

A Bayesian Approach in Pricing Insurance under Stochastic Gompertz Model with Health Risk Factors

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Abstract. This study aims to enhance the Gompertz mortality model by incorporating stochastic parameters and providing a Bayesian framework for pricing term life insurance based on policyholders' survival and familial health history. The Vasicek model is used to describe the temporal evolution of these stochastic parameters, capturing their mean-reverting behavior, with its distribution at a fixed time point serving as the prior in the Bayesian framework. Bayes' rule is used to update these priors using observed survival and death outcomes along with familial health factors. Using illustrative data for 30-year-old policyholders, prior and posterior prices are computed across different health profiles, showing that posterior prices can increase or decrease relative to prior-based prices depending on the observed risk patterns. These results illustrate how incorporating observed survival and health information refines pricing and aligns prices with actual risk across policyholder profiles.

Key words and Phrases: Bayesian inference, Gompertz model, Insurance pricing, Stochastic differential equation, Vasicek model.

1. INTRODUCTION

Mortality modeling plays a significant role in actuarial science, forming the basis for life insurance pricing, pension systems, and risk management. In 1825, Benjamin Gompertz observed that, across specific age intervals, the differences between the common logarithms of the number of surviving individuals remained nearly constant [1]. This observation led to the classical Gompertz law:

$$\mu_x = Ae^{Bx}, \quad (1)$$

where A and B are fixed parameters. The Gompertz model provided a foundational tool for life contingency calculations; however, it does not fully capture mortality dynamics over the entire age range. In 1860, William Makeham extended the

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Gompertz law by adding an age-independent constant term to account for the more rapid increase in mortality at older ages:

$$\mu_x = Ae^{Bx} + C, \quad (2)$$

highlighting limitations in purely age-based mortality laws [2].

While these classical models have been successful at the population level, contemporary actuarial practice increasingly demands models that account for *both systematic mortality trends and individual-level heterogeneity*. Advances in medical diagnostics, electronic health records, and underwriting have revealed substantial variation in mortality risk among individuals of the same age, driven by health conditions, family history, and lifestyle factors. Classical deterministic models like Gompertz and Makeham cannot adequately capture this heterogeneity, leading to potential mispricing and cross-subsidization in insurance portfolios, particularly for short-term products where premiums are highly sensitive to individual risk characteristics [3, 4].

In response to these limitations, this study proposes a **hybrid Bayesian mortality modeling framework** that extends the classical Gompertz law by incorporating *time-varying stochastic parameters* and *individual health risk factors*. Specifically, the Gompertz parameters are modeled as stochastic processes to capture systematic mortality evolution over time, while health indicators enter multiplicatively to account for individual-level heterogeneity. Formally, we define the extended mortality function as:

$$\mu_{x,t} = A_t e^{B_t x} + \lambda_1 D_1 + \lambda_2 D_2, \quad (3)$$

where $D_i \in \{0, 1\}$ indicate the presence of specific health conditions, and λ_i quantify their effect on mortality. The parameters A_t and B_t are modeled as *time-varying stochastic processes*, allowing the model to reflect historical trends and forecast future mortality more accurately.

The **novelty** of this approach lies in the integration of stochastic mortality evolution, health covariates, and Bayesian inference into a unified framework. This addresses three critical gaps in existing mortality modeling literature:

- (1) **Accounting for individual health heterogeneity:** By incorporating health-related covariates, the model captures variations in mortality risk that deterministic models cannot [5].
- (2) **Stochastic mortality :** The time-varying parameters, modeled via the Vasicek mean-reverting process, enable the model to incorporate uncertainty and temporal dynamics in mortality trends, which deterministic approaches cannot address [6, 7].
- (3) **Bayesian inference:** By specifying prior distributions for all unknown parameters, the Bayesian framework allows prior actuarial and medical knowledge to be formally incorporated and updated as new data become available, improving inference under limited or heterogeneous data conditions [8, 9].

Compared to population-level models such as Lee–Carter [10] and its extensions, including the Renshaw–Haberman age–period–cohort model [11], the proposed framework offers a *direct and interpretable link between health factors and individual mortality*, while maintaining analytical tractability suitable for practical insurance pricing. Unlike models that rely heavily on latent factors and complex parameter estimation, our approach is particularly suitable for datasets where individual-level information is sparse or heterogeneous.

This study applies the hybrid Bayesian framework to the pricing of **one-year term life insurance policies**, demonstrating how new health information or recent mortality observations can dynamically update risk estimates and improve premium calculations. By capturing both *systematic mortality uncertainty* and *individual risk heterogeneity*, the model facilitates more *equitable, personalized, and data-informed insurance pricing*, bridging the gap between classical actuarial models and modern requirements for precision and personalization.

In summary, this research extends the classical Gompertz law by:

- Incorporating *time-varying stochastic parameters* to reflect evolving mortality trends.
- Including *health-related risk factors* to capture individual-level heterogeneity.
- Employing a *Bayesian framework* to formally integrate prior knowledge and handle parameter uncertainty in limited data settings.

This hybrid approach represents a meaningful advancement in mortality modeling, providing a *practical, interpretable, and robust framework* for modern actuarial applications, particularly in personalized life insurance pricing.

2. PRELIMINARIES

In this section, we will define important terms and introduce theorems that will be useful throughout the rest of the chapters, taken mainly from [12], [13], [14] and [15].

Definition 2.1. Let (x) denote a life aged $x \geq 0$, and let T_x be the random variable that represents the future lifetime of (x) , i.e., the time until death, with $T_x \geq 0$. The **Gompertz law of mortality** at age x , denoted by μ_x , is defined as:

$$\mu_x = Ae^{Bx},$$

where:

- $A > 0$ is the baseline mortality rate (age-independent),
- $B > 0$ is the rate of aging (governs the exponential increase of mortality with age).

Moreover, the cumulative distribution function for the Gompertz model is given by

$${}_tq_x = Pr[T_x \leq t] = 1 - \exp \left\{ -\frac{A}{B} \left(e^{B(x+t)} - e^{Bx} \right) \right\},$$

and the survival function under the Gompertz model is given by

$${}_t p_x = 1 - {}_t q_x = \exp \left\{ -\frac{A}{B} \left(e^{B(x+t)} - e^{Bx} \right) \right\}.$$

Definition 2.2. Let $\{X_{t_i}\}_{i=1}^N$ be the data observed over consistent time intervals with transition densities $p(t_{i+1}, X_{t_{i+1}}; t_i, X_{t_i} | \theta)$. The joint density of these observations is

$$p(X_{t_1}, \dots, X_{t_N} | \theta) = p(X_{t_1} | \theta) \prod_{i=1}^{N-1} p(t_{i+1}, X_{t_{i+1}}; t_i, X_{t_i} | \theta),$$

where $p(X_{t_1} | \theta)$ is some prior density for X_{t_1} . The **likelihood function** is given by

$$L(\theta) = \prod_{i=1}^{N-1} p(t_{i+1}, X_{t_{i+1}}; t_i, X_{t_i} | \theta),$$

and the **maximum likelihood estimator** for θ is given by

$$\hat{\theta} = \arg \max_{\theta \in \Omega_\theta} L(\theta),$$

where Ω_θ is the set of all possible values of θ .

Definition 2.3. The **prior distribution** is a probability distribution over the space of possible parameter values. It is denoted by $\pi(\theta)$ and represents our belief about the relative likelihood that different values of θ are the true value of the parameter.

Definition 2.4. The **model distribution** is the probability distribution of the collected data given a specific value of the parameter. Its probability density function (pdf) is denoted by $f_{X|\theta}(x|\theta)$, where vector notation for x is used to emphasize that all the data are included.

If the vector of observations $x = (x_1, \dots, x_q)$ consists of independent and identically distributed (i.i.d.) random variables, then

$$f_{X|\theta}(x|\theta) = f_{X|\theta}(x_1|\theta) \times \dots \times f_{X|\theta}(x_q|\theta).$$

We use concepts from multivariate statistics to introduce two definitions. In both cases, as well as in what follows, integrals should be replaced by sums if the distributions are discrete.

Definition 2.5. The **joint distribution** of the data and the parameter has the pdf

$$f_{X,\theta}(x, \theta) = f_{X|\theta}(x|\theta) \pi(\theta)$$

where $f_{X|\theta}(x|\theta)$ is the conditional pdf of the data given the parameter, and $\pi(\theta)$ is the prior distribution of the parameter.

Definition 2.6. The *marginal distribution* of x has the pdf

$$f_X(x) = \int f_{X|\theta}(x|\theta)\pi(\theta) d\theta$$

where the integral is taken over all possible values of θ .

Note that if there is more than one parameter, this equation will be a multiple integral.

Definition 2.7. The *posterior distribution* is the conditional probability distribution of the parameters given the observed data. It is denoted $\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x})$.

Definition 2.8. The *predictive distribution* is the conditional probability distribution of a new observation y given the data $\mathbf{x}$. It is denoted $f_{Y|\mathbf{X}}(y|\mathbf{x})$.

Theorem 2.9 (Bayes' Theorem). Let A and B be events defined on a common probability space. Suppose $Pr(B) > 0$. Then, the conditional probability of A given B is given by:

$$Pr(A | B) = \frac{Pr(B | A) Pr(A)}{Pr(B)},$$

where the left side $Pr(A | B)$ is the posterior probability, B is the event for the observations, and A is the event for the prior distribution.

Bayes' Theorem provides a principled mechanism for updating beliefs about a random variable A in light of new evidence B , and serves as a foundational tool in Bayesian statistics.

Theorem 2.10. The posterior distribution can be computed as

$$\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) = \frac{f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta)\pi(\theta)}{\int f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta)\pi(\theta) d\theta},$$

while the predictive distribution can be computed as

$$f_{Y|\mathbf{X}}(y|\mathbf{x}) = \int f_{Y|\Theta}(y|\theta)\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) d\theta,$$

where $f_{Y|\Theta}(y|\theta)$ is the pdf of the new observation, given the parameter value.

The derivation of important results later on will require understanding of concepts in Stochastic Calculus. As such, we lay out important related definitions and theorems.

Definition 2.11. A stochastic process $B = (B_t, t \in [0, \infty))$ is called a (**standard**) **Brownian motion** or a **Wiener process** if the following conditions are satisfied:

- It starts at zero, i.e., $B_0 = 0$.
- It has independent increments, that is, for every $t_1 < t_2 < \dots < t_n$ with $t_i \in [0, \infty)$ and $n \neq 1$, the increments

$$B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \dots, B_{t_n} - B_{t_{n-1}}$$

are independent random variables.

- For $t > 0$, B_t has a normal distribution with mean 0 and variance t , i.e., $B_t \sim \mathcal{N}(0, t)$.
- It has continuous sample paths, i.e., for all random outcomes $\omega \in \Omega$, the function $t \mapsto B_t(\omega)$ is continuous.

Theorem 2.12. *Given a Brownian motion $X = (X_t, t \geq 0)$, for $0 \leq s < t$ we have*

- (1) $\mathbb{E}[X_t - X_s] = 0$
- (2) $\mathbb{E}[(X_t - X_s)^2] = t - s$
- (3) $\text{Var}[X_t - X_s] = t - s$
- (4) The random variables X_s and $X_t - X_s$ are independent

Definition 2.13. *A **stochastic differential equation** (SDE) is an equation of the form*

$$dX_u = \beta(u, X(u)) du + \gamma(u, X(u)) dW_u,$$

where $\beta(u, X(u))$ is the drift term, $\gamma(u, X(u))$ is the diffusion term, and $W = (W_u, u \in [0, \infty))$ is a Wiener process.

Definition 2.14. *The **Vasicek Model** is a stochastic differential equation given by*

$$dr_t = a(b - r_t) dt + \sigma_1 dW_t,$$

where $W = (W_t, t \in [0, \infty))$ is a Wiener process, b is the mean-reverting parameter, that is, future values of r_t will gather around b in the long run. On the other hand, a is the speed by which future values of r_t gather around b . Lastly, σ_1 is associated with the volatility of values of r_t .

The Vasicek model is a mean-reverting stochastic process commonly used to model interest rates and other quantities like mortality improvement factors.

3. MODELING FRAMEWORK:

The Gompertz model is a widely used tool for modeling human mortality, particularly in long-term data analysis, as the probability of dying increases exponentially with age. However, a limitation is its assumption that all causes of death are age-dependent, which may not always hold true (e.g., accidents).

We introduce non-age-dependent parameters to account for mortality risks unrelated to aging, allowing the model to more accurately reflect the impact of diseases and other non-age factors. Moreover, it can be extended to account for evolving risk factors like diseases. Yashin et al. [16] notes that the additional parameters can be added to adapt the model to emerging health risks, maintaining its interpretability and empirical strength.

3.1. Stochastic Gompertz Model.

To capture the uncertainty and dynamic nature of mortality over time, we introduce a stochastic version of the Gompertz model in which the parameters A and B evolve according to Vasicek-type processes. We model the force of mortality as

$$\mu_{x,t} = A_t e^{B_t x}, \quad (4)$$

where A_t and B_t are stochastic processes governed by the following stochastic differential equations:

$$dA_t = c_A(\bar{A} - A_t) dt + \nu_A dW_t^{(1)} \quad (5)$$

$$dB_t = c_B(\bar{B} - B_t) dt + \nu_B dW_t^{(2)} \quad (6)$$

with

- c_A, c_B as the speed of mean reversion for A_t, B_t ,
- $\bar{A}, \bar{B}$ as the long-term mean level for A_t, B_t ,
- ν_A, ν_B as the volatility parameter for A_t, B_t ,
- $W_t^{(1)}, W_t^{(2)}$ as independent Wiener processes.

This formulation allows both the baseline mortality level and its rate of increase with age to vary randomly over time, providing a more flexible and realistic framework for mortality modeling under uncertainty.

3.2. Stochastic Gompertz Model with Familial Health Risk Factors.

Incorporating additional mortality factors to (4) such as family history of health, we have

$$\mu_{x,t} = A_t e^{B_t x} + \lambda_1 D_1 + \lambda_2 D_2, \quad (7)$$

such that (5) and (6) are satisfied with

- λ_1, λ_2 : *Familial health risk coefficients* (e.g., diabetes, hypertension, lung disease),
- D_1, D_2 : *Indicator variables* for presence of specific familial diseases.

When the indicator variables D_1 and D_2 are binary-valued (0 or 1), they function as discrete covariates representing the presence or absence of specific familial health conditions. Specifically for $i = 1, 2$, $D_i = 1$ denotes the presence of familial disease i , contributing an additive risk factor λ_i to the baseline Gompertz force of mortality. On the other hand, $D_i = 0$ indicates no familial disease risk for that condition, resulting in no incremental mortality effect.

Following this, we have

Case 1: Presence of Disease 1 only

$$\mu_{x,t} = A_t e^{B_t x} + \lambda_1.$$

Case 2: Presence of Disease 2 only

$$\mu_{x,t} = A_t e^{B_t x} + \lambda_2.$$

Case 3: Presence of both Disease 1 and Disease 2

$$\mu_{x,t} = A_t e^{B_t x} + \lambda_1 + \lambda_2.$$

Case 4: No Disease

$$\mu_{x,t} = A_t e^{B_t x}.$$

This formulation allows the mortality model to incorporate individual-specific health risk factors as shifts in the force of mortality model, facilitating personalized mortality projections and risk stratification.

3.3. Bayesian Modeling with Familial Health Risk Factors.

In the Bayesian framework, all model parameters are treated as random variables with associated prior distributions that reflect our initial beliefs before observing any data. For this study, we adopt a stochastic Gompertz model augmented with fixed health risk factors. The survival time of an individual aged x at time $t \geq 0$ is modeled as

$$T_x \mid A_t, B_t, \lambda_1, \lambda_2 \sim \text{SGFHRF}(A_t, B_t, \lambda_1, \lambda_2),$$

where SGFHRF denotes the **Stochastic Gompertz model with Fixed Health Risk Factors**. In this model,

- A_t and B_t govern baseline mortality,
- λ_1 and λ_2 represent additional risk weights associated with disease groups D_1 and D_2 .

3.4. Prior Distributions and Likelihood Function.

Since the Vasicek process assumes normally distributed dynamics, we model A_t and B_t as normally distributed random variables. Specifically, their means and variances are obtained from the analytical solutions of the Vasicek process

$$\begin{aligned} A_t &\sim \mathcal{N}(\mathbb{E}[A_t], \text{Var}[A_t]) \\ B_t &\sim \mathcal{N}(\mathbb{E}[B_t], \text{Var}[B_t]) \end{aligned} \tag{8}$$

where

- for A_t :

$$\begin{aligned} \mathbb{E}[A_t] &= A_0 e^{-c_A t} + \bar{A}(1 - e^{-c_A t}) \\ \text{Var}(A_t) &= \frac{\nu_A^2}{2c_A}(1 - e^{-2c_A t}) \end{aligned}$$

- for B_t :

$$\begin{aligned} \mathbb{E}[B_t] &= B_0 e^{-c_B t} + \bar{B}(1 - e^{-c_B t}) \\ \text{Var}(B_t) &= \frac{\nu_B^2}{2c_B}(1 - e^{-2c_B t}) \end{aligned}$$

For the disease-related weights λ_1 and λ_2 , we use uniform distributions due to limited prior knowledge. The uniform prior reflects a non-informative stance, assigning equal likelihood to all values within a specified interval

$$\lambda_1, \lambda_2 \sim \mathcal{U}(a, b).$$

We experimented with intervals $(0, 1)$, $(0, 10)$, and $(0, 100)$, and observed that both $(0, 10)$ and $(0, 100)$ yielded similar values for the price. Therefore, $(0, 10)$ was deemed sufficient and adapted for both parameters. Hence, we have the following prior distributions for A_t, B_t, λ_1 , and λ_2 .

$$A_t \sim \mathcal{N}(\mathbb{E}[A_t], \text{Var}[A_t]), \quad B_t \sim \mathcal{N}(\mathbb{E}[B_t], \text{Var}[B_t]), \quad \lambda_1, \lambda_2 \sim \mathcal{U}(a, b).$$

Since the parameters are independent random variables, their joint prior distribution is given by the product of their individual prior pdfs expressed as

$$P(A_t, B_t, \lambda_1, \lambda_2) = f_{A_t}(A_t) \cdot f_{B_t}(B_t) \cdot f_{\lambda_1}(\lambda_1) \cdot f_{\lambda_2}(\lambda_2),$$

where the component prior distributions are defined as follows:

$$f_{A_t}(A_t) = \frac{1}{\sqrt{2\pi \text{Var}(A_t)}} \exp \left(-\frac{1}{2} \left(\frac{A_t - \mathbb{E}[A_t]}{\sqrt{\text{Var}(A_t)}} \right)^2 \right),$$

$$f_{B_t}(B_t) = \frac{1}{\sqrt{2\pi \text{Var}(B_t)}} \exp \left(-\frac{1}{2} \left(\frac{B_t - \mathbb{E}[B_t]}{\sqrt{\text{Var}(B_t)}} \right)^2 \right),$$

and for $i = 1, 2$,

$$f_{\lambda_i}(\lambda_i) = \begin{cases} \frac{1}{b-a}, & a \leq \lambda_i \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

3.5. Rationale for Distribution Choices.

Knowing that the stochastic processes A_t and B_t are random variables for a fixed time $t > 0$, we took advantage of this information and used their distributions for a fixed $t > 0$ as their prior distributions.

On the other hand, the disease group weights λ_1 and λ_2 lack strong prior information and are assumed to lie within a plausible range. The uniform distribution allows for flexible modeling with minimal assumptions, while still being bounded.

3.6. Posterior Distribution.

Once data are observed, Bayesian inference updates our beliefs about the model parameters using the Bayes' theorem. Letting $\theta = (A_t, B_t, \lambda_1, \lambda_2)$ and denoting the observed death or survival times as $T_1, \dots, T_n$, the posterior distribution of the parameters is given by

$$P(\theta \mid T_1, \dots, T_n) = \frac{P(T_1, \dots, T_n \mid \theta) \cdot P(\theta)}{\int \dots \int P(T_1, \dots, T_n \mid \tilde{\theta}) \cdot P(\tilde{\theta}) d\tilde{\theta}}.$$

This expression states that the posterior distribution is proportional to the likelihood of the observed data given the parameters, multiplied by the prior distribution. The denominator ensures normalization over the parameter space. Through this process, Bayesian updating incorporates new information from data into our uncertainty about parameter values.

3.7. Parameter Estimation for Stochastic Gompertz Model with Familial Health Risk Factors.

Our Bayesian setup for the SGHRF requires the parameters A_t and B_t to be estimated over time before we are able to estimate the Vasicek model parameters.

These parameters are estimated using **nonlinear least squares (NLS)**, a method that minimizes the sum of squared differences between observed and model-predicted values when the model is nonlinear in its parameters. The objective function is

$$\min_{A, B > 0} \sum_{i=1}^n (y_i - Ae^{Bx})^2,$$

where y_i are observed values. MATLAB's `nlinfit` function was used to perform this estimation, iteratively adjusting the parameters to minimize the error.

3.8. Vasicek Model: Parameter Estimation.

To estimate the parameters of the Vasicek model, we use a method based on Quasi-Maximum Likelihood Estimation (QMLE) in SOA Research Institute [17]. Let $\{r_i\}_{i=0}^n$ represent the observed values, which may correspond to either $\{A_i\}$ or $\{B_i\}$, depending on the process being estimated.

The Vasicek model is given by the discrete-time approximation

$$r_i = \alpha^* + \beta^* r_{i-1} + \varepsilon_i,$$

where $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$. Using equidistant time periods of 1 year, we use the estimates for $\hat{\alpha}^*$, $\hat{\beta}^*$, and $\hat{\sigma}^2$ from [17]:

$$\hat{\beta}^* = \frac{\sum_{i=1}^n r_i r_{i-1} - \frac{1}{n} \left(\sum_{i=0}^{n-1} r_i \right) \left(\sum_{i=1}^n r_i \right)}{\sum_{i=0}^{n-1} r_i^2 - \frac{1}{n} \left(\sum_{i=0}^{n-1} r_i \right)^2}$$

$$\hat{\alpha}^* = \frac{1}{n} \left(\sum_{i=1}^n r_i - \hat{\beta}^* \sum_{i=0}^{n-1} r_i \right)$$

$$\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^n \left(r_i - \hat{\alpha}^* - \hat{\beta}^* r_{i-1} \right)^2.$$

These parameters are then transformed into the Vasicek parameters c , $\bar{r}$, and ν using the formulas from [17]:

$$\begin{aligned} c &= -\ln(\hat{\beta}^*) \\ \bar{r} &= \frac{\hat{\alpha}^*}{1 - \hat{\beta}^*} \\ \nu &= \sqrt{\frac{2c\hat{\sigma}^2}{1 - (\hat{\beta}^*)^2}}. \end{aligned}$$

The parameter estimates were computed using *Mathematica*, following the procedure outlined by the SOA Research Institute [17]. The resulting estimates are then substituted into Equation (8) to construct the prior distributions for the mortality-affecting processes A_t and B_t .

3.9. Standard Error.

Once the estimates have been obtained, their corresponding standard errors are derived from the observed Fisher Information Matrix, denoted by $\mathcal{J}(\hat{\theta})$. This matrix is of dimension $D \times D$, with entries defined as

$$[\mathcal{J}(\hat{\theta})]_{ij} = -\frac{\partial^2 l(\theta)}{\partial \theta_i \partial \theta_j} \Big|_{\theta=\hat{\theta}},$$

where $l(\theta)$ denotes the log-likelihood function.

The standard error associated with the estimate $\hat{\theta}_i$ is given by the square root of the (i, i) -th diagonal element of the inverse Fisher Information Matrix:

$$\text{SE}(\hat{\theta}_i) = \sqrt{[\mathcal{J}^{-1}(\hat{\theta})]_{ii}}.$$

From this, the variance associated with the estimate $\hat{\theta}_i$ is simply the square of $\text{SE}(\hat{\theta}_i)$, i.e., $\text{Var}(\hat{\theta}_i) = (\text{SE}(\hat{\theta}_i))^2$.

Moreover, assuming asymptotic normality of the maximum likelihood estimator $\hat{\theta}_i$, a $100(1 - \alpha)\%$ confidence interval for θ_i can be constructed as

$$\left(\hat{\theta}_i - z_{1-\alpha/2} \text{SE}(\hat{\theta}_i), \hat{\theta}_i + z_{1-\alpha/2} \text{SE}(\hat{\theta}_i) \right),$$

where $z_{1-\alpha/2}$ is the $(1 - \alpha/2)$ quantile of the standard normal distribution. In particular, a 95% confidence interval for θ_i can be constructed as

$$\left(\hat{\theta}_i - 1.96 \text{SE}(\hat{\theta}_i), \hat{\theta}_i + 1.96 \text{SE}(\hat{\theta}_i) \right).$$

In cases where $\mathcal{J}(\hat{\theta})$ is singular and thus non-invertible, specifically, the Moore–Penrose generalized inverse from [18] is adopted, expressed as

$$\mathcal{J}^+(\hat{\theta}) = \left(\mathcal{J}(\hat{\theta})^\top \mathcal{J}(\hat{\theta}) \right)^{-1} \mathcal{J}(\hat{\theta}).$$

Li and Yeh formally established the validity of this approach by demonstrating that the Moore–Penrose pseudoinverse satisfies the Cramér–Rao lower bound even in the case of singular Fisher information matrices [18]. Consequently, the

pseudoinverse provides an asymptotically efficient measure of variance, ensuring that the derived standard errors attain the minimum variance among all admissible estimators under linear constraints.

3.10. Quasi-Log-Likelihood Function.

We follow the approach in [19] for the case of the Vasicek model for A_t given by

$$dA_t = c_A(\bar{A} - A_t)dt + \nu_A dW_t,$$

where $c_A > 0$ is the speed of mean reversion, $\bar{A}$ is the long-run mean level, and $\nu_A > 0$ is the volatility.

Discretizing using the Euler-Maruyama method with $\Delta t = t_{k+1} - t_k = 1$ for $k = 0, \dots, n-1$, we obtain:

$$A_{t_{k+1}} = A_{t_k} + c_A(\bar{A} - A_{t_k}) + \nu_A \epsilon_k, \quad \epsilon_k \sim \mathcal{N}(0, 1).$$

This implies that $A_{t_{k+1}}$ conditional on A_{t_k} is normally distributed:

$$A_{t_{k+1}} | r_{t_k} \sim \mathcal{N}(r_{t_k} + c_A(\bar{A} - A_{t_k}), \nu_A^2).$$

The corresponding log-likelihood function for observations $\{A_0, A_1, \dots, A_n\}$ is:

$$\begin{aligned} \ell(c_A, \bar{A}, \nu_A | A_0, \dots, A_n) &= \sum_{k=0}^{n-1} \ln f(A_{t_{k+1}} | A_{t_k}) \\ &= -\frac{n}{2} \ln(2\pi) - n \ln \nu_A - \frac{1}{2\nu_A^2} \sum_{k=0}^{n-1} (A_{t_{k+1}} - A_{t_k} - c_A(\bar{A} - A_{t_k}))^2. \end{aligned}$$

This allows for efficient estimation of the model parameters $(c_A, \bar{A}, \nu_A)$ by using maximum likelihood techniques. Note that the same method applies directly to a B_t . The discretization and quasi-log-likelihood formulation are structurally identical, with parameters c_B , $\bar{B}$, and ν_B replacing their A -counterparts.

3.11. Bayesian Pricing of 1-Year Term Insurance.

We consider a 1-year term life insurance policy under a stochastic mortality model incorporating health risk factors. The mortality intensity is modeled using a Gompertz function augmented with disease weights

$$\mu_{x,t}(\theta) = A_t e^{B_t x} + \lambda_1 D_1 + \lambda_2 D_2 \tag{9}$$

Here, A_t and B_t are stochastic parameters following Vasicek processes, while λ_1 and λ_2 are weights assigned to the disease indicators D_1 and D_2 , respectively. The full set of parameters in (9) is denoted by

$$\theta = (A_t, B_t, \lambda_1, \lambda_2).$$

3.11.1. Conditional Expectation of Present Value.

Given a specific realization of the parameters θ , the conditional expected present value (EPV) of the 1-year term insurance follows from Slud [20]. The EPV of a 1-year term insurance for a life aged x is given by

$$\mathbb{E} [v^{T_x} \mathbf{1}_{[0,1)}(T_x)] = \int_0^1 v^s \cdot {}_s p_{x,t} \cdot \mu_{x+s,t+s} ds, \quad (10)$$

where:

- ${}_s p_{x,t} = \exp\left(-\int_0^s \mu_{x+r,t+r} dr\right)$ is the probability that a life aged x at time t survives to age $x+s$,
- $v^s = (1+i)^{-s} = e^{-\delta s}$ is the discount factor for a payment at time s .

Under the Constant Force of Mortality (CFM) assumption between integral ages, we assume for a given $t \geq 0$,

$$\mu_{x+s,t+s} = \mu_{x,t} \quad \text{for all } s \in [0, 1].$$

This simplifies the integral in (10) as

$$\begin{aligned} \mathbb{E} [v^{T_x} \mathbf{1}_{[0,1)}(T_x)] &= \int_0^1 e^{-\delta s} \cdot e^{-\mu_{x,t}s} \cdot \mu_{x,t} ds \\ &= \mu_{x,t} \int_0^1 e^{-(\delta + \mu_{x,t})s} ds. \end{aligned}$$

Evaluating the integral, the expectation of a 1 year term insurance is

$$\mathbb{E} [v^{T_x} \mathbf{1}_{[0,1)}(T_x)] = \frac{\mu_{x,t} (1 - e^{-(\delta + \mu_{x,t})})}{\delta + \mu_{x,t}}. \quad (11)$$

Moreover, the expectation of a 1 year term insurance given $\theta = (A_t, B_t, \lambda_1, \lambda_2)$ is

$$\mathbb{E} [v^{T_x} \mathbf{1}_{[0,1)}(T_x) \mid \theta] = \frac{\mu_{x,t}(\theta) (1 - e^{-\delta - \mu_{x,t}(\theta)})}{\delta + \mu_{x,t}(\theta)} \quad (12)$$

This expression arises from the CFM assumption and continuous-time valuation of a 1-year term insurance with force of interest δ .

3.11.2. Unconditional Pricing Based on the Prior.

To compute the unconditional price of the insurance policy, we integrate the conditional expectation over the prior distribution of the parameter θ

$$\mathbb{E} [v^{T_x} \mathbf{1}_{[0,1)}(T_x)] = \int \mathbb{E} [v^{T_x} \mathbf{1}_{[0,1)}(T_x) \mid \theta] P(\theta) d\theta.$$

This represents the EPV using only prior beliefs about the model parameters, before observing any policyholder data.

3.11.3. Bayesian Pricing Based on Posterior Distribution.

Once policyholder data $T_x^{(1)}, \dots, T_x^{(n)}$ are observed, we update our beliefs using Bayes' theorem. The Bayesian price incorporates this information by integrating over the posterior distribution

$$\mathbb{E} [v^{T_x} \mathbf{1}_{[0,1)}(T_x) \mid T_x^{(1)}, \dots, T_x^{(n)}] = \int \mathbb{E} [v^{T_x} \mathbf{1}_{[0,1)}(T_x) \mid \theta] P(\theta \mid T_x^{(1)}, \dots, T_x^{(n)}) d\theta.$$

This is the Bayesian price under posterior distribution, which reflects updated uncertainty in the parameters after accounting for observed data. It offers a more informed and personalized valuation of the policy, particularly useful in cases where the insured belongs to a high-risk health category or where only limited mortality data are available.

4. MAIN RESULTS

This chapter presents the method used to compute the expected premium for a 1-year term life insurance policy using the Gompertz formula and adding familial health risk factor parameter in the existing model, leading to a more informed decision-making. Online software such as Matlab and Mathematica were also used to arrive at the random parameter estimations and to evaluate the expected insurance premium given the new pricing formula.

4.1. Parameter Estimates for A_t, B_t .

We obtained mortality data for Taiwan from the *Human Mortality Database* and fitted the random parameters A and B for each year from 1970 to 2023. Clearly, the data exhibit a clear mean-reverting behavior — they fluctuate over time but tend to return toward a long-term average. This is visualized in the plots below, where the horizontal lines represent the estimated long-term mean levels obtained from the Vasicek model. The estimation procedure is described in the next subsection.

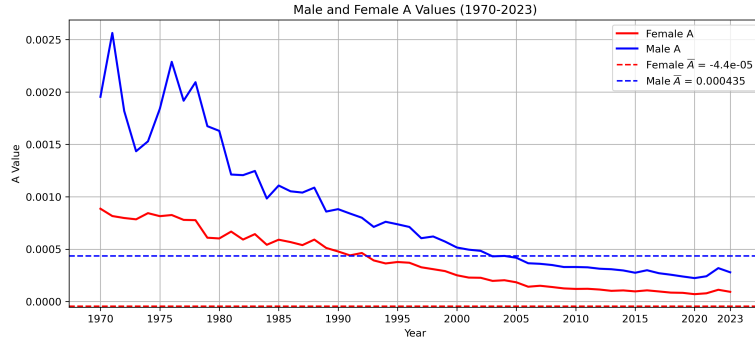


FIGURE 1. Fitted values of random parameter A for years 1970–2023.

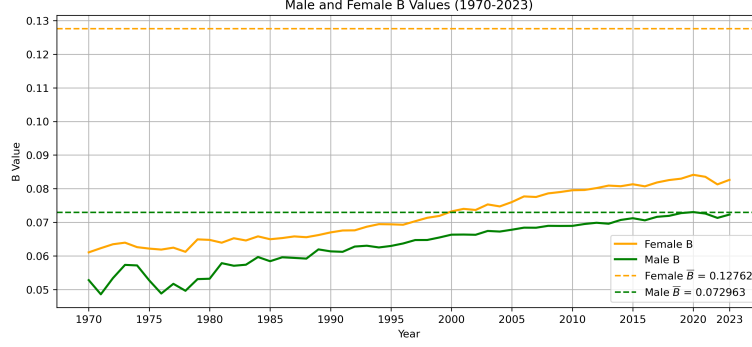


FIGURE 2. Fitted values of random parameter B for years 1970–2023.

Given this observed mean-reverting nature, the Vasicek model emerges as a natural framework to describe the dynamics of these random parameters. This formulation allows the Gompertz parameters in the model $\mu_x = Ae^{Bx}$ to evolve over time, capturing uncertainty and randomness, while still being anchored to a long-term average. Thus, the Vasicek model is both realistic and mathematically tractable, making it a robust choice for modeling mortality parameters in a stochastic setting.

4.2. Vasicek Parameter Estimates for A_t and B_t by Gender.

The continuous-time dynamics of each parameter $X_t \in \{A_t, B_t\}$ follow the stochastic differential equation (SDE)

$$dX_t = c_X(\bar{X} - X_t)dt + \nu_X dW_t,$$

where:

- c_X is the speed of mean reversion,
- $\bar{X}$ is the long-term mean (level to which X_t reverts),
- ν_X is the instantaneous volatility,
- W_t is a standard Brownian motion.

The Vasicek model ensures that the process tends to drift toward the long-term mean $\bar{X}$, with volatility controlled by ν_X . Faster reversion is indicated by a higher c_X , and larger values of ν_X indicate greater uncertainty or fluctuation in the parameter path.

Tables 1–4 show the estimated parameters, the computed standard errors, the variances, and the 95% confidence intervals for the Vasicek models for A_t and B_t for male and female populations:

Interpretation:

- For both sexes, the parameters A_t and B_t exhibit mean-reverting behavior toward their respective long-term means $\bar{A}$ and $\bar{B}$.

TABLE 1. Vasicek Parameter Estimates for A_t (Men)

Parameter	Estimate	Std. Error	Variance	95% CI
c_A	0.077087	0.046222	0.002136	(-0.013508, 0.167682)
$\bar{A}$	0.000435	0.000447	2.00×10^{-7}	(-0.000441, 0.001311)
ν_A	0.000202	0.000021	4.41×10^{-10}	(0.000161, 0.000243)

TABLE 2. Vasicek Parameter Estimates for A_t (Women)

Parameter	Estimate	Std. Error	Variance	95% CI
c_A	0.035050	0.022116	0.000489	(-0.008297, 0.078397)
$\bar{A}$	-0.000044	0.000320	1.024×10^{-7}	(-0.000671, 0.000583)
ν_A	0.000040	0.000004	1.60×10^{-11}	(0.000032, 0.000048)

TABLE 3. Vasicek Parameter Estimates for B_t (Men)

Parameter	Estimate	Std. Error	Variance	95% CI
c_B	0.037771	0.037128	0.001378	(-0.035000, 0.110542)
$\bar{B}$	0.072963	0.011905	0.000142	(0.049629, 0.096297)
ν_B	0.001800	0.000185	3.42×10^{-8}	(0.001437, 0.002163)

TABLE 4. Vasicek Parameter Estimates for B_t (Women)

Parameter	Estimate	Std. Error	Variance	95% CI
c_B	0.007280	0.023867	0.000570	(-0.039499, 0.054059)
$\bar{B}$	0.127620	0.185539	0.034425	(-0.236036, 0.491276)
ν_B	0.000962	0.000097	9.41×10^{-9}	(0.000772, 0.001152)

- The speed of reversion c_A is higher for men than women, indicating that the male baseline mortality level adjusts more quickly to its long-term mean.
- Volatilities ν_A and ν_B suggest that women have more stable (less volatile) baseline mortality (A_t), but slightly more uncertainty in the age effect (B_t) compared to men.
- It may thus be concluded that the QMLE procedure yields stable and reliable parameter estimates, as evidenced by the corresponding standard errors and variances.

4.3. Expectation and Variance Computation.

Given the Vasicek estimations above and the long-term mean values, we proceed to compute the expectations and variances of A_t and B_t . Note that we considered year 2023 to be time $t = 0$. We are looking for the expectation and variances

for year 2025 with $t = 2$.

For Men: Consider $A_0 = 0.000279$ & $B_0 = 0.0723$.

Using (8), the resulting Vasicek parameter estimates are

$$c_A = 0.077087, \quad \bar{A} = 0.000435$$

$$c_B = 0.037771, \quad \bar{B} = 0.072963$$

For Women: Consider $A_0 = 0.000092$ & $B_0 = 0.082626$.

The resulting Vasicek parameter estimates are

$$c_A = 0.035050, \quad \bar{A} = -0.000044$$

$$c_B = 0.007280, \quad \bar{B} = 0.127620$$

Using these estimates, we can solve for the expectation and variances using (8), as shown in the following table:

TABLE 5. Summary of Expected Values and Variances for year 2025 $t = 2$

Gender	$\mathbb{E}[A_2]$	$\text{Var}(A_2)$	$\mathbb{E}[B_2]$	$\text{Var}(B_2)$
Men	0.000301	7.03×10^{-8}	0.072348	6.01×10^{-6}
Women	0.000083	3.01×10^{-9}	0.083276	1.83×10^{-6}

Deterministic EPV (Classical Baseline): Using the expected values of A_2 and B_2 for 2025 and $\delta = 0.1$, we can compute the EPV for a 1-year term insurance policy at age 30. Under a constant force of mortality over the year, the EPV is

$$\mathbb{E} \left[v^{T_{30}} \mathbf{1}_{[0,1)}(T_{30}) \right] = \frac{A_2 e^{B_2 \cdot 30} \left(1 - e^{-\delta - A_2 e^{B_2 \cdot 30}} \right)}{\delta + A_2 e^{B_2 \cdot 30}}.$$

- For males:

$$\mathbb{E}[v^{T_{30}}] \approx 0.00250618$$

- For females:

$$\mathbb{E}[v^{T_{30}}] \approx 0.00096110$$

This calculation provides a baseline net single premium using the expected parameter values for 2025 and does not incorporate uncertainty, i.e., it is *deterministic*. It gives a closed-form expression for the EPV of a 1-year term insurance policy, reflecting the increasing risk with age and gender-based mortality differences. Since the EPV corresponds to the net single premium, it provides a reference for pricing without additional loadings, serving as an intuitive lead-in to the Bayesian framework in the following section.

4.4. Bayesian Pricing of 1-Year Term Insurance.

We aim to compute the EPV of a 1-year term insurance with benefit 1, under a stochastic Gompertz mortality model incorporating familial health risk factors using a Bayesian model. The force of mortality with Familial Health Risk Factors $\mu_{x,t}(\theta)$ is modeled as in (9). Using (12), the conditional EPV given parameters θ is

$$\mathbb{E} \left[v^{T_x} \mathbf{1}_{[0,1)}(T_x) \mid \theta \right] = \frac{\mu_{x,t}(\theta) \cdot (1 - e^{-(\delta + \mu_{x,t}(\theta))})}{\delta + \mu_{x,t}(\theta)}.$$

Explicitly, we can write this as a function of D_1 and D_2 to emphasize that we can compute the price for all 4 of the possible configurations of the familial health risk factors.

$$\begin{aligned} & \mathbb{E}_{D_1, D_2} \left[v^{T_x} \mathbf{1}_{[0,1)}(T_x) \mid \theta \right] \\ &= \frac{(A_t e^{B_t x} + \lambda_1 D_1 + \lambda_2 D_2) \cdot (1 - e^{-(\delta + A_t e^{B_t x} + \lambda_1 D_1 + \lambda_2 D_2)})}{\delta + A_t e^{B_t x} + \lambda_1 D_1 + \lambda_2 D_2}. \end{aligned}$$

4.5. Bayesian Pricing Framework.

Given observed death times $T_x^{(1)}, \dots, T_x^{(n)}$, the Bayesian price is the expectation of the conditional EPV under the posterior distribution

$$\begin{aligned} & \mathbb{E}_{D_1, D_2} \left[v^{T_x} \mathbf{1}_{[0,1)}(T_x) \mid T_x^{(1)}, \dots, T_x^{(n)} \right] \\ &= \int \mathbb{E}_{D_1, D_2} \left[v^{T_x} \mathbf{1}_{[0,1)}(T_x) \mid \theta \right] \cdot P(\theta \mid T_x^{(1)}, \dots, T_x^{(n)}) d\theta \end{aligned}$$

where by Bayes' theorem

$$P(\theta \mid T_x^{(1)}, \dots, T_x^{(n)}) = \frac{P(T_x^{(1)}, \dots, T_x^{(n)} \mid \theta) \cdot P(\theta)}{\int P(T_x^{(1)}, \dots, T_x^{(n)} \mid \tilde{\theta}) \cdot P(\tilde{\theta}) d\tilde{\theta}}.$$

4.6. Prior vs Posterior Pricing.

- **Unconditional Price (Prior-based):**

$$\mathbb{E}_{D_1, D_2} \left[v^{T_x} \mathbf{1}_{[0,1)}(T_x) \right] = \int \mathbb{E}_{D_1, D_2} \left[v^{T_x} \mathbf{1}_{[0,1)}(T_x) \mid \theta \right] \cdot P(\theta) d\theta,$$

which reflects pricing using only prior knowledge about parameters without observed data.

- **Bayesian Price (Posterior-based):**

$$\begin{aligned} & \mathbb{E}_{D_1, D_2} \left[v^{T_x} \mathbf{1}_{[0,1)}(T_x) \mid T_x^{(1)}, \dots, T_x^{(n)} \right] \\ &= \int \mathbb{E}_{D_1, D_2} \left[v^{T_x} \mathbf{1}_{[0,1)}(T_x) \mid \theta \right] \cdot P(\theta \mid T_x^{(1)}, \dots, T_x^{(n)}) d\theta, \end{aligned}$$

which updates prices by incorporating observed mortality data, typically yielding more accurate estimates.

Suppose we observe data for policyholders aged $x = 30$ over a one-year period, classified by disease status as shown in the following table:

TABLE 6. Distribution of survival and death outcomes by disease status

	D ₁ Only	D ₂ Only	D ₁ and D ₂	No Disease
Survived	4	2	5	20
Died	3	4	2	10

The table is interpreted as follows:

- **Case 1: D₁ Only:** Among individuals who were diagnosed with disease D_1 but not D_2 , 4 survived while 3 died.
- **Case 2: D₂ Only:** Among those diagnosed with D_2 only, 2 survived while 4 died.
- **Case 3: D₁ and D₂:** For individuals diagnosed with both diseases simultaneously, 5 survived while 2 died.
- **Case 4: No Disease:** Among individuals with neither D_1 nor D_2 , 20 survived and 10 died.

Remark. The data in Table 4 **do not consist of exact death times** $T_{30}^{(i)}$, but only indicate survival or death outcomes over the one-year period $[30, 31)$. Hence, **the observations are censored** in the sense that we only know whether $T_{30}^{(i)} > 1$ (survived) or $T_{30}^{(i)} \leq 1$ (died). Consequently, the appropriate likelihood contributions are given by the one-year survival probability p_{30} for those who survived and the one-year death probability q_{30} for those who died, rather than the density of $T_{30}^{(i)}$.

Based on this observed data, the likelihood function can be written as:

$$P(T_x^{(1)}, \dots, T_x^{(n)} \mid \theta) = (p_{30}^{\text{none}})^{20} (p_{30}^{D_1})^4 (p_{30}^{D_2})^2 (p_{30}^{D_1 \wedge D_2})^5 (q_{30}^{\text{none}})^{10} (q_{30}^{D_1})^3 (q_{30}^{D_2})^4 (q_{30}^{D_1 \wedge D_2})^2,$$

where p_x denotes the probability of survival at age x for the corresponding disease group, and $q_x = 1 - p_x$ denotes the probability of death. In particular, under CFM assumption and Equation (7),

$$\begin{aligned} p_{30}^{\text{none}} &= P(T_{30} > 1 \mid D_1 = 0, D_2 = 0) = e^{-A_t e^{30B_t}} \\ q_{30}^{\text{none}} &= P(T_{30} \leq 1 \mid D_1 = 0, D_2 = 0) = 1 - e^{-A_t e^{30B_t}} \\ p_{30}^{D_1} &= P(T_{30} > 1 \mid D_1 = 1, D_2 = 0) = e^{-A_t e^{30B_t} - \lambda_1} \\ q_{30}^{D_1} &= P(T_{30} \leq 1 \mid D_1 = 1, D_2 = 0) = 1 - e^{-A_t e^{30B_t} - \lambda_1} \\ p_{30}^{D_2} &= P(T_{30} > 1 \mid D_1 = 0, D_2 = 1) = e^{-A_t e^{30B_t} - \lambda_2} \end{aligned}$$

$$\begin{aligned}
q_{30}^{D_2} &= P(T_{30} \leq 1 | D_1 = 0, D_2 = 1) = 1 - e^{-A_t e^{30B_t} - \lambda_2} \\
p_{30}^{D_1 \wedge D_2} &= P(T_{30} > 1 | D_1 = 1, D_2 = 1) = e^{-A_t e^{30B_t} - \lambda_1 - \lambda_2} \\
q_{30}^{D_1 \cap D_2} &= P(T_{30} \leq 1 | D_1 = 1, D_2 = 1) = 1 - e^{-A_t e^{30B_t} - \lambda_1 - \lambda_2}
\end{aligned}$$

For the computations that follows, $\delta = 0.1$ and $t = 2$ to reflect the predicted prices of the 1-year term insurance for a 30-year old in the year 2025. Note that earlier, we set $t = 0$ to indicate the last available parameters of A_t and B_t in the year 2023.

4.7. Comparison of Pricing under Prior and Posterior Distribution.

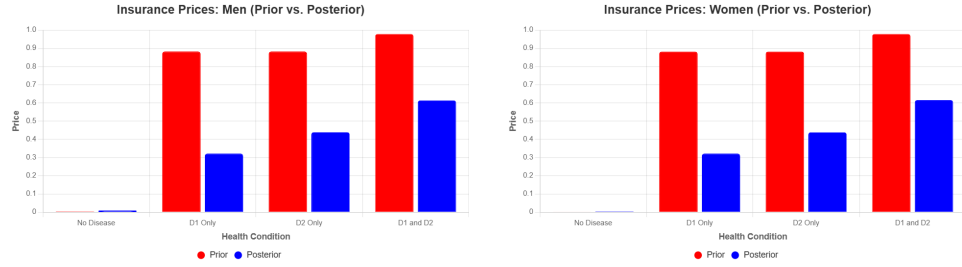


FIGURE 3. Comparison of 1-Year Term Insurance Prices under Prior and Posterior Distributions. Blue: Pricing reflecting updated beliefs after observing mortality data (posterior). Red: Pricing based solely on prior assumptions.

TABLE 7. Insurance prices under prior and posterior distributions for men.

Health Condition	Men Prior	Men Posterior
No Disease	0.002511	0.008865
D1 Only	0.881860	0.321485
D2 Only	0.881860	0.439133
D1 and D2	0.978333	0.613664

TABLE 8. Insurance prices under prior and posterior distributions for women.

Health Condition	Women Prior	Women Posterior
No Disease	0.000961	0.002619
D1 Only	0.881698	0.321207
D2 Only	0.881698	0.438723
D1 and D2	0.978315	0.615527

Note that Figure 3, Table 7, and Table 8 illustrate how incorporating observed data via the posterior distribution refines pricing, often reducing uncertainty and adjusting prices closer to empirical mortality experience. In contrast, the prior-based pricing relies solely on initial assumptions and may be less precise. Specifically,

- For data showing higher survival rates, the **Bayesian prices under the posterior distribution are generally lower than the unconditional prices based on the prior**. This indicates that the prior has been updated to reflect improved survival, leading to cheaper prices. Thus, the observed data plays a crucial role in determining posterior-based pricing.
- **Bayesian pricing ensures fairness** by incorporating actual risk data and health risks, leading to more precise and individualized premium adjustments.

The pricing framework presented not only enhances **pricing accuracy** and **fairness** but also shows strong potential for **actuarial modeling** in **data-scarce environments**.

Data Availability Statement. The data used to support the findings of this study are available from the first author upon request.

Declarations. The authors declare no conflict of interest.

Author Contributions. Anjali Lomugdang (First Author): Conceptualization, methodology, writing-original draft, data curation, resources, software, formal analysis, visualization, validation, writing-review, and editing. Jude Martin Grozen (Second Author): Conceptualization, methodology, resources, software, formal analysis, supervision, and validation.

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REFERENCES

- [1] B. Gompertz, "On the nature of the function expressive of the law of human mortality and on a new mode of determining life contingencies," *Philosophical Transactions of the Royal Society of London*, vol. 115, pp. 513–585, 1825. <https://doi.org/10.1098/rspl.1815.0271>.
- [2] W. M. Makeham, "On the law of mortality and the construction of annuity tables," *Journal of the Institute of Actuaries*, vol. 6, pp. 301–310, 1860. <https://www.jstor.org/stable/41134925>.

- [3] E. Pitacco, M. Denuit, S. Haberman, and A. Olivieri, *Modelling longevity dynamics for pensions and annuity business*. Oxford University Press, 2009. <https://doi.org/10.1093/oso/9780199547272.001.0001>.
- [4] M. Denuit, P. Devolder, and A. Goderniaux, “Securitization of longevity risk: Pricing survivor bonds with wang transform in the lee-carter framework,” *Journal of Risk and Insurance*, vol. 74, pp. 87–113, 2007. <https://www.jstor.org/stable/4138426>.
- [5] K. Antonio and E. A. Valdez, “Statistical concepts of a priori and a posteriori risk classification in insurance,” *Advances in Statistical Analysis*, vol. 96(2), pp. 187–224, 2012. <https://doi.org/10.1007/s10182-011-0152-7>.
- [6] O. Vasicek, “An equilibrium characterization of the term structure,” *Journal of Financial Economics*, vol. 5(2), pp. 177–188, 1977. [https://doi.org/10.1016/0304-405X\(77\)90016-2](https://doi.org/10.1016/0304-405X(77)90016-2).
- [7] E. Biffis, “Affine processes for dynamic mortality and actuarial valuations,” *Insurance: Mathematics and Economics*, vol. 37(3), pp. 443–468, 2005. <https://doi.org/10.1016/j.insmatheco.2005.05.003>.
- [8] B. P. Carlin and T. A. Louis, *Bayesian Methods for Data Analysis*. CRC Press, 3rd ed., 2009. <https://books.google.com.ph/books?id=GTJUt8fcFx8C>.
- [9] K. Barigou, P.-O. Goffard, S. Loisel, and Y. Salhi, “Bayesian model averaging for mortality forecasting using leave-future-out validation,” *International Journal of Forecasting*, vol. 39(2), pp. 674–690, 2023. <https://doi.org/10.1016/j.ijforecast.2022.01.011>.
- [10] R. D. Lee and L. R. Carter, “Modeling and forecasting u.s. mortality,” *Journal of the American Statistical Association*, vol. 87(419), pp. 659–671, 1992. <https://doi.org/10.1080/01621459.1992.10475265>.
- [11] A. E. Renshaw and S. Haberman, “A cohort-based extension to the lee-carter model for mortality reduction factors,” *Insurance: Mathematics and Economics*, vol. 38(3), pp. 556–570, 2006. <https://doi.org/10.1016/j.insmatheco.2005.12.001>.
- [12] S. A. Klugman, H. H. Panjer, and G. E. Willmot, *Loss Models: From Data to Decisions*. John Wiley & Sons, Inc., 3rd ed., 2008. <https://doi.org/10.1017/S1748499500000531>.
- [13] T. Mikosch, *Elementary stochastic calculus with finance in view*. World scientific, 1998. http://lib.ysu.am/open_books/417513.pdf.
- [14] S. Ross, *A First Course in Probability*. London, United Kingdom: Pearson Education, Inc., 9th, illustrated edition, 2014. <https://chengzhaoxi.xyz/download/pdf/book/A-First-Course-in-Probability.pdf>.
- [15] S. E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*, vol. 11 of *Springer Finance Textbooks*. New York: Springer, 2004. <https://www.scribd.com/document/397066094/Steven-E-Shreve-Stochastic-calculus-for-finance-II-Continuous-time-models-pdf>.
- [16] A. I. Yashin, I. A. Iachine, and A. Z. Begun, “Mortality modeling: A review,” *Mathematical Population Studies*, vol. 8(4), pp. 305–332, 2000. <https://doi.org/10.1080/08898480009525489>.
- [17] R. Ambagaspitiya and C. Ford, “Calibrating interest rate models,” *Society of Actuaries Research Report*, October 2023. <https://www.soa.org/globalassets/assets/files/resources/research-report/2023/interest-rate-model-calibration-study.pdf>.
- [18] Y. Li and P. Yeh, “An interpretation of the moore–penrose generalized inverse of a singular fisher information matrix,” *IEEE Transactions on Signal Processing*, vol. 60(10), pp. 5532–5536, 2012. <https://doi.org/10.1109/TSP.2012.2208105>.
- [19] J. James and N. Webber, *Interest Rate Modeling*. John Wiley & Sons, 2000. <https://api.semanticscholar.org/CorpusID:153632252>.
- [20] E. V. Slud, *Actuarial Mathematics and Life-Table Statistics*. University of Maryland, College Park, 2001. <https://www.math.umd.edu/~slud/s470/BookChaps/01Book.pdf>.