

Merton's Portfolio Optimization with Multiple Risky Assets Based on Risk Preference

Chasna Zahara^{1*}, Endah Rokhmati Merdika Putri²

¹Department of Actuarial Science, Institut Teknologi Sepuluh Nopember, Indonesia,

²Department of Mathematics, Institut Teknologi Sepuluh Nopember, Indonesia.

Abstract. This study addresses the issue of Merton portfolio optimization based on the Hyperbolic Absolute Risk Aversion (HARA) utility function within a finite time interval. The portfolio is constructed using two risky assets and one risk-free asset, ensuring that investments are not limited to a single risky asset in order to achieve optimal investment and consumption strategies. The mathematical model of the Merton portfolio problem will be analyzed using the Hamilton-Jacobi-Bellman (HJB) equation. The simulation shows that the optimal investment proportion in multi-risk assets decreases as wealth increases. This is influenced by the investor's risk preference relative to their wealth. The results also show that optimal consumption increases over time in order to obtain maximum utility.

Key words and Phrases: Portfolio Optimization, HARA Utility Function, Multiple Risky Assets, HJB Equation.

1. INTRODUCTION

Investors are faced with two choices in allocating their wealth in a portfolio. They must divide their wealth between consumption and investment. Consumption is used to meet daily needs such as purchasing basic necessities and other requirements, while investment is used to accumulate wealth. Investors hope that the steps taken will optimize their potential profit in the future. The discussion about portfolio selection for investment and consumption attracts not only financial practitioners but also academics [1], [2], [3]. The portfolio selection problem is demonstrated by Merton's initial paper on the optimization of investment and consumption [4], known as Merton portfolio problem. This problem is studied by [5] in a continuous-time framework using dynamic programming. Moreover, the result

*Corresponding author : chasnazhr@its.ac.id

2020 Mathematics Subject Classification: Primary 91G10; Secondary 93E20, 49L20.

Received: 31-07-2025, accepted: 26-06-2026.

shows an optimal portfolio strategy in allocating wealth between consumption and investment in financial markets composed of risky and risk-free assets.

Utility functions are applied to the optimization model to optimize the expected utility in total wealth [6]. One such function is a Hyperbolic Absolute Risk Aversion (HARA) utility function. The HARA utility function was first introduced by Merton [7] as a general form of the Constant Relative Risk Aversion (CRRA), Constant Absolute Risk Aversion (CARA), and quadratic utility functions. We can create CRRA and CARA from general form of HARA by changing several parameter values. In 2021, Ma and Zhu [8] applied the HARA utility function to the Merton portfolio problem. This is done to approximate the solution of the CARA utility function. However, Ma and Zhu only considered one risky asset and one risk-free asset. In practice, investors often have portfolios consisting of more than one risky asset.

This study proposes an extension Merton's portfolio problem by considering more than one risky asset and applying the HARA utility function. This study aims to maximize the expected utility of investors in consumption and investment problems based on their risk preferences. This study presents the analytical solution of the HJB equation [9] and simulations, to obtain the optimal solution. The solution contribute new insights to the financial literature by offering a broader and more realistic approach to wealth management. We apply the HARA utility function in the Merton problem because of its flexibility.

There are sections that structure of the paper is as follows. The mathematical model of Merton's portfolio optimization and its solution are explained in section 2 and section 3. The numerical model is then presented in section 4 and the results of simulation is discussed in section 5. The final section outlines the conclusions of all things in this study.

2. MATHEMATICAL MODEL

We consider a Merton model created continuous-time in financial market. An investor has a portfolio during a limited interval $[0, T]$ which consists of one risk-free asset and two risky assets. The risk-free asset is a bond whose price is expressed as follows [10]

$$dB(t) = rB(t) dt, \quad (1)$$

where r is the risk-free interest rate. The other assets are two risky assets in the form of stocks, which are expressed in the stock price model as follows

$$dS_1(t) = \mu_1 S_1(t) dt + \sigma_1 S_1(t) dW_1(t), \quad (2)$$

$$dS_2(t) = \mu_2 S_2(t) dt + \sigma_2 S_2(t) dW_2(t), \quad (3)$$

where μ_1, μ_2 are the expected returns of $S_1(t)$ and $S_2(t)$; σ_1, σ_2 are the volatilities of $S_1(t)$ and $S_2(t)$; and $W_1(t), W_2(t)$ are Wiener processes that have a correlation $\rho \in [-1, 1]$ such that [11]

$$dW_1(t) dW_2(t) = \rho dt. \quad (4)$$

The investor has a total wealth derived from risky assets and risk-free assets. The investor invests $u_1(t)X(t)$ of wealth in a risky asset $S_1(t)$ and $u_2(t)X(t)$ in a risky asset $S_2(t)$ with proportions $u_1(t)$ in a risky asset $S_1(t)$ and $u_2(t)$ in a risky asset $S_2(t)$. $(1 - u_1(t) - u_2(t))X(t)$ has been invested in risk-free assets. In addition, the investor also spends wealth on consumption. The amount of consumption at time t is denoted by $c(t)$, which reduces the amount of wealth. As a result, the total wealth $(X(T))$ of investor with multiple risky assets can be expressed as

$$\begin{aligned}
dX(t) &= \{\text{portfolio income}\} - \{\text{consumption}\} \\
&= r(1 - u_1(t) - u_2(t))X(t) dt + u_1(t)X(t)(\mu_1 dt + \sigma_1 dW_1(t)) + \\
&\quad u_2(t)X(t)(\mu_2 dt + \sigma_2 dW_2(t)) - c(t) dt \\
&= \left[(r + u_1(t)(\mu_1 - r) + u_2(t)(\mu_2 - r))X(t) - c(t) \right] dt + \\
&\quad (\sigma_1 u_1(t) dW_1(t) + \sigma_2 u_2(t) dW_2(t))X(t).
\end{aligned} \tag{5}$$

The investor aims to achieve an optimal balance between investment and consumption so that the expected utility value of consumption over time and final wealth, is maximized based on their risk preferences. The utility function for consumption over time is represented by $U(c(t))$ and the utility function for wealth at time T is stated by $B(X(T))$. The objective function of the Merton's portfolio optimization problem with multiple risky assets is expressed by

$$\max \mathbf{E} \left[\int_0^T e^{-\delta s} U(c(s)) ds + e^{-\delta T} B(X(T)) \right], \tag{6}$$

with a dynamical system in Eq. (5) and boundary values

$$c(t) \geq 0, \quad X(t) > 0, \tag{7}$$

where $\mathbf{E}$ is the expectation operator and δ is the discount rate.

3. MAIN RESULTS

In the Merton portfolio problem, we use dynamic programming principle to solve the mathematical model. Define a function $V(t, x)$ for time interval $[t, T]$ expressed by

$$V(t, x) = \max_{(u_1, u_2, c) \in \mathcal{A}} J(u_1(\cdot), u_2(\cdot), c(\cdot); t, x), \tag{8}$$

where $J(u_1(\cdot), u_2(\cdot), c(\cdot); t, x) = \mathbf{E} \left(\int_t^T e^{-\delta(s-t)} U(c(s)) ds + e^{-\delta(T-t)} B(X(T)) \middle| \mathcal{F}_t \right)$. Here, $\mathcal{A}$ is the class of all acceptable investment and consumption strategies (u_1, u_2, c) , while $\mathcal{F}_t$ is the filtration that describes market's history and contains all information about event that has occurred up to time t . It is assumed that the optimal

strategy is developed dynamically during a very small time interval dt so that based on the Dynamic Programming Principle, we obtain

$$V(t, x) = \max_{(u_1, u_2, c) \in \mathcal{A}} \mathbf{E} \left(\int_t^{t+dt} e^{-\delta(s-t)} U(c(s)) ds + e^{-\delta dt} V(t+dt, X(t+dt)) \right). \quad (9)$$

Jacobsson [12] constructs the HJB equation from Ito's Lemma. This is similar to this study, in which the approach was carried out between t and $t+dt$ as in Eq. (9). Then, we obtain HJB equation as

$$\begin{cases} \max_{(u_1, u_2, c) \in \mathcal{A}} \mathcal{L}^{u_1, u_2, c} V(t, x) = 0, \\ V(T, x) = B(x), \end{cases} \quad (10)$$

where

$$\begin{aligned} \mathcal{L}^{u_1, u_2, c} V(t, x) = & V_t + \left[(r + u_1(t)(\mu_1 - r) + u_2(t)(\mu_2 - r))X(t) - c(t) \right] V_x + \\ & \frac{1}{2} \left(\sigma_1^2 u_1^2(t) + 2\rho\sigma_1\sigma_2 u_1(t)u_2(t) + \sigma_2^2 u_2^2(t) \right) X^2(t) V_{xx} + U(c(t)) - \delta V(t, x). \end{aligned} \quad (11)$$

The optimal investment and consumption strategy (u_1^*, u_2^*, c^*) are gained if first derivative of $\mathcal{L}^{u_1, u_2, c} V(t, x)$ is valid. The detail of derivation is presented in 7.1 Appendix 1, subsequently we obtain,

$$u_1^* = \frac{(\rho\sigma_1(\mu_2 - r) - \sigma_2(\mu_1 - r))V_x}{(1 - \rho^2)\sigma_1^2\sigma_2 x V_{xx}}, \quad (12)$$

$$u_2^* = \frac{(\rho\sigma_2(\mu_1 - r) - \sigma_1(\mu_2 - r))V_x}{(1 - \rho^2)\sigma_1\sigma_2^2 x V_{xx}}, \quad (13)$$

$$c^* = \frac{(1 - \gamma) \left(V_x^{\frac{1}{\gamma-1}} - \alpha \right)}{\eta}. \quad (14)$$

The utility function $U(c)$ is assumed to be a HARA utility function expressed by [8], [13]

$$U(c) = \frac{1 - \gamma}{\eta\gamma} \left(\alpha + \frac{\eta c}{1 - \gamma} \right)^\gamma. \quad (15)$$

In addition, the bequest function $B(x)$ is also defined following the HARA form so that the HJB equation obtained from Merton's portfolio optimization problem with mutiple risky assets is

$$\begin{cases} V_t + rxV_x + Y \frac{V_x^2}{V_{xx}} + \frac{1 - \gamma}{\eta} \alpha V_x + \frac{(1 - \gamma)^2}{\eta\gamma} V_x^{\frac{\gamma}{\gamma-1}} - \delta V = 0, \\ V(T, x) = \frac{1 - \gamma}{\eta\gamma} \left(\alpha + \frac{\eta x}{1 - \gamma} \right)^\gamma, \end{cases} \quad (t, x) \in [0, T] \times \mathbb{R}. \quad (16)$$

with $Y = \frac{2\rho\sigma_1\sigma_2(\mu_1 - r)(\mu_2 - r) - \sigma_1^2(\mu_2 - r)^2 - \sigma_2^2(\mu_1 - r)^2}{2(1 - \rho^2)\sigma_1^2\sigma_2^2}$. To verify the consistency of the HJB function and the optimal control of the Merton's portfolio optimization problem with multiple risky assets, we provide a comparison with the single risky asset model in 7.2 Appendix 2.

Based on the HARA utility function, the solution value $V(t, x)$ is explored in parametric form in accordance with the nature of investment consumption optimization. It is assumed that the solution is in the form of a power function as proposed by Ma and Zhu [8] as well as Nikoonejad and Heydari [14], i.e

$$V(t, x) = -[a_1(t)x + b_1(t)]^\gamma, \quad (17)$$

where $a_1(t)$ is a time-dependent function for wealth x and $b_1(t)$ is a time-dependent function that describes the additional contribution of non-wealth factors. The form of the power function $V(t, x)$ in Eq. (17) is based on the general form of the HARA function in Eq. (15) and the terminal conditions of the HJB function in Eq. (16), both of which are powers of the affine function. Then, a substitution of the derivative of $V(t, x)$ into the HJB equation is performed, resulting in the following system of ordinary differential equations for $a_1(t)$ and $b_1(t)$

$$\begin{cases} \dot{a}_1(t) &= \left(\frac{\delta}{\gamma} - \frac{Y}{(\gamma-1)} - r \right) a_1(t) + \frac{1}{\eta} \left(\frac{1-\gamma}{\gamma} \right)^2 (-\gamma)^{\frac{\gamma}{\gamma-1}} (a_1(t))^{\frac{2\gamma-1}{\gamma-1}}, \\ \dot{b}_1(t) &= \left(\frac{\delta}{\gamma} - \frac{Y}{(\gamma-1)} + \frac{1}{\eta} \left(\frac{1-\gamma}{\gamma} \right)^2 (-\gamma)^{\frac{\gamma}{\gamma-1}} \right) b_1(t) + \frac{\alpha(\gamma-1)}{\eta} a_1(t), \end{cases} \quad (18)$$

where the following final conditions $a_1(T) = \frac{\eta}{1-\gamma} \left(\frac{\gamma-1}{\eta\gamma} \right)^{\frac{1}{\gamma}}$ and $b_1(T) = \alpha \left(\frac{\gamma-1}{\eta\gamma} \right)^{\frac{1}{\gamma}}$.

The solution of the system of ordinary differential equations is performed separately for $a_1(t)$ and $b_1(t)$. The form of $a_1(t)$ is a non-linear differential equation. Let $h(t) = (a_1(t))^{\frac{\gamma}{1-\gamma}}$ so that the ordinary differential equation for $a_1(t)$ becomes

$$\dot{h}(t) = \left(\frac{\delta}{1-\gamma} + \frac{\gamma Y}{(1-\gamma)^2} - \frac{r\gamma}{1-\gamma} \right) h(t) + \frac{1-\gamma}{\eta\gamma} (-\gamma)^{\frac{\gamma}{\gamma-1}}. \quad (19)$$

Suppose that $A_1 = \frac{\delta}{1-\gamma} + \frac{\gamma Y}{(1-\gamma)^2} - \frac{r\gamma}{1-\gamma}$ and $B_1 = \frac{1-\gamma}{\eta\gamma} (-\gamma)^{\frac{\gamma}{\gamma-1}}$, then we obtain the following ordinary differential equation

$$\dot{h}(t) = A_1 h(t) + B_1, \quad (20)$$

with final conditions $h(T) = \frac{1}{\eta} \left(\frac{1}{1-\gamma} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{1-\gamma}}$. The solution to this ordinary differential equation is

$$h(t) = e^{-A_1(T-t)} \left[\frac{1}{\eta} \left(\frac{1}{1-\gamma} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{1-\gamma}} + \frac{B_1}{A_1} (1 - e^{A_1(T-t)}) \right]. \quad (21)$$

Consequently, the following equation represents the differential equation system's solution for $a_1(t)$

$$a_1(t) = e^{-A_1(T-t)\frac{1-\gamma}{\gamma}} \left[\frac{1}{\eta} \left(\frac{1}{1-\gamma} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{1-\gamma}} + \frac{B_1}{A_1} (1 - e^{A_1(T-t)}) \right]^{\frac{1-\gamma}{\gamma}}. \quad (22)$$

After obtaining the solution $a_1(t)$, the next step is to determine the solution $b_1(t)$. Based on Eq. (18), the form of $b_1(t)$ is a linear differential equation and can be expressed as

$$\dot{b}_1(t) = A_2(t) b_1(t) + B_2(t) \quad (23)$$

with $A_2(t) = \frac{\delta}{\gamma} - \frac{Y}{(\gamma-1)} + \frac{1}{\eta} \left(\frac{1-\gamma}{\gamma} \right)^2 (-\gamma a_1(t))^{\frac{\gamma}{\gamma-1}}$ and $B_2(t) = \frac{\alpha(\gamma-1)}{\eta} a_1(t)$. This differential equation is a linear differential equation that has the following solutions

$$b_1(t) = e^{\int_t^T A_2(s) ds} \left(b_1(T) + \int_t^T B_2(s) e^{-\int_s^T A_2(u) du} ds \right) \quad (24)$$

where $b_1(T) = \alpha \left(\frac{\gamma-1}{\eta\gamma} \right)^{\frac{1}{\gamma}}$.

After obtaining the solution to the HJB system of equations and the optimal conditions, the final step is to ensure that the optimal conditions are maximum points that satisfy the condition that the Hessian matrix must be negative definite. This Hessian matrix is shown as

$$H = \begin{bmatrix} \frac{\partial^2 \mathcal{L}^{u_1, u_2, c}}{\partial u_1^2} & \frac{\partial^2 \mathcal{L}^{u_1, u_2, c}}{\partial u_1 \partial u_2} & \frac{\partial^2 \mathcal{L}^{u_1, u_2, c}}{\partial u_1 \partial c} \\ \frac{\partial^2 \mathcal{L}^{u_1, u_2, c}}{\partial u_2 \partial u_1} & \frac{\partial^2 \mathcal{L}^{u_1, u_2, c}}{\partial u_2^2} & \frac{\partial^2 \mathcal{L}^{u_1, u_2, c}}{\partial u_2 \partial c} \\ \frac{\partial^2 \mathcal{L}^{u_1, u_2, c}}{\partial c \partial u_1} & \frac{\partial^2 \mathcal{L}^{u_1, u_2, c}}{\partial c \partial u_2} & \frac{\partial^2 \mathcal{L}^{u_1, u_2, c}}{\partial c^2} \end{bmatrix} \quad (25)$$

Consider the following part of the Hessian matrix

$$|H_1| = -\gamma(\gamma-1)\sigma_1^2 x^2 a_1^2(t) (a_1(t)x + b_1(t))^{\gamma-2}, \quad (26)$$

$$|H_2| = (1-\rho^2)\sigma_1^2 \sigma_2^2 \gamma^2 (\gamma-1)^2 x^4 a_1^4(t) (a_1(t)x + b_1(t))^{2(\gamma-2)}, \quad (27)$$

$$|H_3| = (\rho^2 - 1)\eta\gamma^2 (\gamma-1)^2 \sigma_1^2 \sigma_2^2 x^4 a_1^4(t) (a_1(t)x + b_1(t))^{2(\gamma-2)} \left(\alpha + \frac{\eta c}{1-\gamma} \right)^{\gamma-2}. \quad (28)$$

If $\gamma < 0$, $\eta > 0$, and $(a_1(t)x + b_1(t)) > 0$, then the values $|H_1| < 0$, $|H_2| > 0$, and $|H_3| < 0$, so the Hessian matrix is negative definite. From the above discussion, it can be concluded that the solution to the HJB system of equations for the Merton

portfolio problem with multiple risky assets in Eq. (16), given $\gamma < 0$ and $\eta > 0$, is as follows

$$V(t, x) = -[a_1(t)x + b_1(t)]^\gamma,$$

$$\text{where } \begin{cases} a_1(t) = e^{-A_1(T-t)\frac{1-\gamma}{\gamma}} \left[\frac{1}{\eta} \left(\frac{1}{1-\gamma} \right)^{\frac{1}{1-\gamma}} \left(\frac{\gamma-1}{\gamma} \right)^{\frac{1}{1-\gamma}} + \frac{B_1}{A_1} (1 - e^{A_1(T-t)}) \right]^{\frac{1-\gamma}{\gamma}}, \\ b_1(t) = e^{\int_t^T A_2(s) ds} \left(\alpha \left(\frac{\gamma-1}{\eta\gamma} \right)^{\frac{1}{\gamma}} + \int_t^T B_2(s) e^{-\int_s^T A_2(u) du} ds \right), \\ A_1 = \frac{\delta}{1-\gamma} + \frac{\gamma Y}{(1-\gamma)^2} - \frac{r\gamma}{1-\gamma}, \quad B_1 = \frac{1-\gamma}{\eta\gamma} (-\gamma)^{\frac{\gamma}{\gamma-1}}, \\ A_2(t) = \frac{\delta}{\gamma} - \frac{Y}{(\gamma-1)} + \frac{1}{\eta} \left(\frac{1-\gamma}{\gamma} \right)^2 (-\gamma a_1(t))^{\frac{\gamma}{\gamma-1}}, \quad B_2(t) = \frac{\alpha(\gamma-1)}{\eta} a_1(t), \\ Y = \frac{2\rho\sigma_1\sigma_2(\mu_1-r)(\mu_2-r) - \sigma_1^2(\mu_2-r)^2 - \sigma_2^2(\mu_1-r)^2}{2(1-\rho^2)\sigma_1^2\sigma_2^2}. \end{cases} \quad (29)$$

4. SIMULATION MODEL

As an illustration, we simulate the model that involves several numerical methods, including the Euler method and the trapezoidal method. The model parameters required for the simulation are defined in advance. For each of the simulations, a time grid is constructed with $t \in [0, T]$. Then, the constants Y , A_1 , and B_1 , as well as the function $a_1(t)$, are calculated based on Eq. (29). However, the functions $A_2(t)$ and $B_2(t)$ are involved in the function $b_1(t)$, which contain the integration of the function $a_1(t)$. Since there are boundary conditions as in Eq. (24) and a simple numerical integration method is required, the backward trapezoidal method is chosen to obtain the value of function $a_1(t)$ from $t = T$ to $t = 0$. For the time partition $[t_i, t_{i+1}]$ divided into n sub-intervals, the integration result of the function $a_1(t_i)$, denoted by $I(t_i)$, is obtained using the following equation

$$I(t_i) \approx \frac{h}{2} \left(a_1(t_i) + 2 \sum_{j=1}^{n-1} a_1(t_j) + a_1(t_{i+1}) \right), \quad (30)$$

where $h = \frac{\Delta t}{n}$ and $\Delta t = t_{i+1} - t_i$.

Next, the wealth dynamics model is discretized using the Euler Maruyama method. Eq. (5) is approached using the left Riemann integral for the partition $[t_i, t_{i+1}]$ for $i = 0, 1, 2, \dots, N-1$ so that we obtain

$$X(t_{i+1}) = X(t_i) + \left[(r + u_1(t_i)(\mu_1 - r) + u_2(t_i)(\mu_2 - r))X(t_i) - c(t_i) \right] \Delta t + \left(\sigma_1 u_1(t_i) \Delta W_1 + \sigma_2 u_2(t_i) \Delta W_2 \right) X(t_i), \quad (31)$$

where $\Delta W_1 = W_1(t_{i+1}) - W_1(t_i)$ and $\Delta W_2 = W_2(t_{i+1}) - W_2(t_i)$. An analysis of the convergence of the Euler Maruyama model in Eq. (31) is presented in 7.3 Appendix

3. Let $\tilde{W}_2(t)$ is another independent Wiener process and is defined

$$\Delta W_2 = \rho \Delta W_1 + \sqrt{1 - \rho^2} \Delta \tilde{W}_2. \quad (32)$$

In terms of correlation as in Eq. (4), we derive the correlation between ΔW_1 and ΔW_2 is ρ , which is suitable with what we defined in the beginning. Hull [15] states that the Wiener process has a variable change in a small time period of $Z\sqrt{\Delta t}$, with Z distributed normally $(0, 1)$ so that Eq. (32) can be expressed as

$$\begin{aligned} \Delta W_1 &= Z_1(t_i) \sqrt{\Delta t}, \\ \Delta W_2 &= \sqrt{\Delta t} \left(\rho Z_1(t_i) + \sqrt{1 - \rho^2} \tilde{Z}_2(t_i) \right) \end{aligned} \quad (33)$$

where $Z_1(t_i)$ and $\tilde{Z}_2(t_i)$ are random number with a standard normal distribution $N(0, 1)$. Therefore, we simulate with the following discretization

$$\begin{aligned} X(t_{i+1}) &= X(t_i) + \left[(r + u_1(t_i)(\mu_1 - r) + u_2(t_i)(\mu_2 - r))X(t_i) - c(t_i) \right] \Delta t + \\ &\quad (\sigma_1 u_1(t_i) Z_1(t_i) + \sigma_2 u_2(t_i) \tilde{Z}_2(t_i)) X(t_i) \sqrt{\Delta t}, \\ Z_2(t_i) &= \rho Z_1(t_i) + \sqrt{1 - \rho^2} \tilde{Z}_2(t_i). \end{aligned} \quad (34)$$

The Monte Carlo simulation was conducted by generating 10^6 wealth paths and generating standard normally distributed random numbers $Z_1(t_i)$ and $\tilde{Z}_2(t_i)$, for $i = 0, 1, 2, \dots, N - 1$. The wealth value at time t was obtained from the average of all wealth paths. This can be written by the following equation

$$\mathbf{E}[X(t)] = \frac{1}{M} \sum_{k=1}^M X_k(t_i). \quad (35)$$

After that, the value of each optimal controls is obtained by Eq. (12), Eq. (13), and Eq. (14). The final step is to create several plots of results, including the value function $V(t, x)$ and optimal controls $(u_1(t), u_2(t), c(t))$ movements against wealth to visualize the numerical simulation results.

5. ANALYSIS

The optimal condition analytical solution in Eq. (12) and Eq. (13) shows that the optimal investment proportions u_1 and u_2 are inversely proportional to wealth x . Mathematically, this shows that current wealth affects the optimal investment proportions for both, the first risky asset and the second risky asset. This aligns with the financial meaning where an investor's total wealth influences their risk aversion level. The larger the total wealth an investor holds, the higher their risk aversion level. Eq. (12) and Eq. (13) also represent that the optimal solution changes over time. To further illustrate how the optimal policy changes over time t , the optimal investment proportions u_1 and u_2 are shown in Figure 1 with the

following parameters whose values are determined based on data from Erb, Harvey, and Viskanta [16],

$$\begin{aligned} \mu_1 = 0.1, \quad \mu_2 = 0.07, \quad r = 0.05, \quad \sigma_1 = 0.35, \quad \sigma_2 = 0.2, \quad \rho = 0.5, \quad \delta = 0.05, \\ \gamma = -0.15, \quad \eta = 1, \quad \alpha = 0.01, \quad T = 1. \end{aligned} \quad (36)$$

Given that the optimal investment proportions u_1 and u_2 have a singular point at $x = 0$ due to the form Eq. (12) and (13), Figure 1 has a domain of $[0.02, 1]$. This range of values was chosen as a unit of investor wealth so that the interpretation of the results is flexible to various total wealth amounts. Investors with low wealth tend to be very willing to take risks to increase their wealth. In this context, they are prepared to invest more in risky assets that can provide high returns, even though they also carry high risks. The goal is to achieve higher returns so that their wealth can grow more significantly. On the other hand, investors with large wealth may choose to reduce risk by shifting more of their wealth into safer and more stable investments over the long term. Additionally, with substantial wealth, investors feel their financial goals are already met. They are instead more focused on asset management and risk avoidance.

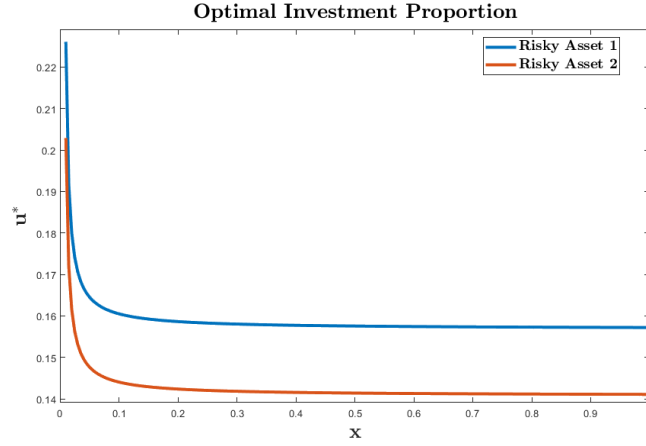


FIGURE 1. Optimal Investment Proportion

As can be seen in Figure 1, there is a difference in the amount of wealth invested in risky asset 1 and risky asset 2. This is based on the return and volatility values of each risky asset. If a risky asset has a higher return and volatility, then the proportion of investment for that risky asset is greater than other risky assets. This is in line with the simulation results, which show that risky asset 1 has a higher return and volatility than risky asset 2, resulting in a larger proportion of investment in risky asset 1 than in risky asset 2. On the other hand, the correlation between risky asset 1 and risky asset 2 may encourage investors to use a risky asset diversification strategy. This is one of strategy for reducing investment risk,

especially in risky assets. When losses occur in one risky asset, there are still other risky assets that can recover the loss.

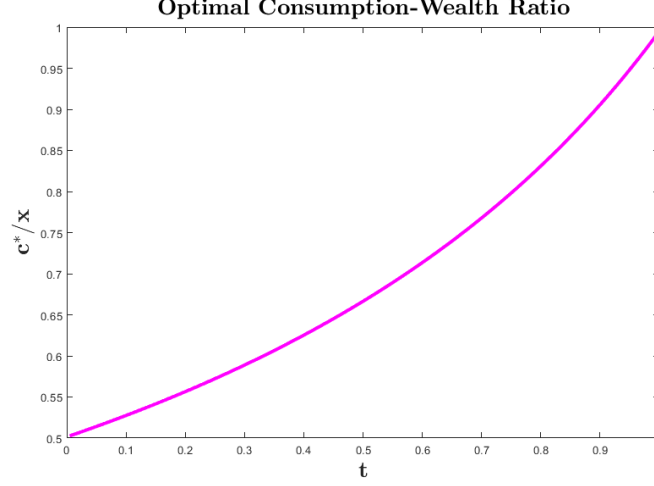


FIGURE 2. Optimal Consumption-Wealth Ratio

After discussing the optimal investment proportion, another component of optimal conditions is optimal consumption, which is presented in Figure 2. This graph shows that optimal consumption increases slightly over time. This indicates that investors use their wealth for consumption continuously over time, even at terminal time $t = T$. Investors choose to enjoy more consumption than before in order to increase their utility from consumption. A investor's utility can be achieved by fulfilling their needs. This is due to the definition of utility in consumption, which represents the value of the investor's wealth for consumption. Therefore, it makes sense for investors to continuously increase their consumption to maximize the value of utility in consumption over time.

The Merton portfolio model has an objective function that aims to maximize the expected utility of investors from their final wealth. The relationship between the objective function value, time, and wealth is presented in Figure 3. This graph shows that as wealth increases, the value of utility obtained also increases. This aligns with the principle of the time value of money, where greater wealth yields greater value. Additionally, individuals with more wealth have a greater capacity to consume, thereby enhancing their utility. In the optimization model, the closer to the end point, the clearer the goal to be achieved, especially since there is a substantial final value function of realized wealth. This reflects how investors optimize their consumption and investment to maximize utility over their lifetime, taking into account the time remaining. Figure 3 also represents the movement of the total utility value over time t . Based on the graph, the total utility value continues to increase as time progresses. Since the investor's final wealth has a

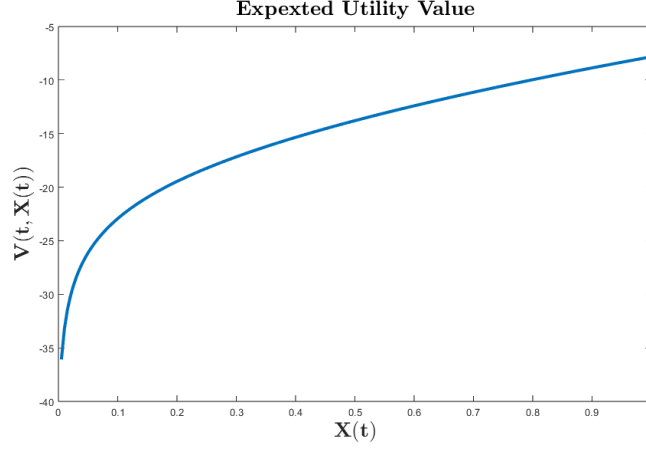


FIGURE 3. Expected Utility Value

greater value than the initial wealth, the investor tends to be more conservative and cautious in long-term investments. It is because the investor wants to avoid losses resulting from the dynamics of risky assets. Therefore, the investor allocates their wealth for consumption as the end of the period approaches rather than investing in risky assets. This is done to achieve maximum utility.

6. CONCLUSIONS

The simulation results of the optimal solution to Merton's multi-asset portfolio problem show that the optimal investment proportion decreases as wealth increases, with low-wealth investors tending to be more willing to take risks to increase their wealth by investing more in risky assets. Conversely, investors with large wealth prefer to reduce risk by shifting their investments to safer assets, and focus more on wealth management and risk avoidance. Additionally, optimal consumption increases sharply over time, with consumption initially lower due to a focus on investment. Then, it begins to increase dramatically as the time horizon increases. This can happen when consumption becomes more important than investment since investors feel financially secure and no longer delay consumption.

This study has several limitations that open up opportunities for further research. First, a perfect market is used in this study, without transaction costs or other taxes. Second, the time considered is finite at the interval $[0, T]$ so that the model constructed only applies to finite time. Since there are many parameters, further study is required to analyze the sensitivity of these parameters. In addition, for further development, this model can be expanded to other alternative assets such as gold, foreign exchange, or property.

7. APPENDIX

7.1. Appendix 1.

To obtain the optimal strategy (u_1^*, u_2^*, c^*) , we use the function $\mathcal{L}^{u_1, u_2, c}V(t, x)$ and involve its first derivative with respect to each strategy. First, we derive the function for the optimal investment in risky asset 1 (u_1) so that we obtain

$$\begin{aligned} \frac{\partial \mathcal{L}^{u_1, u_2, c}V}{\partial u_1} &= 0, \\ (\mu_1 - r)xV_x + \sigma_1^2 u_1 x^2 V_{xx} + \rho \sigma_1 \sigma_2 u_2 x^2 V_{xx} &= 0, \\ u_1 &= -\frac{(\mu_1 - r)V_x}{\sigma_1^2 x V_{xx}} - \frac{\rho \sigma_2 u_2}{\sigma_1}, \\ u_1 &= \frac{(\rho \sigma_1 (\mu_2 - r) - \sigma_2 (\mu_1 - r))V_x}{(1 - \rho^2) \sigma_1^2 \sigma_2 x V_{xx}}. \end{aligned} \quad (37)$$

Then, in order to get the the optimal investment risky asset 2, we derive the function $\mathcal{L}^{u_1, u_2, c}V(t, x)$ with respect to u_2 as

$$\begin{aligned} \frac{\partial \mathcal{L}^{u_1, u_2, c}V}{\partial u_2} &= 0, \\ (\mu_2 - r)xV_x + \rho \sigma_1 \sigma_2 u_1 x^2 V_{xx} + \sigma_2^2 u_2 x^2 V_{xx} &= 0, \\ u_2 &= -\frac{(\mu_2 - r)V_x}{\sigma_2^2 x V_{xx}} - \frac{\rho \sigma_1 u_1}{\sigma_2}, \\ u_2 &= \frac{(\rho \sigma_2 (\mu_1 - r) - \sigma_1 (\mu_2 - r))V_x}{(1 - \rho^2) \sigma_1 \sigma_2^2 x V_{xx}}. \end{aligned} \quad (38)$$

Last, we continue to derive the function $\mathcal{L}^{u_1, u_2, c}V(t, x)$ for c so that we get the optimal consumption as

$$\begin{aligned} \frac{\partial \mathcal{L}^{u_1, u_2, c}V}{\partial c} &= 0, \\ U_c &= V_x, \\ c &= (U_c)^{-1}(V_x) \\ c &= \frac{(1 - \gamma) \left(V_x^{\frac{1}{\gamma-1}} - \alpha \right)}{\eta}. \end{aligned} \quad (39)$$

7.2. Appendix 2.

Set $\mu_1 = \mu_2 = \mu$, $\sigma_1 = \sigma_2 = \sigma$, $\rho = 1$, and $dW_1(t) = dW_2(t) = dW(t)$ so we have

$$\begin{aligned} dX(t) &= \left[(r + u_1(t)(\mu - r) + u_2(t)(\mu - r))X(t) - c(t) \right] dt + \\ &\quad (\sigma u_1(t) dW(t) + \sigma u_2(t) dW(t))X(t) \\ &= \left[(r + u(t)(\mu - r))X(t) - c(t) \right] dt + \sigma u(t) dW(t)X(t). \end{aligned} \quad (40)$$

with $u(t) = u_1(t) + u_2(t)$. Eq. (40) shows. Later, HJB equation in Eq. (10) satisfy with one risky asset model where

$$\begin{aligned}\mathcal{L}^{u_1, u_2, c}V(t, x) &= V_t + \left[(r + u_1(t)(\mu - r) + u_2(t)(\mu - r))X(t) - c(t) \right] V_x + \\ &\quad \frac{1}{2} \left(\sigma^2 u_1^2(t) + 2\sigma^2 u_1(t)u_2(t) + \sigma^2 u_2^2(t) \right) X^2(t) V_{xx} + U(c(t)) - \delta V(t, x) \\ &= V_t + \left[(r + u(t)(\mu - r))X(t) - c(t) \right] V_x + \frac{1}{2} \sigma^2 u^2(t) X^2(t) V_{xx} + U(c(t)) - \delta V(t, x).\end{aligned}\tag{41}$$

The optimal control in Eq. (12) and Eq. (13) fulfills one risky asset model as follows

$$\begin{aligned}u^*(t) &= u_1^*(t) + u_2^*(t) \\ &= \frac{(\rho\sigma_1(\mu_2 - r) - \sigma_2(\mu_1 - r))V_x}{(1 - \rho^2)\sigma_1^2\sigma_2xV_{xx}} + \frac{(\rho\sigma_2(\mu_1 - r) - \sigma_1(\mu_2 - r))V_x}{(1 - \rho^2)\sigma_1\sigma_2^2xV_{xx}} \\ &= -\frac{(\sigma_1^2(\mu_2 - r) + \sigma_2^2(\mu_1 - r) - \rho\sigma_1\sigma_2[(\mu_1 - r) + (\mu_2 - r)])V_x}{(1 - \rho^2)\sigma_1^2\sigma_2^2xV_{xx}} \\ &= -\frac{(\sigma^2(\mu - r) + \sigma^2(\mu - r) - \rho\sigma^2[(\mu - r) + (\mu - r)])V_x}{(1 - \rho^2)\sigma^4xV_{xx}} \\ &= -\frac{2\sigma^2(\mu - r)(1 - \rho)V_x}{(1 - \rho)(1 + \rho)\sigma^4xV_{xx}} \\ &= -\frac{(\mu - r)V_x}{\sigma^2xV_{xx}}.\end{aligned}\tag{42}$$

HJB equation in Eq. (16) also meet with one risky asset model where

$$\begin{aligned}Y &= \frac{2\rho\sigma\sigma_2(\mu - r)(\mu - r) - \sigma^2(\mu - r)^2 - \sigma^2(\mu - r)^2}{2(1 - \rho^2)\sigma^4} \\ &= -\frac{2\sigma^2(\mu - r)^2(1 - \rho)}{2(1 - \rho)(1 + \rho)\sigma^4} \\ &= -\frac{(\mu - r)^2}{2\sigma^2}.\end{aligned}\tag{43}$$

Note that one risky asset model discussed here is refers to the model in [8].

7.3. Appendix 3.

Set $f(X(t_i)) = \left[(r + u_1(t_i)(\mu_1 - r) + u_2(t_i)(\mu_2 - r))X(t_i) - c(t_i) \right]$ and matrix function $g(X(t_i)) = \begin{pmatrix} \sigma_1 u_1(t_i)X(t_i) & \sigma_2 u_2(t_i)X(t_i) \end{pmatrix}$. Then, let ΔW is 2-dimensional Brownian Motion described by $\Delta W = (\Delta W_1 \quad \Delta W_2)^T$ so Eq. (31) can be expressed by

$$X(t_{i+1}) = X(t_i) + f(X(t_i))\Delta t + g(X(t_i))\Delta W.\tag{44}$$

It can be confirmed that the function f is globally continuous Lipschitz function because there exists a constant $K_1 = r + u_1(t)(\mu_1 - r) + u_2(t)(\mu_2 - r)$ such that $|f(x) - f(y)| \leq K_1|x - y|$ as

$$\begin{aligned} |f(x) - f(y)| &= \left| (r + u_1(t)(\mu_1 - r) + u_2(t)(\mu_2 - r))(x - y) \right| \\ &\leq \left| r + u_1(t)(\mu_1 - r) + u_2(t)(\mu_2 - r) \right| |x - y|. \end{aligned} \quad (45)$$

Furthermore, using the Euclidean norm, we can show in the same way that the function g is globally continuous Lipschitz function.

$$\begin{aligned} \|g(x) - g(y)\| &= \sqrt{(x - y)^2 (\sigma_1^2 u_1^2(t) + \sigma_2^2 u_2^2(t))} \\ &\leq |x - y| \sqrt{(\sigma_1^2 u_1^2(t) + \sigma_2^2 u_2^2(t))} \end{aligned} \quad (46)$$

so there is a constant $K_2 = \sqrt{(\sigma_1^2 u_1^2(t) + \sigma_2^2 u_2^2(t))}$ such that $|g(x) - g(y)| \leq K_2|x - y|$. Since f and g are globally continuous Lipschitz functions, Eq. (31) can be shown that it has strong convergence order one, i.e. $\gamma = 1$, as demonstrated by [17] and [18].

Data Availability Statement

No new data were created or analyzed in this study.

Declarations. The authors state that they have no known financial conflicts of interest or personal relationships that could be considered to influence the research reported in this paper.

Author Contributions. Chasna Zahara: conceptualization, methodology, software, visualization, writing - original draft. Endah RM Putri: supervision, writing - review & editing, funding acquisition. All authors discussed the results and contributed to the final manuscript.

Funding Statement. The authors gratefully acknowledge financial support from The Directorate General of Higher Education, Research, and Technology of the Ministry of Education, Culture, Research, and Technology (MOECRT) of the Republic of Indonesia for this work, under Master Thesis Research Scheme with main contract number 017/C3/DT.05.00/PL/2025 and derivative contract number 1116/PKS/ITS/2025.

Acknowledgment. The authors would like to thank Institut Teknologi Sepuluh Nopember as a campus affiliate, particularly the Laboratory of Mathematics for Industry and Finance. Special thanks go to Mr. Subchan, Mrs. Valeriana Lukitosari, and Mrs. Sena Safarina for their valuable suggestions regarding the process and results of this research.

REFERENCES

- [1] S.-P. Zhu and G. Ma, “An Analytical Solution for The HJB Equation Arising from The Merton Problem,” *International Journal of Financial Engineering*, vol. 5, no. 1, p. 1850008, 2018. <https://doi.org/10.1142/S2424786318500081>.
- [2] D. Hobson, A. S. Tse, and Y. Zhu, “A Multi-Asset Investment and Consumption Problem with Transaction Costs,” *Finance and Stochastics*, vol. 23, pp. 641–676, 2019. <https://doi.org/10.1007/s00780-019-00391-6>.
- [3] N. Wang and T. K. Siu, “Investment-Consumption Optimization with Transaction Cost and Learning about Return Predictability,” *European Journal of Operational Research*, vol. 318, no. 3, pp. 877–891, 2024. <https://doi.org/10.1016/j.ejor.2024.06.024>.
- [4] R. C. Merton, “Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case,” *The Review of Economics and Statistics*, pp. 247–257, 1969. <https://doi.org/10.2307/1926560>.
- [5] Z. Bodie, “Robert C. Merton and The Science of Finance,” *Annual Review of Financial Economics*, vol. 12, no. 1, pp. 19–38, 2020. <https://doi.org/10.1146/annurev-financial-100520-074656>.
- [6] H. U. Gerber and G. Pafum, “Utility Functions: from Risk Theory to Finance,” *North American Actuarial Journal*, vol. 2, no. 3, pp. 74–91, 1998. <https://doi.org/10.1080/10920277.1998.10595728>.
- [7] R. C. Merton, “Optimum Consumption and Portfolio Rules in A Continuous-Time Model,” in *Stochastic Optimization Models in Finance*, pp. 621–661, Elsevier, 1971. <https://doi.org/10.1016/B978-0-12-780850-5.50052-6>.
- [8] G. Ma and S.-P. Zhu, “Revisiting The Merton Problem: from HARA to CARA Utility,” *Computational Economics*, pp. 1–36, 2021. <https://doi.org/10.1007/s10614-021-10102-z>.
- [9] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, vol. 43. Springer Science & Business Media, 1999. <https://doi.org/10.1007/978-1-4612-1466-3>.
- [10] L. Jiang, *Mathematical Modeling and Methods of Option Pricing*. World Scientific Publishing Company, 2005. <https://doi.org/10.1142/5855>.
- [11] U. F. Wiersema, *Brownian Motion Calculus*. John Wiley & Sons, 2008. www.wiley.com/go/brownianmotioncalculus.
- [12] H. Jacobsson, “On Merton’s Portfolio Problem: A Stochastic Optimal Control Problem,” 2022.
- [13] M. Nguyen, “Finite and Mean Field Games for Optimal Investment with HARA Utility Function and The Presence of Risk-Seeking Agents,” *Research Square*, 2024. <https://doi.org/10.21203/rs.3.rs-3861413/v1>.
- [14] Z. Nikooeinejad and M. Heydari, “Numerical solution of hamilton–jacobi–bellman pdes in stochastic optimal control problems using fractional-order legendre collocation method,” *Journal of Vibration and Control*, vol. 30, no. 21–22, pp. 5078–5096, 2024. <https://doi.org/10.1177/10775463231217531>.

- [15] J. C. Hull, “Options, Futures, and Other Derivatives, Global Edition,” *Harlow: United Kingdom: Pearson*, 2018.
- [16] C. B. Erb, C. R. Harvey, and T. E. Viskanta, “Expected Returns and Volatility in 135 Countries,” *Available at SSRN 871253*, 1996.
- [17] Z. Zhang and G. E. Karniadakis, “Numerical Methods for Stochastic Partial Differential Equations with White Noise,” *Applied mathematical sciences*, vol. 196, 2017. <https://link.springer.com/book/10.1007/978-3-319-57511-7>.
- [18] E. Rahman, M. Subhan, and H. Maulana, “Convergence Analysis of Euler-Maruyama Method on Ornstein-Uhlenbeck Equation,” in *Journal of Physics: Conference Series*, vol. 1554, p. 012004, IOP Publishing, 2020. <https://iopscience.iop.org/article/10.1088/1742-6596/1554/1/012004>.