

Distinguishing Number of All Trees with Radius Two

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Abstract. Let G be a connected graph. The distinguishing number of G , denoted by $D(G)$, is the least integer d such that G has a vertex d -labeling that is preserved only by the trivial automorphism. In this paper, we will show that given an integer d , we can find all trees of radius at most two with distinguishing number d . Moreover, we provide an algorithm to find the distinguishing labeling of any tree with radius at most two.

Keywords: automorphism, distinguishing number, graph labeling, radius, tree

1. INTRODUCTION

For any positive integer d , a vertex labeling on a graph $G(V, E)$ is a function from $V(G)$ to $\{1, 2, \dots, d\}$. A vertex labeling is said to be distinguishing if the labels are preserved only by the identity automorphism. The distinguishing number of graph G is defined as the least integer d such that G has a distinguishing d -labeling and it is denoted by $D(G)$.

The concept of distinguishing labeling in graphs was introduced by Albertson and Collins [1] in 1996. Since then, several results are found regarding this topic. The distinguishing number of highly symmetric graphs, like complete graphs K_n or complete bipartite graphs $K_{m,n}$, tends to be higher because of the large number of automorphisms. For Cartesian products of graphs, the distinguishing number often increases. Imrich and Klavzar [5] studied this and found that for Cartesian products of trees, the distinguishing number is related to the distinguishing number of the factors. Bogstad and Cowen [3] showed that the distinguishing number of an n -dimensional hypercube Q_n is $D(Q_n) = 2$ for all $n \geq 4$. Hypercubes have a high level of symmetry, but two labels are enough to distinguish them. Rahadi, Baskoro, and Saputro [7] characterize all connected graphs of order n with distinguishing number

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$n - 1$ or $n - 2$. However, there are not much results regarding the distinguishing number of trees.

A tree is a connected graph that contains no subgraph isomorphic to a cycle. For instance, a path of length n , P_n , is a tree. A star $K_{1,n}$ is also a tree. In this paper, we use standard terms to describe a tree. A vertex in a tree that has no parent is called the root of the tree. The degree of the root shows how many children it has. A vertex without any child is called a leaf.

To make a clear definition of the radius of a graph, we will start with the definition of distance and eccentricity. Let G be a simple connected graph. The distance between two distinct vertices in G is the length of the shortest path between those vertices. The maximum distance between a vertex to all other vertices in G is considered as the eccentricity of the vertex. The radius of G is defined as the minimum eccentricity of all vertices in G , and denoted as $rad(G)$.

Some results regarding the distinguishing labeling and distinguishing number of trees can be found in Cheng [4] and Alikhani & Soltani [2]. Cheng presents an efficient algorithm to compute the distinguishing numbers of trees and forests. Cheng develops an $O(n \log n)$ -time algorithm, which relies on the combinatorial structure of trees rather than their automorphism groups, a departure from traditional methods. Cheng's method begins by converting any tree into a rooted tree, ensuring the distinguishing number remains unchanged. She further introduces a recursive formula for computing the number of distinguishing labelings and an algorithm to efficiently identify isomorphic subtrees. In this paper, we will also convert trees into rooted trees. However, instead of use a recursive formula, we will develop what we called the tuple notation tree to characterize all two radius trees with certain distinguishing number.

In [2], Alikhani and Soltani provide a thorough characterization of all trees that possess a distinguishing number of two. Focusing initially on trees with a radius of at most two, they identify and depict all such trees, highlighting their distinct structural features. Moreover, they demonstrate that trees with a radius greater than two must meet specific conditions to have a distinguishing number of two: they must be self-centered, connected, and non-bipartite. Moreover, it was proved that no trees with a diameter of three can have this property. Their results offer a clear classification of trees with distinguishing number two, uncovering the intricate structural requirements these trees must satisfy. However, when characterize all trees with distinguishing number two, they have missed one tree and double counted a tree as shown in Figure 5.

2. MAIN RESULTS

In this paper, we shall determine the distinguishing number of all trees with a radius of at most two. We also characterize all such trees with a certain distinguishing number.

Let T be a tree of radius at most two with a central vertex x_0 . Let $N(x_0)$ be the set of all neighbors of x_0 . Let k be the maximum degree of all vertices in $N(x_0)$. Partition the set $N(x_0)$ as follows:

$$N(x_0) = H_1 \cup H_2 \cup \cdots \cup H_k,$$

where $H_i = \{x_{i,j} \in N(x_0) : \deg(x_{i,j}) = i\}$ for each $i = 1, 2, \dots, k$. If $t_i = |H_i|$, for $i = 1, 2, \dots, k$, then such a T of radius at most two can be denoted as a tuple of an $[t_1, t_2, \dots, t_k]$ -tree. Note that the root (center) of T has $t_1 + t_2 + \cdots + t_k$ children. From now on, we use this tuple notation for any tree of radius at most two. We formalize this notation in the following definition.

Definition 2.1. A tree of radius at most two with a center x_0 is denoted by $[t_1, t_2, \dots, t_n]$ -tree if vertex x_0 is adjacent to exactly t_i vertices of degree i for each $i \in [1, n]$.

For example, a $[2, 1, 3, 1]$ -tree, as depicted in Figure 1, has the root x_0 and its children are $N(x_0) = \{x_{1,1}, x_{1,2}, x_{2,1}, x_{3,1}, x_{3,2}, x_{3,3}, x_{4,1}\}$, where $\deg(x_{1,1}) = \deg(x_{1,2}) = 1$, $\deg(x_{2,1}) = 2$, $\deg(x_{3,1}) = \deg(x_{3,2}) = \deg(x_{3,3}) = 3$, and $\deg(x_{4,1}) = 4$. Thus, $t_1 = 2$, $t_2 = 1$, $t_3 = 3$, and $t_4 = 1$. The $[2, 1, 3, 1]$ -tree is an example of a single-center tree. It is well known that a tree has only one or two centers. In the following lemma, we show that this condition can be determined from the values of $t_2, t_3, \dots, t_n$.

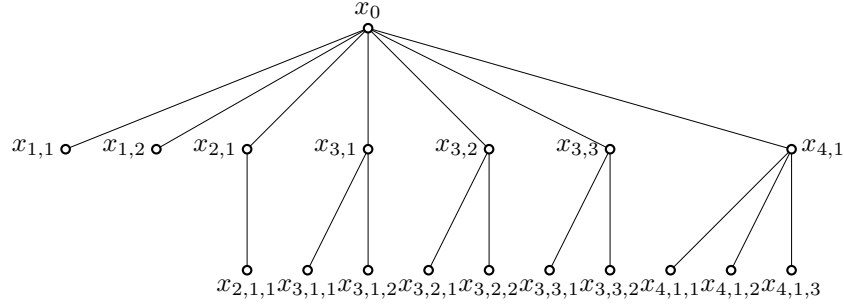


FIGURE 1. The $[2, 1, 3, 1]$ -tree. Here we introduce the indexing rules for each vertex. The first generation vertices have i, j index, meaning it has degree i and is the j -th children. The second generation vertices have i, j, k index, to infer that they are the children of respected $x_{i,j}$.

Lemma 2.2. Let T be a $[t_1, t_2, \dots, t_k]$ -tree with $t_1 \geq 1$ and $\sum_{i=2}^k t_i = \sigma$. Then, the following statements hold.

- (i) If $\sigma \geq 2$ then T has a single center.
- (ii) If $\sigma = 1$ then T has two centers.
- (iii) If $\sigma = 0$ then T is a star S_{1,t_1} .

Proof. Let T be a $[t_1, t_2, \dots, t_k]$ -tree with $t_1 \geq 1$ and x_0 be a center of T . Let $\sigma = t_2 + t_3 + \dots + t_k$.

- (i) Let $u, v \in N(x_0)$ be of degree i and j , respectively, with $i, j \geq 2$ (note that i and j are not necessarily different). Then, there is a path of length 4 connecting a child of u and a child of v in T . This implies that x_0 is the only center of T .
- (ii) If u is the only vertex in $N(x_0)$ of degree $i \geq 2$, then the vertices x_0 and u will be the vertices with minimum eccentricity in T . Therefore, T has two centers.
- (iii) Since all children of x_0 are leaves, T is a star.

□

Lemma 2.3. An $[a_1, a_2, a_3, \dots, a_k]$ -tree is isomorphic to a $[b_1, b_2, b_3, \dots, b_k]$ -tree if one of the following holds.

- (1) ($a_1 = 0, a_t = 1$ for some $t \geq 2$ and $a_i = 0$ for all $i \neq t$) and ($b_1 = t, b_i = 0$ for all $i \neq 1$).
- (2) ($a_1 \geq 1, a_t = 1$ for some $t \geq 2$ and $a_i = 0$ for all $i \notin \{1, t\}$) and ($b_1 = t - 1, b_{a_1+1} = 1, b_i = 0$ for all $i \notin \{1, a_1 + 1\}$)

Proof. Let T be an $[a_1, a_2, a_3, \dots, a_k]$ -tree with a rooted vertex x_0 . If $a_1 = 0, a_t = 1$ for some fixed $t \geq 2$, $a_i = 0$ for $i \notin \{1, t\}$, then T is a star with t leaves. Therefore, T can be represented as an $[b_1, b_2, \dots, b_k]$ -tree with $b_1 = t$ and $b_i = 0$ for $i \neq 1$.

Now, if $a_1 = s \geq 1, a_t = 1$ for some fixed $t \geq 2$, and $a_i = 0$ for $i \notin \{1, t\}$, then T has two centers, namely x_0 and the vertex of degree t in T , call this vertex y_0 . Therefore, if we choose y_0 as a new root vertex of T , then T can be represented as an $[b_1, b_2, \dots, b_k]$ -tree with $b_1 = t - 1, b_{s+1} = 1$ and $b_i = 0$ for all $i \notin \{1, s + 1\}$. □

In Figure 2, we provide an example of Lemma 2.3 (i); that is, the $[0, 0, 0, 1]$ -tree is isomorphic to the $[4, 0, 0, 0]$ -tree. In Figure 3, we provide an example of Lemma 2.3 (ii), that is, the $[3, 0, 1, 0]$ -tree is isomorphic to the $[2, 0, 0, 1]$ -tree.

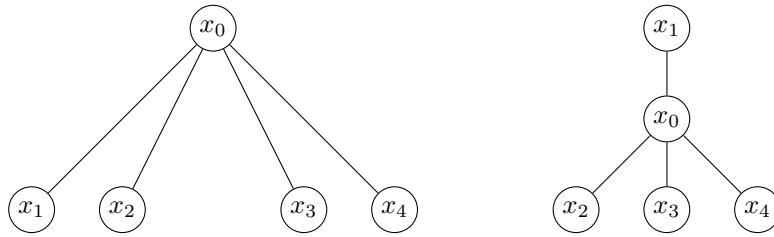


FIGURE 2. The star graph S_4 can be denoted as tuple $[4, 0, 0, 0]$ -tree rooted at the vertex x_0 or tuple $[0, 0, 0, 1]$ -tree rooted at the vertex x_1 .

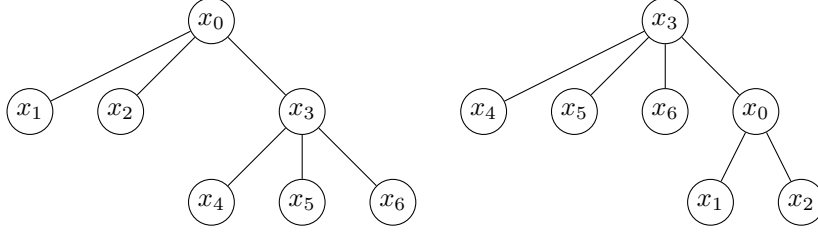


FIGURE 3. The double star $S_{2,3}$ can be denoted as tuple $[2, 0, 0, 1]$ -tree rooted at the vertex x_0 **or** the tuple $[3, 0, 1, 0]$ -tree rooted at the vertex x_3 .

When we denote a tree of radius at most two using a tuple, we can practically omit the last zero entries. For example, the $[2, 3, 0, 0]$ -tree has the same meaning as the $[2, 3]$ -tree, which is a tree where the root has two children of degree 1 and three children of degree 2. Therefore, throughout this paper, we will omit the last zero entries in any tuple-tree to simplify the proof processes.

Theorem 2.4. *Let n be a positive integer. Let T be a $[t_1, t_2, \dots, t_n]$ -tree with $t_n \neq 0$. $D(T) = m$ if and only if m is the least integer such that $t_i \leq m \cdot \binom{m}{i-1}$, for all $i \in [1, n]$ and $m \geq n - 1$.*

Proof. Let n be a positive integer. Let T be a $[t_1, t_2, \dots, t_n]$ -tree rooted at x_0 with $t_n \neq 0$.

($\Leftarrow$) Let m be the least integer such that $t_i \leq m \cdot \binom{m}{i-1}$ for all $i \in [1, n]$, and $m \geq n - 1$. Let $V(T) = \{x_0\} \cup \{x_{i,j}, x_{i,j,k} | i \in [1, n], j \in [1, t_i], k \in [1, i-1]\}$ where $x_{i,j}$ is the j -th child vertex of x_0 with degree i , and $x_{i,j,k}$ is the k -th child of $x_{i,j}$. Let $M = \{1, 2, \dots, m\}$. For $i = 1, 2, \dots, n$, define $H_i = \{x_{i,j} | j \in [1, t_i]\}$, and $\mathcal{M}^i$ be the collection of all subsets of M with i elements. Let μ_i be the cardinality of $\mathcal{M}^i$, that is $\mathcal{M}^i = \{L_1, L_2, \dots, L_{\mu_i}\}$. It is obvious that $\mu_i = \binom{m}{i}$.

Now, define a vertex-labeling $\alpha : V(T) \rightarrow \{1, 2, \dots, m\}$ on T with the property that:

- $\alpha(x_0) = 1$,
- $\alpha(x_{1,j}) = j$, for $j \in [1, t_1]$,
- $\alpha(x_{i,j}) = \left\lceil \frac{j}{\binom{m}{i-1}} \right\rceil$, for $i \in [2, n]$, $j \in [1, t_i]$.

Note that since $t_i \leq m \cdot \binom{m}{i-1}$ for all $i \in [1, n]$, then $\alpha(x_{i,j}) \leq m$ for all $i \in [1, n], j \in [1, t_i]$.

Next we will define the labeling α on all children of $x_{i,j}$. For $i = 2, \dots, n$ and $j = 1, 2, \dots, t_i$, let $A_{i,j} = \{x_{i,j,k} | k \in [1, i-1]\}$ be the set of all children of $x_{i,j}$. Now define $\alpha(A_{i,j}) = L_j$, $L_j \in \mathcal{M}^{i-1}$, that is we label all the vertices in $A_{i,j}$ with

distinct numbers in L_j such that $\alpha(A_{i,j}) = L_j \neq L_{j'} = \alpha(A_{i,j'})$ for any $j \neq j'$. We can do this assignment since $t_i \leq m \cdot \binom{m}{i-1}$, for all $i \in [1, n]$.

Now we will show that α is a distinguishing labeling on T . Let f be any non-identity automorphism on $V(T)$. We will show that f does not preserve labeling α . Since f is an automorphism, it preserves degrees, and so $f(x_0) = x_0$ and $f(H_i) = H_i$. If there are two vertices $x_{i,j}$ and $x_{i,j'}$ with $j \neq j'$ such that $f(x_{i,j}) = x_{i,j'}$ with $\alpha(x_{i,j}) \neq \alpha(x_{i,j'})$ then f does not preserve α . Therefore, we focus on the case of $\alpha(x_{i,j}) = \alpha(x_{i,j'})$. It follows that $f(A_{i,j}) = A_{i,j'}$. However, $\alpha(A_{i,j}) \neq \alpha(A_{i,j'})$. Therefore, f does not preserve α . Since m is the smallest integer such that $t_i \leq m \cdot \binom{m}{i-1}$, for all $i \in [1, n]$, then we obtain $D(T) = m$.

($\Rightarrow$) Let $D(T) = m$. Then, every child $x_{i,j}$ of x_0 has a degree of at most $m+1$. This implies that $n \leq m+1$. Now, assume that for some $i_0 \in [1, n]$, we have that $t_{i_0} > m \cdot \binom{m}{i_0-1}$. Let β be any m -labeling of T . For each i, j , let $(x_{i,j}, A_{i,j})$ represent the vertex $x_{i,j}$ together with the associated set of its children $A_{i,j}$. In particular, when $i = i_0$, there are precisely t_{i_0} such pairs. By the pigeonhole principle, there exist two distinct pairs $(x_{i_0,j}, A_{i_0,j})$ and $(x_{i_0,j'}, A_{i_0,j'})$ whose corresponding labels under β are identical. Therefore, there exists a non-identity automorphism f that maps $x_{i_0,j}$ to $x_{i_0,j'}$ while preserving the labeling β , a contradiction. This concludes the proof. $\square$

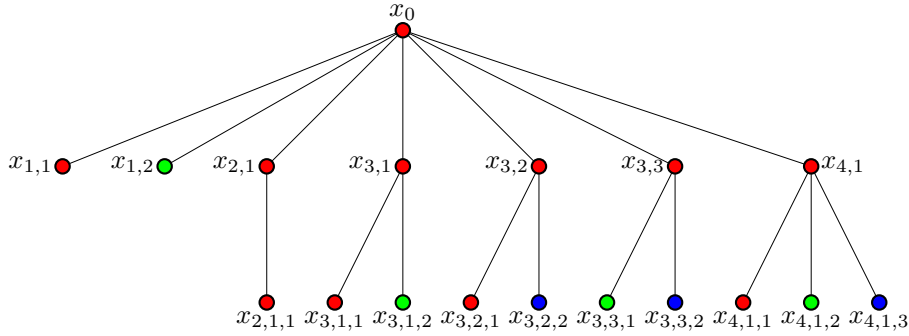


FIGURE 4. The $[2, 1, 3, 1]$ -tree assigned with 3 colors.

As an example, in Figure 4 we present a distinguishing coloring of $[2, 1, 3, 1]$ -tree with 3 colors. In the following corollary, we give an upper bound for the maximum degree of a tree with a distinguishing number m .

Corollary 2.5. *Let T be a rooted tree at x_0 with $\text{rad}(T) \leq 2$. If $D(T) = m$, then*

$$\deg(x_0) \leq m \cdot 2^m.$$

Proof. Let T be a $[t_1, t_2, \dots, t_k]$ -tree, with $D(T) = m$ and $k \leq m + 1$. The proof is immediate from the Theorem 2.4, that is

$$\deg(x_0) = \sum_{i=1}^k t_i \leq m \cdot \sum_{i=1}^{m+1} \binom{m}{i-1} = m \cdot 2^m.$$

□

$\deg(r(T))$	Trees with distinguishing number 2 and radius at most two.
$\deg(r(T)) = 1$	
$\deg(r(T)) = 2$	
$\deg(r(T)) = 3$	
$\deg(r(T)) = 4$	
$\deg(r(T)) = 5$	
$\deg(r(T)) = 6$	
$\deg(r(T)) = 7$	
$\deg(r(T)) = 8$	

FIGURE 5. All trees with radius at most 2 and distinguishing number 2 as investigated by Alikhani and Soltani [2]. The trees enclosed by the blue boxes are isomorphic, so, they must be counted once.

Theorem 2.6. Let T be a tree with $\text{rad}(T) \leq 2$, then

- (i) $[0]$ -tree is the only tree with $D(T) = 1$,
- (ii) $[2, 4, 2]$ -tree is the largest tree with $D(T) = 2$.
- (iii) $[3, 9, 9, 3]$ -tree is the largest tree with $D(T) = 3$.

Proof. Let T be a tree with $\text{rad}(T) \leq 2$. If T has at least two vertices, then there are always two leaves that can be exchanged by a nontrivial automorphism. This implies that $[0]$ -tree is the only tree T with $D(T) = 1$. The statement (i) holds.

The statements (ii) and (iii) follow directly from Theorem 2.4 for the cases of $m = 2$ and $m = 3$, respectively, where we take the maximum value of t_i for each i . Thus, it concludes the proof. $\square$

Alikhani and Soltani [2] has discovered 41 non-isomorphic trees of radius at most 2 with $D(T) = 2$. Among those trees, there are two trees that are considered different, but they are isomorphic, as shown in Figure 5. In the following theorem, we will revise their result using the tuple tree notation approach. Note that Alikhani and Soltani double-counts the $[1, 0, 1]$ -tree and its isomorphic tree, the $[2, 1, 0]$ -tree. Moreover, the paper did not include the $[2, 0, 1]$ -tree.

Theorem 2.7. *There are exactly 41 non-isomorphic trees T with $\text{rad}(T) \leq 2$ and $D(T) = 2$.*

Proof. Let T be a $[t_1, t_2, t_3]$ -tree with $\text{rad}(T) \leq 2$ and $D(T) = 2$. By Theorem 2.6, every such tree must be a subtree of the $[2, 4, 2]$ -tree, which is the largest tree with distinguishing number 2.

Since $0 \leq t_1, t_3 \leq 2$ and $0 \leq t_2 \leq 4$, the total number of $[t_1, t_2, t_3]$ -trees with radius at most 2 is

$$3 \cdot 5 \cdot 3 = 45.$$

From these, we must exclude the tree $[0, 0, 0]$, since it has distinguishing number 1, and the star $[0, 0, 1]$, since it has distinguishing number 3.

Next, by Lemma 2.3, the $[2, 1, 0]$ -tree is isomorphic to the $[1, 0, 1]$ -tree, and the $[2, 0, 0]$ -tree is isomorphic to the $[0, 1, 0]$ -tree. These account for two duplicate isomorphism classes. On the other hand, although the $[1, 1, 0]$ -tree and the $[2, 0, 1]$ -tree are double-center trees, each is isomorphic only to itself and hence does not create any further duplication.

Therefore, the number of non-isomorphic trees T with $\text{rad}(T) \leq 2$ and $D(T) = 2$ is

$$45 - 2 - 2 = 41.$$

This completes the proof. $\square$

Theorem 2.8. *There are exactly 1552 non-isomorphic trees T with $\text{rad}(T) \leq 2$ and $D(T) = 3$.*

Proof. Let T be a $[t_1, t_2, t_3, t_4]$ -tree with $\text{rad}(T) \leq 2$ and $D(T) = 3$. By Theorem 2.6, every such tree is a subtree of the $[3, 9, 9, 3]$ -tree, which is the maximal tree with distinguishing number 3.

Hence, $0 \leq t_1, t_4 \leq 3$ and $0 \leq t_2, t_3 \leq 9$, giving a total of

$$4 \cdot 10 \cdot 10 \cdot 4 = 1600$$

$[t_1, t_2, t_3, t_4]$ -trees of radius at most 2 (counted up to labeling, not isomorphism).

Next, we exclude those trees whose distinguishing number is not equal to 3. In particular, all subtrees of the $[2, 4, 2, 0]$ -tree have distinguishing number at

most 2, and must therefore be removed, except for the star $[0, 0, 1, 0]$, which has distinguishing number 3. In addition, the tree $[0, 0, 0, 1]$ has distinguishing number 4 and must also be excluded. After these removals, 1555 trees remain.

Finally, we account for isomorphic duplicates using Lemma 2.3. The following pairs are isomorphic:

$$[1, 0, 0, 1] \cong [3, 1, 0, 0], \quad [2, 0, 0, 1] \cong [3, 0, 1, 0], \quad [3, 0, 0, 0] \cong [0, 0, 1, 0].$$

Each pair contributes one duplication, so we subtract 3 from the count. Note that the $[3, 0, 0, 1]$ -tree is a double-center tree but is only isomorphic to itself, and hence does not introduce any further duplication.

Therefore, the number of non-isomorphic trees T with $\text{rad}(T) \leq 2$ and $D(T) = 3$ is

$$1555 - 3 = 1552.$$

This completes the proof. $\square$

Now we generalize Theorems 2.7 and 2.8 to enumerate all non-isomorphic trees of radius at most 2 with a prescribed distinguishing number.

Theorem 2.9. *Let d be a positive integer. The number of non-isomorphic trees T with $\text{rad}(T) \leq 2$ and $D(T) = d$ is*

$$\prod_{i=1}^{d+1} \left(d \binom{d}{i-1} + 1 \right) - \prod_{i=1}^d \left((d-1) \binom{d-1}{i-1} + 1 \right) - d.$$

Proof. Let T be a $[t_1, t_2, \dots, t_n]$ -tree with $\text{rad}(T) \leq 2$ and $D(T) = d$. Then necessarily $n = d + 1$. Indeed, if $n \geq d + 2$ and $t_n \neq 0$, then the root of T has a child of degree at least $d + 2$, implying $D(T) > d$, which is a contradiction.

By Theorem 2.4, every such tree can be represented as a $[t_1, t_2, \dots, t_{d+1}]$ -tree satisfying

$$0 \leq t_i \leq d \binom{d}{i-1}, \quad \text{for } i = 1, \dots, d+1.$$

Hence, the total number of such trees (counted with possible isomorphisms) having a distinguishing number of at most d is

$$\prod_{i=1}^{d+1} \left(d \binom{d}{i-1} + 1 \right).$$

Similarly, the number of $[t_1, t_2, \dots, t_d]$ -trees (again counted with possible isomorphisms) with a distinguishing number of at most $d - 1$ is

$$\prod_{i=1}^d \left((d-1) \binom{d-1}{i-1} + 1 \right).$$

Excluding these trees leaves those with a distinguishing number exactly equal to d .

It remains to account for isomorphic duplicates. By Lemma 2.2 and Lemma 2.3, the trees of radius at most 2 with two centers (double-center trees) correspond precisely to the tuples

$$[1, 0, \dots, 0, 1], [2, 0, \dots, 0, 1], \dots, [d-1, 0, \dots, 0, 1], \text{ and } [d, 0, \dots, 0, 0].$$

Each of these corresponds to a pair of isomorphic trees and hence contributes one duplication. Therefore, we subtract d from the count.

Consequently, the number of non-isomorphic trees T with $\text{rad}(T) \leq 2$ and $D(T) = d$ is

$$\prod_{i=1}^{d+1} \left(d \binom{d}{i-1} + 1 \right) - \prod_{i=1}^d \left((d-1) \binom{d-1}{i-1} + 1 \right) - d.$$

□

TABLE 1. The number of all non-isomorphic trees T of $\text{rad}(T) \leq 2$ with certain distinguishing number.

$D(T)$	Number of non-isomorphic Trees T
1	1
2	41
3	1, 552
4	179021
5	63, 117, 306
6	67, 151, 931, 139
7	2.1202×10^{14}

By using Theorem 2.9 we can compute the total number of all non-isomorphic trees T with $\text{rad}(T) \leq 2$ and distinguishing number d , with $d \in [1, 7]$ as shown in Table 1.

Data Availability Statement. No new data were created or analyzed in this study.

Declarations. The authors declare that they have no known competing interests or personal relationships that could have appeared to influence the work reported in this paper.

Author Contributions. *Andi Pujo Rahadi*: writing-review & editing, writing-original draft, conceptualization, funding acquisition, visualization, project administration. *Edy Tri Baskoro*: writing-review & editing, supervision, conceptualization, funding acquisition, methodology, formal analysis. *Suhadi Widodo Saputro*:

writing-review & editing, conceptualization, supervision, validation, methodology. All authors discussed the results and contributed to the final manuscript.

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