \otimes (L \otimes_{q/n} \mathbf{x}) \\ &= d \otimes (\lambda_q(L) \otimes \mathbf{x}) \\ &= (d \otimes \lambda_q(L)) \otimes \mathbf{x}.\end{aligned}$$

Hence, for all $d \in \mathbb{R}$, it is proven that $\lambda_q(d \otimes L) = d \otimes \lambda_q(L)$ and $\mathbf{x} \in \mathbb{E}_q(d \otimes L)$. □

Corollary 3.6. *Let $S = \{s_1, \ldots, s_n\}$ be a set such that $s_1 < s_2 < \ldots < s_n$, and let $\omega = q/n$ for $q \in [n]$. If $L \in \mathbb{L}_n(S)$ and $\mathbf{x} \in \mathbb{E}_q(L)$, then $\lambda_q(-s_q \otimes L) = 0$ and $\mathbf{x} \in \mathbb{E}_q(-s_q \otimes L)$.*

Proof. By Theorem 3.1, we have $\lambda_q(L) = s_q$. Therefore, applying Proposition 3.5 with $d = -s_q$ yields $\lambda_q(-s_q \otimes L) = -s_q \otimes \lambda_q(L) = -s_q + s_q = 0$. □

Theorem 3.7. *Let $L \in \mathbb{L}_n(S)$ be a Latin square with $S = \{s_1, \ldots, s_n\}$ such that $s_1 < s_2 < \ldots < s_n$, and let $\omega = q/n$ for $q \in [n]$. For every $\mathbf{x} \in \mathbb{E}_q(L)$, if $L(i,j) = \lambda_q$ and $x_i = 0$, then $x_j = 0$.*

Proof. Assume $L_{\lambda_q} = -\lambda_q(L) \otimes L$, so that $\lambda_q(L_{\lambda_q}) = 0$ for some $q \in [n]$. Since $\lambda_q = s_q$ when $s_1 < s_2 < \cdots < s_n$, then $L_{\lambda_q} \in \mathbb{L}_n(S_\lambda)$ where $S_\lambda = \{s_1 - s_q, s_2 - s_q, \cdots, 0, s_{q+1} - s_q, \cdots, s_n - s_q\}$, which contains $q-1$ negative elements and $n-q$ positive elements.

Let H be a matrix defined by $H(k,m) = L(k,m) + x_m$ for all $k, m \in \{1, 2, \cdots, n\}$. Since $L(i,j) = \lambda_q = s_q$ and $x_i = 0$, consider row $k = i$, so we have

$$\bigoplus_{\omega} \{H(i,m) \mid m = 1, 2, \cdots, n\} = 0 \tag{8}$$

with

$$H(i,m) = \begin{cases} (s_r - s_q) + x_m, & \text{if } m \neq j, \\ x_j, & \text{if } m = j, \end{cases}$$

for $m \in [n]$. The equation (8) has a solution if there are at least q non-positive elements or $n+1-q$ non-negative elements. Since we assume each $x_m \geq 0$, clearly for each $r > q$, $(s_r - s_q) + x_m > 0$, which gives $n - q$ positive elements. Therefore, the remaining q elements in $[H]_{i,-}$ must be non-positive. It follows that $x_j \leq 0$, and since $x_m \geq 0$ for all $m \in [n]$, the only possible solution is $x_j = 0$. □

Definition 3.8. *Let $\sigma \in S_n$ be a permutation with $\sigma = c_1\, c_2 \cdots c_m$ and $c_i = (b_{1i}\ b_{2i}\ \cdots\ b_{ki})$ a cycle in σ for each $i \in \{1, 2, \ldots, m\}$. Define a column vector of size $n \times 1$ associated with the cycle c_i, such that the entries take the value e at the positions b_{ji} and the value f at other positions, for all $j \in \{1, 2, \ldots, k\}$, denoted by $\boldsymbol{v}^n_{c_i}(e, f)$.*

Theorem 3.9. *Let $S = \{s_1, \ldots, s_n\}$ be a set with $s_1 < s_2 < \ldots < s_n$ and let $\omega = q/n$ for some $q \in [n]$. Let $L \in \mathbb{L}_n(S)$ and suppose that the permutation of $\lambda_q(L)$ is $\sigma_{\lambda_q} = c_1\, c_2 \cdots c_m$, where each cycle is of the form $c_i = (b_{1i}\ b_{2i}\ \cdots\ b_{ki})$. If $\mathbf{x} \in \mathbb{E}_q(L)$, then*

$$\mathbf{x} \in \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \boldsymbol{v}_{c_i}(0, [0, s_n - s_1])\}\,.$$

Proof. From Proposition 3.4, it follows that for any $\mathbf{x} \in \mathbb{E}_q(L)$, we have $0 \leq x_i \leq s_n - s_1$. Then, from Proposition 3.7, if for some cycle $c_i = (b_{1i}\ b_{2i}\ \cdots\ b_{ki})$ we choose $x_{b_{1i}} = 0$, it follows that $x_{b_{2i}} = x_{b_{3i}} = \cdots = x_{b_{ki}} = 0$. Hence, the vector $\mathbf{x}$ takes value 0 at the positions b_{ji} and values in the interval $[0, s_n - s_1]$ at the other positions, for all $j \in \{1, 2, \ldots, k\}$, which can be denoted as $\mathbf{v}_{c_i}(0, [0, s_n - s_1])$. □

Definition 3.10. *Let $S = \{s_1, \ldots, s_n\}$ be a set such that $s_1 < s_2 < \ldots < s_n$ and let $\omega = q/n$ for some $q \in [n]$. Define $I_{c_i}(0)$ as the set whose elements are vectors of length n of the form $\boldsymbol{v}_{c_i}(0, [0, s_n - s_1])$, where $c_i = (b_{1i}\ b_{2i}\ \cdots\ b_{ki})$ is a cycle in σ_{λ_q}.*

Next, suppose $I_{c_i}(k) = [\mathbf{u} \mid \mathbf{w}]$, such that for every $\mathbf{x} \in I_{c_i}(k)$, we have $u(j) \leq x(j) \leq w(j)$ for all $j \in [n]$. Then, define

$$\{L \otimes_\omega \mathbf{x} \mid \mathbf{x} \in I_{c_i}(k)\} \subseteq \mathrm{Im}(I_{c_i}(k)),$$

where $\mathrm{Im}(I_{c_i}(k)) = [L \otimes_\omega \mathbf{u} \mid L \otimes_\omega \mathbf{w}]$. This leads to the following equation:

$$I_{c_i}(k+1) = I_{c_i}(k) \cap \mathrm{Im}(I_{c_i}(k)), \tag{9}$$

for some integer $k \geq 0$, $\omega = q/n$, and $L \in \mathbb{L}_n(S)$ with $S = \{s_1, s_2, \ldots, s_n\}$. Next, an algorithm is presented to obtain the non-trivial eigenvector of a *Latin square* $L \in \mathbb{L}_n(S)$.

Algorithm 1. *Non-Trivial Eigenvector of $L \in \mathbb{L}_n(S)$*

1. *For each $q \in [n]$, obtain $\lambda_q(L) = \bigoplus_\omega S = s_q$.*
2. *Obtain the permutation of $\lambda_q(L)$, $\sigma_{\lambda_q} = c_1\, c_2 \cdots c_m$.*
3. *For each c_i with $i \in \{1, 2, \cdots, m\}$, define the set of vectors $I_{c_i}(0) = v_{c_i}(0, [0, s_n - s_1])$.*
4. *Perform the iteration in (9) until a non-negative integer $p \geq 0$ is found such that $I_{c_i}(p+1) = I_{c_i}(p)$.*
5. *Define the set of non-trivial eigenvectors as $I_{c_i}(p)$.*

Then, an example is given on how to obtain a set of non-trivial eigenvectors using Algorithm 1.

Example 3.11. *Define a* 4×4 *Latin square matrix* L *with* $\omega = 2/4$ *and obtain its eigenvalue 0.*

$$L = \begin{bmatrix} 0 & 1 & 2 & -1 \\ -1 & 0 & 1 & 2 \\ 1 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}$$

From the matrix L*, the permutation cycle of element 0 is* $c_1 = (1), c_2 = (2)$*, and* $c_3 = (3\ 4)$*, so based on Theorem 3.9, we obtain* $I_{c_1}(0), I_{c_2}(0), I_{c_3}(0)$ *as follows.*

$$I_{c_1}(0) = \begin{bmatrix} 0 \\ [0,3] \\ [0,3] \\ [0,3] \end{bmatrix}, I_{c_2}(0) = \begin{bmatrix} [0,3] \\ 0 \\ [0,3] \\ [0,3] \end{bmatrix}, I_{c_3}(0) = \begin{bmatrix} [0,3] \\ [0,3] \\ 0 \\ 0 \end{bmatrix}$$

From $I_{c_1}(0)$ *we obtain* $Im(I_{c_1}(0)) = \begin{bmatrix} [0,2] & [0,3] & [0,2] & [0,2] \end{bmatrix}^\top$*, so that it then applies*

$$\{L \otimes_{2/4} \mathbf{x} \mid \mathbf{x} \in I_{c_1}(0)\} \subseteq \begin{bmatrix} [0,2] \\ [0,3] \\ [0,2] \\ [0,2], \end{bmatrix}$$

next, iterate (9) *until an integer* $p \geq 0$ *is found such that* $I_{c_1}(p+1) = I_{c_1}(p)$

$$I_{c_1}(1) = \begin{bmatrix} 0 \\ [0,3] \\ [0,2] \\ [0,2] \end{bmatrix} \to I_{c_1}(2) = \begin{bmatrix} 0 \\ [0,3] \\ [0,1] \\ [0,2] \end{bmatrix} \to I_{c_1}(3) = \begin{bmatrix} 0 \\ [0,2] \\ [0,1] \\ [0,1] \end{bmatrix} \to I_{c_1}(4) = \begin{bmatrix} 0 \\ [0,2] \\ [0,1] \\ [0,1] \end{bmatrix}.$$

We can find $p = 3$ *such that* $I_{c_1}(4) = I_{c_1}(3)$*. Perform the calculation* $L \otimes_{2/4} I_{c_1}(4)$ *as follows.*

$$L \otimes_{2/4} I_{c_1}(4) = \begin{bmatrix} 0 & [1,3] & [2,3] & [-1,0] \\ -1 & [0,2] & [1,2] & [2,3] \\ 1 & [2,4] & [-1,0] & [0,1] \\ 2 & [-1,1] & [0,1] & [1,2] \end{bmatrix}$$

Using the same steps, calculate $I_{c_2}(0)$ *and* $I_{c_3}(0)$*.*

$$\{L \otimes_{2/4} \mathbf{x} \mid \mathbf{x} \in I_{c_2}(0)\} \subseteq \begin{bmatrix} [0,2] \\ [0,2] \\ [0,2] \\ [0,3] \end{bmatrix}$$

$$I_{c_2}(1) = \begin{bmatrix} [0,2] \\ 0 \\ [0,2] \\ [0,3] \end{bmatrix} \to I_{c_2}(2) = \begin{bmatrix} [0,2] \\ 0 \\ [0,2] \\ [0,2] \end{bmatrix} \to I_{c_2}(3) = \begin{bmatrix} [0,1] \\ 0 \\ [0,2] \\ [0,2] \end{bmatrix} \to I_{c_2}(4) = \begin{bmatrix} [0,1] \\ 0 \\ [0,2] \\ [0,2] \end{bmatrix}$$

$$L \otimes_{2/4} I_{c_2}(4) = \begin{bmatrix} [0,1] & 1 & [2,4] & [-1,1] \\ [-1,0] & 0 & [1,3] & [2,4] \\ [1,2] & 2 & [-1,1] & [0,2] \\ [2,3] & -1 & [0,2] & [1,3] \end{bmatrix}$$

$$\{L \otimes_{2/4} \mathbf{x} \mid \mathbf{x} \in I_{c_3}(0)\} \subseteq \begin{bmatrix}[0,2]\\ [0,2]\\ 0\\ [0,1]\end{bmatrix}$$

$$I_{c_3}(1) = \begin{bmatrix}[0,2]\\ [0,2]\\ 0\\ 0\end{bmatrix} \to I_{c_3}(2) = \begin{bmatrix}[0,2]\\ [0,1]\\ 0\\ 0\end{bmatrix} \to I_{c_3}(3) = \begin{bmatrix}[0,2]\\ [0,1]\\ 0\\ 0\end{bmatrix}$$

$$L \otimes_{2/4} I_{c_3}(3) = \begin{bmatrix}[0,2] & [1,2] & 2 & -1\\ [-1,1] & [0,1] & 1 & 2\\ [1,3] & [2,3] & -1 & 0\\ [2,4] & [-1,0] & 0 & 1\end{bmatrix}$$

It can be seen from the previous example that a set of non-trivial eigenvectors is obtained in maxmin-omega in the Latin square. So it can be confirmed that every element of the set $I_{c_i}(p)$ found is an eigenvector, but it is possible that there are other eigenvectors that are not included in the set.

4. CONCLUDING REMARKS

This paper examined the eigenproblem for Latin squares within the maxmin-ω algebraic framework, a generalization of max-plus and min-plus systems parameterized by $0 < \omega \leq 1$. For a Latin square of order n with elements from an ordered set $S = \{s_1, s_2, \ldots, s_n\}$ where $s_1 < s_2 < \ldots < s_n$, we established that the trivial eigenvector is $[k\ k\ \ldots\ k]^\top$ for any $k \in \mathbb{R}$ with the corresponding unique eigenvalue equal to $\bigoplus_\omega S = s_q$ where $q = \lceil \omega n \rceil$. Furthermore, the non-trivial eigenvectors can be systematically derived using an iterative process based on the cycle decomposition of the permutation associated with the eigenvalue.

Future research may focus on extending these techniques to generalized eigenmodes, non-Latin square matrices, and applications such as two-sided linear systems and real-world dynamic systems where maxmin-ω models naturally arise.

Data Availability Statement No data was used for the research described in the article.

Declarations.The authors declare no conflict of interest.

Author Contributions. Nur Millatul Af Idah: Conceptualization, methodology, writing-original draft, validation. Muhammad Syifa'ul Mufid: Formal analysis, validation, writing-review & editing. All authors discussed the results and contributed to the final manuscript.

Funding. This research was funded by the Department of Mathematics, Institut Teknologi Sepuluh Nopember, Surabaya.

Acknowledgment. The authors acknowledge the support from the Department of Mathematics, Institut Teknologi Sepuluh Nopember, Surabaya.

REFERENCES

[1] Subiono, F. Kistosil, and D. Adzkiya, "Generalized public transportation scheduling using max-plus algebra," *Kybernetika*, vol. 54, 2018. http://dx.doi.org/10.14736/kyb-2018-2-0243.

[2] S. Kubo and K. Nishinari, "Applications of max-plus algebra to flow shop scheduling problems," *Discrete Applied Mathematics*, vol. 247, pp. 278–293, 2018. https://www.sciencedirect.com/science/article/pii/S0166218X18301446.

[3] Y. I. Mahjoub, A. Nait-Sidi-Moh, E. C. E. Alaoui, and A. Tajer, "Petri nets conflicts resolution for performance evaluation and control of urban bus networks: a (max, +)-based approach," *Transportmetrica A: Transport Science*, vol. 16, no. 2, pp. 164–193, 2020. https://doi.org/10.1080/23249935.2018.1527864.

[4] C. A. Brackley, D. S. Broomhead, M. C. Romano, and M. Thiel, "A max-plus model of ribosome dynamics during mrna translation," *Journal of Theoretical Biology*, vol. 303, pp. 128–140, 2012. https://doi.org/10.1016/j.jtbi.2012.03.007.

[5] Subiono, "On classes of min-max-plus systems and their applications," 01 2000. https://cir.nii.ac.jp/crid/1971712334827713412.

[6] E. L. Patel, *Maxmin-Plus Models of Asynchronous Computation*. PhD thesis, The University of Manchester (United Kingdom), 2012. https://www.proquest.com/openview/af4b54eca2c46b010ea93084b55b35c0.

[7] M. S. Mufid, E. Patel, and S. Sergeev, "Solving linear equations over maxmin-ω systems," *Linear Algebra and its Applications*, 2023. https://doi.org/10.1016/j.laa.2023.10.012.

[8] M. S. Mufid and Subiono, "Eigenvalues and eigenvectors of latin squares in max-plus algebra," *Journal of the Indonesian Mathematical Society*, vol. 20, p. 37–45, May 2014. https://www.doi.org/10.22342/jims.20.1.178.37-45.

[9] A. D. Keedwell and J. Dénes, *Latin squares and their applications*. Elsevier, 2015.

[10] F. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat, *Synchronization and Linearity*. John Wiley and Sons, Now York, 1992. https://www.rocq.inria.fr/metalau/cohen/documents/BCOQ-book.pdf.

[11] A. Abate, A. Cimatti, A. Micheli, and M. S. Mufid, "Computation of the transient in max-plus linear systems via smt-solving," in *Formal Modeling and Analysis of Timed Systems: 18th International Conference, FORMATS 2020, Vienna, Austria, September 1–3, 2020, Proceedings 18*, pp. 161–177, Springer, 2020. https://link.springer.com/chapter/10.1007/978-3-030-57628-8_10.

[12] Subiono and J. W. v. d. Woude, "Power algorithms for (max,+)-and bipartite(min,max,+)-systems," *Discrete Event Systems*, vol. 10, 2000. https://link.springer.com/article/10.1023/A:1008315821604.