

Multiple Optimal Control Strategies in a Trypanosomiasis-Malaria co-infection Reaction-Diffusion Model

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Abstract. A reaction–diffusion model is formulated to investigate the spatial movement of humans and vectors in the co-transmission dynamics of trypanosomiasis and malaria. The existence, positivity, and boundedness of solutions are rigorously established, thereby ensuring the mathematical well-posedness of the model. Prevention and treatment interventions are incorporated as spatio-temporal control variables to characterise optimal control strategies aimed at minimising the populations of exposed and infected humans and vectors at the least possible cost. The optimal control characterisation is derived through the state and adjoint systems by applying the directional derivative of the objective functional with respect to the control variables. Numerical simulations are presented to validate the analytical results and demonstrate the effectiveness of the proposed control strategies.

Key words and Phrases: Optimal control, Sensitivity system, Adjoint system, Objective functional.

1. INTRODUCTION

Mathematical modelling of infectious diseases has evolved over the years, with the first compartmental epidemic model introduced in 1927 by Kermack and McKendrick. This pioneering framework laid the foundation for numerous extensions and refinements by subsequent researchers [1]. Malaria remains one of the most extensively studied infectious diseases because of its persistent global burden, particularly in sub-Saharan Africa. Several mathematical models describing the transmission dynamics of malaria have been presented in the literature [2, 3, 4]. Similarly, trypanosomiasis thrives in environments where other tropical diseases, particularly malaria, are also prevalent. This overlap exists because both diseases

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are transmitted by vectors that inhabit similar ecological settings [5]. Co-infection with malaria often results in complications and more severe manifestations of parasitic diseases [6]. Gervas et al. [7] formulated a deterministic mathematical model for trypanosomiasis that incorporates control measures. However, disease control remains a major challenge in resource-limited settings. Interventions such as vector reduction, treatment provision, and other preventive measures often compete for the same limited resources.

The foundational framework for the optimal control of partial differential equations was established by J. L. Lions in 1971. Since then, several extensions of Pontryagin's Maximum Principle to systems governed by partial differential equations have been developed, as reported by [8]. Miyaoka et al. [9] applied optimal control to a PDE model for zika virus using vaccination as a control variable with the aim of minimizing the costs associated with the application of vaccine and infections. The optimality system was developed and solved by applying appropriate numerical techniques so as to simulate possible situations. Miyaoka et al. [10] applied multiple control strategies like mosquito elimination, minimization of vector and human contact, and vaccination to a reaction diffusion model to characterize optimal strategy that reduces associated cost with interventions and infections. Adnaoui and Laaroussi [11] presented an optimal control theory of a partial differential form of a two dimensional spatial-temporal SEIR model with the aim of reducing the population of susceptible and infected while maximizing recovered individuals through cost-efficient vaccination strategies. The characterization of the optimal control dynamics was obtained in terms of adjoint and state functions. Liu et al. [12] formulated a spatially diffusive heterogeneous SIVR model under Neumann boundary conditions with partial immunity effects and latent infection. The system is incorporated with patient's treatment as the control parameter before Hamiltonian function with adjoint equation were used to discuss the optimality conditions. Mehdaoui et al. [13] used the adjoint state method to derive a reaction-diffusion SIR dynamics with heterogeneous incidence rate by incorporating treatment and vaccination as controls. Also, the model subdivides each epidemiological compartment into multiple sub-groups, capturing diverse transmission pathways.

The optimal control modelling of malaria and trypanosomiasis have substantial body of literature along ordinary differential equations. While temporal modelling provides treasured insights into the dynamics of malaria over time, optimal control of spatial modelling provides understanding of the disease's geographical aspects, enhancing the effectiveness of interventions and promoting sustainable malaria control strategies. A limited number of studies have investigated the spatio-temporal optimal control of malaria [14, 15, 16], yet research on the spatial optimal control of trypanosomiasis remains scarce. More critically, to the best of our knowledge, no study has yet addressed the spatio-temporal optimal control of trypanosomiasis-malaria co-infection, a deficit that directly motivates the present work.

In this paper, we extend the reaction diffusion trypanosomiasis malaria co-infection model of [17] by incorporating spatiotemporal preventive and treatment

controls. This refinement is essential because intervention coverage and effectiveness often vary across regions and over the course of an outbreak. Our framework accounts for host mobility and spatially varying transmission pressures, which are critical drivers of co-infection dynamics in endemic areas. We subsequently apply Pontryagin's Maximum Principle to derive the necessary optimality conditions for the controlled system, thereby providing a rigorous mathematical basis for designing spatially and temporally adaptive public health strategies. The remainder of this paper is organized as follows. Section 2 presents the model formulation, Section 3 establishes the main analytical results, Section 4 provides numerical simulations and a discussion of the findings, and Section 5 concludes the study.

2. MODEL FORMULATION

This section is devoted to the formulation of the trypanosomiasis–malaria co-infection model. We begin by presenting the uncontrolled reaction–diffusion system, which describes the spatial and temporal spread of both diseases in the absence of intervention. Subsequently, we extend this framework to include preventive and treatment controls, resulting in a controlled model that will be analyzed using optimal control theory in the subsequent section.

2.1. Co-infection model without control.

The proposed model for the co-infection of malaria and trypanosomiasis by [17] divides the total human population, N_h , into several epidemiological compartments. Susceptible humans are denoted by N_{ss} . Humans who are exposed to, infected with, and recovered from trypanosomiasis but remain susceptible to malaria are represented by N_{es} , N_{is} , and N_{rs} , respectively. Similarly, humans who are exposed to, infected with, and recovered from malaria but remain susceptible to trypanosomiasis are represented by N_{se} , N_{si} , and N_{sr} , respectively. A distinct compartment, N_{rr} , represents individuals who are not susceptible to either infection. Transmission occurs when an infectious mosquito or tsetse fly bites a susceptible individual at rates $c_m\beta_h$ and $c_t\phi_h$, respectively.

The susceptible class is increased through the loss of immunity to malaria and trypanosomiasis, occurring at rates ω_m and ω_t , respectively, as well as through recruitment at rate Λ_h . The human population exposed to trypanosomiasis but susceptible to malaria is increased at the rate $\frac{c_t\phi_h N_{ss} I_t}{N_h}$ following contact with infectious tsetse flies. This class is reduced through progression to the infected class at rate σ_t and through interaction with infectious mosquitoes. The human population infected with trypanosomiasis but susceptible to malaria is increased through progression from the exposed class. It is reduced through disease-induced death due to trypanosomiasis at rate δ_a , recovery at rate θ_t , and interaction with infectious mosquitoes. The human population that has recovered from trypanosomiasis but remains susceptible to malaria is increased through progression from

the infected class. It is reduced through loss of immunity at rate ω_t and through interaction with infectious mosquitoes.

The human population exposed to malaria but susceptible to trypanosomiasis is increased at the rate $\frac{c_m \beta_h N_{ss} I_m}{N_h}$ following contact with infectious mosquitoes. This class is reduced through progression to the infected class at rate σ_m and through interaction with infectious tsetse flies. The human population infected with malaria but susceptible to trypanosomiasis is increased through progression from the exposed class. It is reduced through malaria-induced death at rate δ_b , recovery at rate θ_m , and interaction with infectious tsetse flies. The human population that has recovered from malaria but remains susceptible to trypanosomiasis is increased through progression from the infected class. It is reduced through loss of immunity at rate ω_m and through interaction with infectious tsetse flies. The human population that is not susceptible to either disease is increased through the addition of individuals from the exposed, infected, and recovered classes of either infection. It is reduced through loss of immunity. All human compartments experience natural death at rate μ_h .

The tsetse fly compartment is categorized into susceptible, S_t , exposed, E_t , and infected, I_t . Similarly, the mosquito compartment is classified as susceptible, S_m , exposed, E_m , and infected, I_m . The susceptible tsetse fly population becomes exposed through effective contact with infectious human with trypanosomiasis with transmission rate $c_t \phi_t$. This population rises through migration at rate Λ_t and decreases by natural death at rate μ_t . The population of exposed tsetse fly rises at rate $\frac{c_t \phi_t N_{is} S_t}{N_h}$ and decreases through advancement to infected tsetse fly at rate α_t , and natural death at rate μ_t . The population of infected tsetse fly is increased as a result of progression from exposed class and decreased by natural death and diseased induced death at rate μ_t and δ_t respectively.

Susceptible mosquito becomes exposed through effective contact with infectious human with malaria with transmission rate $c_m \beta_t$. This population rises through new recruitment at rate Λ_m and decreased by natural death at rate μ_m . Exposed mosquito is increased at rate $\frac{c_m \beta_m N_{si} S_m}{N_h}$ and decreased through advancement to infected compartment at rate α_m , and natural death at rate μ_m . The population of infected mosquito rises through additions from exposed class and decreased by natural death and diseased induced death at rate μ_m and δ_m respectively. Complex transmission systems of both diseases justifies the complexity of (1).

All human and vector population classes are assumed to move spatially. Let $D_1, D_2, \dots, D_{14}$ denote the diffusion coefficients of $N_{ss}, N_{es}, N_{is}, N_{rs}, N_{se}, N_{si}, N_{sr}, N_{rr}, S_t, E_t, I_t, S_m, E_m$, and I_m , respectively. Hence, the resulting system of fourteen partial differential equations is described in (1). The model parameters are also presented in Table1.

$$\left. \begin{aligned}
\frac{\partial N_{ss}}{\partial t} &= D_1 \Delta N_{ss} + \Lambda_h - \frac{c_t \phi_h N_{ss} I_t}{N_h} - \frac{c_m \beta_h N_{ss} I_m}{N_h} - \mu_h N_{ss} \\
&\quad + \omega_t N_{rs} + \omega_m N_{sr}, \\
\frac{\partial N_{es}}{\partial t} &= D_2 \Delta N_{es} + \frac{c_t \phi_h N_{ss} I_t}{N_h} - (\mu_h + \sigma_t) N_{es} - \frac{\rho_m \lambda_m N_{es} I_m}{N_h}, \\
\frac{\partial N_{is}}{\partial t} &= D_3 \Delta N_{is} + \sigma_t N_{es} - (\mu_h + \delta_a + \theta_t) N_{is} - \frac{\rho_m \lambda_m N_{is} I_m}{N_h}, \\
\frac{\partial N_{rs}}{\partial t} &= D_4 \Delta N_{rs} + \theta_t N_{is} - (\mu_h + \omega_t) N_{rs} + \epsilon_m N_{rr} - \frac{\lambda_m N_{rs} I_m}{N_h}, \\
\frac{\partial N_{se}}{\partial t} &= D_5 \Delta N_{se} + \frac{c_m \beta_h N_{ss} I_m}{N_h} - (\mu_h + \sigma_m) N_{se} - \frac{\rho_t \lambda_t N_{se} I_t}{N_h}, \\
\frac{\partial N_{si}}{\partial t} &= D_6 \Delta N_{si} + \sigma_m N_{se} - (\mu_h + \delta_b + \theta_m) N_{si} - \frac{\rho_t \lambda_t N_{si} I_t}{N_h}, \\
\frac{\partial N_{sr}}{\partial t} &= D_7 \Delta N_{sr} + \theta_m N_{si} - (\mu_h + \omega_m) N_{sr} + \epsilon_t N_{rr} - \frac{\lambda_t N_{sr} I_t}{N_h}, \\
\frac{\partial N_{rr}}{\partial t} &= D_8 \Delta N_{rr} + \frac{\rho_m \lambda_m (N_{es} + N_{is}) I_m}{N_h} + \frac{\rho_t \lambda_t (N_{se} + N_{si}) I_t}{N_h} \\
&\quad - (\mu_h + \epsilon_t + \epsilon_m) N_{rr} + \frac{\lambda_m N_{rs} I_m}{N_h} + \frac{\lambda_t N_{sr} I_t}{N_h}, \\
\frac{\partial S_t}{\partial t} &= D_9 \Delta S_t + \Lambda_t - \frac{c_t \phi_t N_{is} S_t}{N_h} - \mu_t S_t, \\
\frac{\partial E_t}{\partial t} &= D_{10} \Delta E_t + \frac{c_t \phi_t N_{is} S_t}{N_h} - (\alpha_t + \mu_t) E_t, \\
\frac{\partial I_t}{\partial t} &= D_{11} \Delta I_t + \alpha_t E_t - (\delta_t + \mu_t) I_t, \\
\frac{\partial S_m}{\partial t} &= D_{12} \Delta S_m + \Lambda_m - \frac{c_m \beta_m N_{si} S_m}{N_h} - \mu_m S_m, \\
\frac{\partial E_m}{\partial t} &= D_{13} \Delta E_m + \frac{c_m \beta_m N_{si} S_m}{N_h} - (\alpha_m + \mu_m) E_m, \\
\frac{\partial I_m}{\partial t} &= D_{14} \Delta I_m + \alpha_m E_m - (\delta_m + \mu_m) I_m.
\end{aligned} \right\} \quad (1)$$

The model is equipped with a Neumann boundary condition (2).

$$\frac{\partial N_{ss}}{\partial n} = \frac{\partial N_{es}}{\partial n} = \dots = \frac{\partial I_m}{\partial n} = 0, \text{ on } \partial\Omega \times [0, +\infty], \quad (2)$$

where Ω is a bounded set in $\mathbb{R}^n$ and smooth boundary $\partial\Omega$. The initial conditions are given as

$$\begin{aligned}
N_{ss}(\mathbf{x}, 0) &= \phi_1(x), N_{es}(\mathbf{x}, 0) = \phi_2(x), N_{is}(\mathbf{x}, 0) = \phi_3(x), N_{rs}(\mathbf{x}, 0) = \phi_4(x), \\
N_{se}(\mathbf{x}, 0) &= \phi_5(x), N_{si}(\mathbf{x}, 0) = \phi_6(x), N_{sr}(\mathbf{x}, 0) = \phi_7(x), N_{rr}(\mathbf{x}, 0) = \phi_8(x), \\
S_t(\mathbf{x}, 0) &= \phi_9(x), E_t(\mathbf{x}, 0) = \phi_{10}(x), I_t(\mathbf{x}, 0) = \phi_{11}(x), S_m(\mathbf{x}, 0) = \phi_{12}(x), \\
E_m(\mathbf{x}, 0) &= \phi_{13}(x), I_m(\mathbf{x}, 0) = \phi_{14}(x), \forall \mathbf{x} \in \bar{\Omega}.
\end{aligned} \tag{3}$$

TABLE 1. Parameter definitions and values for the system (1)

Definition	Symbols	Value	Source
Recruitment term	Λ_h	0.6/day	Variable
Biting rate of Mosquito	c_m	0.5/day	[18]
Biting rate of Tsetse fly	c_t	0.6/day	[6]
Transmission probability of infectious mosquito bite	β_h	0.8333	[19]
Transmission probability of infectious tsetse fly bite	ϕ_h	0.62	[20]
Transmission probability of parasite from humans to susceptible mosquito	β_m	0.09	[18]
Transmission probability of parasite from humans to susceptible tsetse fly	ϕ_t	0.1	[20]
Progression rate to trypanosomiasis infectious humans from exposed class	σ_t	0.083/day	[6]
Progression rate to malaria infectious humans from exposed class	σ_m	0.101/day	[21]
Natural death rate of humans	μ_h	0.000548/day	[22]
Malaria co-infection parameter	ρ_m	0 – 1	Variable
Trypanosomiasis co-infection parameter	ρ_t	0 – 1	Variable
Malaria immunity loss	ω_m	0.000017	[23]
Trypanosomiasis immunity loss	ω_t	0.02	[24]
Trypanosomiasis-only treatment rate	θ_t	0.0333/day	[6]
Malaria-only treatment rate	θ_m	0.0022/day	[6]
Mosquito recruitment rate	Λ_m	0.071/day	[19]
Tsetse fly recruitment rate	Λ_t	0.1	[6]
Progression rate to infectious mosquito from exposed class	α_m	0.0556	[25]
Progression rate to infectious tsetse fly from exposed class	α_t	0.056	[20]
Natural death rate of mosquito	μ_m	0.067/day	[18]
Natural death rate of tsetse fly	μ_t	0.03/day	[26]
Death rate of humans caused by malaria	δ_b	0.019/day	[27]
Death rate of humans caused by trypanosomiasis	δ_a	0.004/day	[28]
Mosquito death rate	δ_m	0.05/day	[6]
Tsetse fly death rate	δ_t	0.0073/day	[6]
Effective transmission rate for malaria	λ_m	0.045	Estimated
Effective transmission rate for trypanosomiasis	λ_t	0.06	Estimated
Diffusion coefficients	$D_1, \dots, D_{14}$	0.02	

The Pontryagin's maximum principle is extended to determine the necessary conditions for the optimal control of the trypanosomiasis-malaria co-infection spatial model. We extend the model (1) for optimal control analysis by introducing space-time dependent control variables.

2.2. Co-infection model with control.

We introduced control preventive functions $(u_1(t, x), u_3(t, x))$ as treated nets and insecticides respectively and treatment function $u_2(t, x)$ to curtail the spread of trypanosomiasis and malaria into the mass-action version of model (1). The mass-action incidence is justified since the control variables act directly on the contact or transmission rate between susceptible and infected individuals which fits naturally between the mass-action framework. The co-infection model with control becomes:

$$\begin{aligned}
\frac{\partial N_{ss}}{\partial t} - D_1 \Delta N_{ss} &= \Lambda_h - c_t \phi_h N_{ss} I_t (1 - u_1) - c_m \beta_h N_{ss} I_m (1 - u_1) - \mu_h N_{ss} \\
&\quad + \omega_t N_{rs} + \omega_m N_{sr}, \\
\frac{\partial N_{es}}{\partial t} - D_2 \Delta N_{es} &= c_t \phi_h N_{ss} I_t (1 - u_1) - (\mu_h + \sigma_t u_2) N_{es} \\
&\quad - \rho_m \lambda_m N_{es} I_m (1 - u_1), \\
\frac{\partial N_{is}}{\partial t} - D_3 \Delta N_{is} &= \sigma_t u_2 N_{es} - (\mu_h + \delta_a + \theta_t u_2) N_{is} - \rho_m \lambda_m N_{is} I_m (1 - u_1), \\
\frac{\partial N_{rs}}{\partial t} - D_4 \Delta N_{rs} &= \theta_t u_2 N_{is} - (\mu_h + \omega_t) N_{rs} + \epsilon_m N_{rr} - \lambda_m N_{rs} I_m (1 - u_1), \\
\frac{\partial N_{se}}{\partial t} - D_5 \Delta N_{se} &= c_m \beta_h N_{ss} I_m (1 - u_1) - (\mu_h + \sigma_m u_2) N_{se} \\
&\quad - \rho_t \lambda_t N_{se} I_t (1 - u_1), \\
\frac{\partial N_{si}}{\partial t} - D_6 \Delta N_{si} &= \sigma_m u_2 N_{se} - (\mu_h + \delta_b + \theta_m u_2) N_{si} - \rho_t \lambda_t N_{si} I_t (1 - u_1), \\
\frac{\partial N_{sr}}{\partial t} - D_7 \Delta N_{sr} &= \theta_m u_2 N_{si} - (\mu_h + \omega_m) N_{sr} + \epsilon_t N_{rr} - \lambda_t N_{sr} I_t (1 - u_1), \\
\frac{\partial N_{rr}}{\partial t} - D_8 \Delta N_{rr} &= \rho_m \lambda_m (N_{es} + N_{is}) I_m (1 - u_1) + \rho_t \lambda_t (N_{se} + N_{si}) I_t (1 - u_1) \\
&\quad - (\mu_h + \epsilon_t + \epsilon_m) N_{rr} + \lambda_m N_{rs} I_m (1 - u_1) + \lambda_t N_{sr} I_t (1 - u_1), \\
\frac{\partial S_t}{\partial t} - D_9 \Delta S_t &= \Lambda_t - c_t \phi_t N_{is} S_t (1 - u_1) - (\mu_t + u_3) S_t, \\
\frac{\partial E_t}{\partial t} - D_{10} \Delta E_t &= c_t \phi_t N_{is} S_t (1 - u_1) - (\alpha_t + \mu_t + u_3) E_t, \\
\frac{\partial I_t}{\partial t} - D_{11} \Delta I_t &= \alpha_t E_t - (\delta_t + \mu_t + u_3) I_t, \\
\frac{\partial S_m}{\partial t} - D_{12} \Delta S_m &= \Lambda_m - c_m \beta_m N_{si} S_m (1 - u_1) - (\mu_m + u_3) S_m, \\
\frac{\partial E_m}{\partial t} - D_{13} \Delta E_m &= c_m \beta_m N_{si} S_m (1 - u_1) - (\alpha_m + \mu_m + u_3) E_m, \\
\frac{\partial I_m}{\partial t} - D_{14} \Delta I_m &= \alpha_m E_m - (\delta_m + \mu_m + u_3) I_m.
\end{aligned} \tag{4}$$

Let our objective functional be defined as

$$\begin{aligned}
J(u_1, u_2, u_3) &= \int_Q \left(p(N_{es}(\mathbf{x}, t) + N_{se}(\mathbf{x}, t)) + q(N_{is}(\mathbf{x}, t) + N_{si}(\mathbf{x}, t)) \right. \\
&\quad \left. + r(I_t(t, x) + I_m(t, x)) + n_1 u_1^2(\mathbf{x}, t) + n_2 u_2^2(\mathbf{x}, t) + n_3 u_3^2(\mathbf{x}, t) \right) d\mathbf{x} dt.
\end{aligned} \tag{5}$$

The parameters p, q, r, n_1, n_2, n_3 are nonnegative weights to balance the factors. The goal is to reduce the exposed and infectious individuals with infectious vectors at minimum possible cost. Thus, an optimal solution u_1^*, u_2^*, u_3^* are needed such that:

$$J(u_1^*, u_2^*, u_3^*) = \min\{J(u_1, u_2, u_3) : (u_1, u_2, u_3) \in \mathcal{U}\}, \quad (6)$$

subject to the system (4) where,

$$\mathcal{U} = \{(u_1, u_2, u_3) \in L^2(Q), 0 \leq u_i \leq 1, i = 1, 2, 3\}. \quad (7)$$

The optimality system is obtained which comprises of state equations, adjoint equations, and control characterization [29].

3. MAIN RESULTS

3.1. Existence, positivity and boundedness of Solutions.

The existence, positivity and boundedness of solutions of system (1) is established because this model describes population. Hence, the population should remain nonnegative and bounded.

Theorem 3.1. *The system (1)–(3) has a bounded and nonnegative solution defined on $\Omega \times [0, \infty)$ with its initial condition.*

Proof. System (1) can be written as

$$\begin{aligned} u'(t) &= Au(t) + F(u(t)) \quad t > 0 \\ u(0) &= u_0. \end{aligned}$$

where $u = \text{col}(N_{ss}, N_{es}, N_{is}, N_{rs}, N_{se}, N_{si}, N_{sr}, N_{rr}, S_t, E_t, I_t, S_m, E_m, I_m)$, $u_0 = (\phi_1, \phi_2, \dots, \phi_{14})$ and $F(u(t))$ is defined as

$$\begin{pmatrix} \Lambda_h - \frac{c_t \phi_h N_{ss} I_t}{N_h} - \frac{c_m \beta_h N_{ss} I_m}{N_h} - \mu_h N_{ss} + \omega_t N_{rs} + \omega_m N_{sr} \\ \frac{c_t \phi_h N_{ss} I_t}{N_h} - (\mu_h + \sigma_t) N_{es} - \frac{\rho_m \lambda_m N_{es} I_m}{N_h} \\ \sigma_t N_{es} - (\mu_h + \delta_a + \theta_t) N_{is} - \frac{\rho_m \lambda_m N_{is} I_m}{N_h} \\ \theta_t N_{is} - (\mu_h + \omega_t) N_{rs} + \epsilon_m N_{rr} - \frac{\lambda_m N_{rs} I_m}{N_h} \\ \frac{c_m \beta_h N_{ss} I_m}{N_h} - (\mu_h + \sigma_m) N_{se} - \frac{\rho_t \lambda_t N_{se} I_t}{N_h} \\ \sigma_m N_{se} - (\mu_h + \delta_b + \theta_m) N_{si} - \frac{\rho_t \lambda_t N_{si} I_t}{N_h} \\ \theta_m N_{si} - (\mu_h + \omega_m) N_{sr} + \epsilon_t N_{rr} - \frac{\lambda_t N_{sr} I_t}{N_h} \\ \frac{\rho_m \lambda_m (N_{es} + N_{is}) I_m}{N_h} + \frac{\rho_t \lambda_t (N_{se} + N_{si}) I_t}{N_h} - (\mu_h + \epsilon_t + \epsilon_m) N_{rr} + \frac{\lambda_m N_{rs} I_m}{N_h} + \frac{\lambda_t N_{sr} I_t}{N_h} \\ \Lambda_t - \frac{c_t \phi_t N_{is} S_t}{N_h} - \mu_t S_t \\ \frac{c_t \phi_t N_{is} S_t}{N_h} - (\alpha_t + \mu_t) E_t \\ \alpha_t E_t - (\delta_t + \mu_t) I_t \\ \Lambda_m - \frac{c_m \beta_m N_{si} S_m}{N_h} - \mu_m S_m \\ \frac{c_m \beta_m N_{si} S_m}{N_h} - (\alpha_m + \mu_m) E_m \\ \alpha_m E_m - (\delta_m + \mu_m) I_m \end{pmatrix}.$$

It is manifest that F is locally Lipschitz in X . We deduce that system (1) admits a unique local solution on $[0, T_m)$, where T_m is the maximal existence time for solution of system (1) using [30, 31].

The first state equation of system (1) can be written as

$$\frac{\partial N_{ss}}{\partial t} - D_1 \triangle N_{ss} = F_1(N_{ss}, N_{es}, N_{is}, N_{rs}, N_{se}, N_{si}, N_{sr}, N_{rr}, S_t, E_t, I_t, S_m, E_m, I_m).$$

It is easy to see that the function

$F_1(N_{ss}, N_{es}, N_{is}, N_{rs}, N_{se}, N_{si}, N_{sr}, N_{rr}, S_t, E_t, I_t, S_m, E_m, I_m)$ is continuously differentiable and satisfies

$F_1(0, N_{es}, N_{is}, N_{rs}, N_{se}, N_{si}, N_{sr}, N_{rr}, S_t, E_t, I_t, S_m, E_m, I_m) = \Lambda_h \geq 0$. Similar procedure can be used for other state variables. Since initial data of system (1) are nonnegative, we deduce the positivity of the local solution.

Let the total human population be $N_h = N_{ss} + N_{es} + N_{is} + N_{rs} + N_{se} + N_{si} + N_{sr} + N_{rr}$.

Let $D_h = \min\{D_1, D_2, \dots, D_8\}$. We obtain

$$\left. \begin{aligned} \frac{\partial N_h(x, t)}{\partial t} - D_h \triangle N_h(x, t) &\leq \Lambda_h - \mu_h N_h(x, t) \\ \frac{\partial N_h(x, t)}{\partial n} &= 0 \\ N_h(x, 0) &= \phi_1(x) + \phi_2(x) + \dots + \phi_8(x) \end{aligned} \right\} \quad (8)$$

Let $N_h(t)$ be the solution of the ODE

$$\left. \begin{aligned} \frac{dN_h(t)}{dt} &= \Lambda_h - \mu_h N_h(t) \\ N_h(0) &= \max_{x \in \Omega} N_h(x, 0). \end{aligned} \right\} \quad (9)$$

Hence, $N_h(t) \leq \max \left\{ \frac{\lambda_h}{\mu_h}, \max_{x \in \Omega} N_h(x, 0) \right\}$. The application of comparison principle [32] shows that $N_h(x, t) \leq N_h(t)$. Hence we have that

$$N_h(x, t) \leq \max \left\{ \frac{\lambda_h}{\mu_h}, \max_{x \in \Omega} N_h(x, 0) \right\}. \quad (10)$$

Thus, $N_{ss}(x, t), N_{es}(x, t), N_{is}(x, t), N_{rs}(x, t), N_{se}(x, t), N_{si}(x, t), N_{sr}(x, t), N_{rr}(x, t)$ are bounded. Similar procedure can be used to establish the boundedness of $S_t(x, t), E_t(x, t), I_t(x, t), S_m(x, t), E_m(x, t)$ and $I_m(x, t)$. $\square$

3.2. Adjoint PDE.

The adjoint PDE is necessary for the characterization of the optimal control.

Theorem 3.2. (*Adjoint System*) Given system (4), there exists adjoint variables $\lambda_1, \lambda_2, \dots, \lambda_{14}$ satisfying

$$\begin{aligned} -(\lambda_1)_t - D_1 \triangle \lambda_1 + [(c_t \phi_h I_t + c_m \beta_h I_m)(1 - u_1) + \mu_h] \lambda_1 \\ - (c_t \phi_h I_t)(1 - u_1) \lambda_2 + (c_m \beta_h I_m)(1 - u_1) \lambda_5 = 0, \end{aligned} \quad (11)$$

$$\begin{aligned} & -(\lambda_2)_t - D_2 \triangle \lambda_2 + [(\mu_h + \sigma_t u_2) + \rho_m \lambda_m I_m (1 - u_1)] \lambda_2 - \sigma_t u_2 \lambda_3 \\ & - \rho_m \lambda_m I_m (1 - u_1) \lambda_8 = p, \end{aligned} \quad (12)$$

$$\begin{aligned} & -(\lambda_3)_t - D_3 \triangle \lambda_3 + [(\mu_h + \delta_a + \theta_t u_2) + \rho_m \lambda_m I_m (1 - u_1)] \lambda_3 - \theta_t u_2 \lambda_4 \\ & - \rho_m \lambda_m I_m (1 - u_1) \lambda_8 + c_t \phi_t S_t (1 - u_1) \lambda_9 - c_t \phi_t S_t (1 - u_1) \lambda_{10} = q, \end{aligned} \quad (13)$$

$$\begin{aligned} & -(\lambda_4)_t - D_4 \triangle \lambda_4 - \omega_t \lambda_4 + [(\mu_h + \omega_t) + \lambda_m I_m (1 - u_1)] \lambda_5 \\ & - \lambda_m I_m (1 - u_1) \lambda_8 = 0, \end{aligned} \quad (14)$$

$$\begin{aligned} & -(\lambda_5)_t - D_5 \triangle \lambda_5 + [(\mu_h + \sigma_m u_2) + \rho_t \lambda_t I_t (1 - u_1)] \lambda_5 - \sigma_m u_2 \lambda_6 \\ & - \rho_t \lambda_t I_t (1 - u_1) \lambda_8 = p, \end{aligned} \quad (15)$$

$$\begin{aligned} & -(\lambda_6)_t - D_6 \triangle \lambda_6 + [(\mu_h + \delta_b + \theta_m u_2) + \rho_t \lambda_t I_t (1 - u_1)] \lambda_6 - \theta_m u_2 \lambda_7 \\ & \rho_t \lambda_t I_t (1 - u_1) \lambda_8 + c_m \beta_m S_m (1 - u_1) \lambda_{12} - c_m \beta_m S_m (1 - u_1) \lambda_{13} = q, \end{aligned} \quad (16)$$

$$\begin{aligned} & -(\lambda_7)_t - D_7 \triangle \lambda_7 - \omega_m \lambda_1 + [(\mu_h + \omega_m) + \lambda_t I_t (1 - u_1)] \lambda_7 \\ & - \lambda_t I_t (1 - u_1) \lambda_8 = 0, \end{aligned} \quad (17)$$

$$-(\lambda_8)_t - D_8 \triangle \lambda_8 - \epsilon_m \lambda_4 - \epsilon_t \lambda_7 + (\mu_h + \epsilon_t + \epsilon_m) \lambda_8 = 0, \quad (18)$$

$$-(\lambda_9)_t - D_9 \triangle \lambda_9 + [c_t \phi_t N_{is} (1 - u_1) + \mu_t + u_3] \lambda_9 - c_t \phi_t N_{is} (1 - u_1) \lambda_{10} = 0, \quad (19)$$

$$-(\lambda_{10})_t - D_{10} \triangle \lambda_{10} + (\alpha_t + \mu_t + u_3) \lambda_{10} - \alpha_t \lambda_{11} = 0, \quad (20)$$

$$\begin{aligned} & -(\lambda_{11})_t - D_{11} \triangle \lambda_{11} + c_t \phi_h N_{ss} (1 - u_1) (\lambda_1 - \lambda_2) + \rho_t \lambda_t N_{se} (1 - u_1) \lambda_5 \\ & + \rho_t \lambda_t N_{si} (1 - u_1) \lambda_6 + \lambda_t N_{sr} (1 - u_1) \lambda_7 \\ & - [\rho_t \lambda_t (N_{se} + N_{si}) (1 - u_1) + \lambda_t (N_{sr} (1 - u_1))] \lambda_8 - (\delta_t + \mu_t + u_3) \lambda_{11} = r, \end{aligned} \quad (21)$$

$$\begin{aligned} & -(\lambda_{12})_t - D_{12} \triangle \lambda_{12} + [c_m \beta_m N_{si} (1 - u_1) + \mu_m + u_3] \lambda_{12} \\ & - c_m \beta_m N_{si} (1 - u_1) \lambda_{13} = 0, \end{aligned} \quad (22)$$

$$-(\lambda_{13})_t - D_{13} \triangle \lambda_{13} + (\alpha_m + \mu_m + u_m) \lambda_{13} - \alpha_m \lambda_{14} = 0, \quad (23)$$

$$\begin{aligned} & -(\lambda_{14})_t - D_{14} \triangle \lambda_{14} + c_m \beta_h N_{ss} (1 - u_1) (\lambda_1 - \lambda_5) + \rho_m \lambda_m N_{es} (1 - u_1) \lambda_2 \\ & + \rho_m \lambda_m N_{is} (1 - u_1) \lambda_3 + \lambda_m N_{rs} (1 - u_1) \lambda_4 \\ & - [\rho_m \lambda_m (N_{es} + N_{is}) (1 - u_1) + \lambda_m (N_{rs} (1 - u_1))] \lambda_8 - (\delta_m + \mu_m + u_m) \lambda_{14} = r. \end{aligned} \quad (24)$$

with transversality conditions:

$$\lambda_1(T, x) = 0, \lambda_2(T, x) = 0, \dots, \lambda_{14}(T, x) = 0,$$

and boundary conditions:

$$\frac{\partial \lambda_1}{\partial n} = 0, \frac{\partial \lambda_2}{\partial n} = 0, \dots, \frac{\partial \lambda_{14}}{\partial n} = 0.$$

Proof. To prove theorem 3.2, sensitivity system is needed. The sensitivity system is the derivative of the control-to-state map. It satisfies a partial differential equation which produces the linearized version of the state system. Given control $u = (u_1, u_2, u_3) \in \mathcal{U}$ and another control $u + \epsilon l = (u_1 + \epsilon l_1, u_2 + \epsilon l_2, u_3 + \epsilon l_3) \in \mathcal{U}$ for small ϵ with $l = (l_1, l_2, l_3) \in L^2(Q)$, the Gateaux derivative of the control-to-state map which is given below gives the sensitivity function:

$$\begin{aligned} \frac{N_{ss}^\epsilon - N_{ss}}{\epsilon} &\rightharpoonup \psi_1, \frac{N_{es}^\epsilon - N_{es}}{\epsilon} \rightharpoonup \psi_2, \frac{N_{is}^\epsilon - N_{is}}{\epsilon} \rightharpoonup \psi_3, \frac{N_{rs}^\epsilon - N_{rs}}{\epsilon} \rightharpoonup \psi_4, \\ \frac{N_{se}^\epsilon - N_{se}}{\epsilon} &\rightharpoonup \psi_5, \frac{N_{si}^\epsilon - N_{si}}{\epsilon} \rightharpoonup \psi_6, \frac{N_{sr}^\epsilon - N_{sr}}{\epsilon} \rightharpoonup \psi_7, \frac{N_{rr}^\epsilon - N_{rr}}{\epsilon} \rightharpoonup \psi_8, \\ \frac{S_t^\epsilon - S_t}{\epsilon} &\rightharpoonup \psi_9, \frac{E_t^\epsilon - E_t}{\epsilon} \rightharpoonup \psi_{10}, \frac{I_t^\epsilon - I_t}{\epsilon} \rightharpoonup \psi_{11}, \frac{S_m^\epsilon - S_m}{\epsilon} \rightharpoonup \psi_{12}, \\ \frac{E_m^\epsilon - E_m}{\epsilon} &\rightharpoonup \psi_{13}, \frac{I_m^\epsilon - I_m}{\epsilon} \rightharpoonup \psi_{14}. \end{aligned}$$

Likewise, $N_{ss}^\epsilon \rightarrow N_{ss}, N_{es}^\epsilon \rightarrow N_{es}, N_{is}^\epsilon \rightarrow N_{is}, N_{rs}^\epsilon \rightarrow N_{rs}, N_{se}^\epsilon \rightarrow N_{se}, N_{si}^\epsilon \rightarrow N_{si}, N_{sr}^\epsilon \rightarrow N_{sr}, N_{rr}^\epsilon \rightarrow N_{rr}, S_t^\epsilon \rightarrow S_t, E_t^\epsilon \rightarrow E_t, I_t^\epsilon \rightarrow I_t, S_m^\epsilon \rightarrow S_m, E_m^\epsilon \rightarrow E_m, I_m^\epsilon \rightarrow I_m$ as $\epsilon \rightarrow 0$.

The sensitivity function $\Psi = \psi_1, \psi_2, \dots, \psi_{14}$ satisfy a system that corresponds to the linearized version of the state equations.

Consider the first equation of (4) given as

$$\begin{aligned} \frac{\partial N_{ss}}{\partial t} - D_1 \Delta N_{ss} &= \Lambda_h - c_t \phi_h N_{ss} I_t (1 - u_1) - c_m \beta_h N_{ss} I_m (1 - u_1) - \mu_h N_{ss} \\ &+ \omega_t N_{rs} + \omega_m N_{sr}. \end{aligned}$$

It can be expressed as

$$\begin{aligned} \left(\frac{N_{ss}^\epsilon - N_{ss}}{\epsilon} \right)_t - D_1 \Delta \left(\frac{N_{ss}^\epsilon - N_{ss}}{\epsilon} \right) &= -c_t \phi_h \left(\frac{N_{ss}^\epsilon I_t^\epsilon - N_{ss} I_t}{\epsilon} \right) \\ &+ c_t \phi_h \left(\frac{N_{ss}^\epsilon I_t^\epsilon (u_1 + \epsilon l_1) - N_{ss} I_t u_1}{\epsilon} \right) - c_m \beta_h \left(\frac{N_{ss}^\epsilon I_m^\epsilon - N_{ss} I_m}{\epsilon} \right) \\ &+ c_m \beta_h \left(\frac{N_{ss}^\epsilon I_m^\epsilon (u_1 + \epsilon l_1) - N_{ss} I_t u_1}{\epsilon} \right) - \mu_h \left(\frac{N_{ss}^\epsilon - N_{ss}}{\epsilon} \right) \\ &+ \omega_t \left(\frac{N_{rs}^\epsilon - N_{rs}}{\epsilon} \right) + \omega_m \left(\frac{N_{sr}^\epsilon - N_{sr}}{\epsilon} \right). \end{aligned}$$

Expanding and simplifying gives

$$\begin{aligned}
& \left(\frac{N_{ss}^\epsilon - N_{ss}}{\epsilon} \right)_t - D_1 \Delta \left(\frac{N_{ss}^\epsilon - N_{ss}}{\epsilon} \right) = -c_t \phi_h \left(\frac{N_{ss}^\epsilon I_t^\epsilon - N_{ss} I_t}{\epsilon} \right) \\
& + c_t \phi_h u_1 \left(\frac{N_{ss}^\epsilon I_t^\epsilon - N_{ss} I_t}{\epsilon} \right) + c_t \phi_h N_{ss}^\epsilon I_t^\epsilon l_1 - c_m \beta_h \left(\frac{N_{ss}^\epsilon I_m^\epsilon - N_{ss} I_m}{\epsilon} \right) \\
& + c_m \beta_h u_1 \left(\frac{N_{ss}^\epsilon I_m^\epsilon - N_{ss} I_m}{\epsilon} \right) + c_m \beta_h N_{ss}^\epsilon I_m^\epsilon l_1 - \mu_h \left(\frac{N_{ss}^\epsilon - N_{ss}}{\epsilon} \right) \\
& + \omega_t \left(\frac{N_{rs}^\epsilon - N_{rs}}{\epsilon} \right) + \omega_m \left(\frac{N_{sr}^\epsilon - N_{sr}}{\epsilon} \right).
\end{aligned}$$

Further simplification gives

$$\begin{aligned}
& \left(\frac{N_{ss}^\epsilon - N_{ss}}{\epsilon} \right)_t - D_1 \Delta \left(\frac{N_{ss}^\epsilon - N_{ss}}{\epsilon} \right) = \\
& - c_t \phi_h (1 - u_1) \left(\frac{N_{ss}^\epsilon I_t^\epsilon - N_{ss}^\epsilon I_t + N_{ss}^\epsilon I_t - N_{ss} I_t}{\epsilon} \right) + c_t \phi_h N_{ss}^\epsilon I_t^\epsilon l_1 \\
& - c_m \beta_h (1 - u_1) \left(\frac{N_{ss}^\epsilon I_m^\epsilon - N_{ss}^\epsilon I_m + N_{ss}^\epsilon I_m - N_{ss} I_m}{\epsilon} \right) + c_m \beta_h N_{ss}^\epsilon I_m^\epsilon l_1 \\
& - \mu_h \left(\frac{N_{ss}^\epsilon - N_{ss}}{\epsilon} \right) + \omega_t \left(\frac{N_{rs}^\epsilon - N_{rs}}{\epsilon} \right) + \omega_m \left(\frac{N_{sr}^\epsilon - N_{sr}}{\epsilon} \right).
\end{aligned}$$

As $\epsilon \rightarrow 0$, we have

$$\begin{aligned}
& L_1 \psi_1 + c_t \phi_h (N_{ss} \psi_{11} + I_t \psi_1) (1 - u_1) + c_m \beta_h (N_{ss} \psi_{14} + I_m \psi_1) (1 - u_1) \\
& + \mu_h \psi_1 - \omega_t \psi_4 - \omega_m \psi_7 = l_1 N_{ss} (c_t \phi_h I_t + c_m \beta_h I_m),
\end{aligned} \tag{25}$$

where $L_1 \psi_1 = \frac{\partial \psi_1}{\partial t} - \Delta(D_1 \psi_1)$.

Others can be obtained using the same procedure as seen in (26)–(38).

$$\begin{aligned}
& L_2 \psi_2 - c_t \phi_h (N_{ss} \psi_{11} + I_t \psi_1) (1 - u_1) + (\mu_h + \sigma_t u_2) \psi_2 \\
& + \rho_m \lambda_m (N_{es} \psi_{14} + I_m \psi_2) (1 - u_1) = -l_1 (c_t \phi_h N_{ss} I_t - \rho_m \lambda_m N_{es} I_m) \\
& - l_2 \sigma_t N_{es},
\end{aligned} \tag{26}$$

$$\begin{aligned}
& L_3 \psi_3 - \sigma_t u_2 \psi_2 + (\mu_h + \delta_a + \theta_t u_2) \psi_3 + \rho_m \lambda_m (N_{is} \psi_{14} + I_m \psi_3) (1 - u_1) \\
& = l_1 \rho_m \lambda_m N_{is} I_m + l_2 (\sigma_t N_{es} - \theta_t N_{is}),
\end{aligned} \tag{27}$$

$$\begin{aligned}
& L_4 \psi_4 - \theta_t u_2 \psi_3 + (\mu_h + \omega_t) \psi_4 - \epsilon_m \psi_8 + \lambda_m (N_{rs} \psi_{14} + I_m \psi_4) (1 - u_1) \\
& = l_1 \lambda_m N_{rs} I_m + l_2 \theta_t N_{is},
\end{aligned} \tag{28}$$

$$\begin{aligned}
& L_5 \psi_5 - c_m \beta_h (N_{ss} \psi_{14} + I_m \psi_1) (1 - u_1) + (\mu_h + \sigma_m u_2) \psi_5 \\
& + \rho_t \lambda_t (N_{se} \psi_{11} + I_t \psi_5) (1 - u_1) = -l_1 (c_m \beta_h N_{ss} I_m - \rho_t \lambda_t N_{se} I_t) - l_2 \sigma_m N_{se},
\end{aligned} \tag{29}$$

$$\begin{aligned}
& L_6 \psi_6 - \sigma_m u_2 \psi_5 + (\mu_h + \delta_b + \theta_m u_2) \psi_6 + \rho_t \lambda_t (N_{si} \psi_{11} + I_t \psi_6) (1 - u_1) \\
& = l_1 \rho_t \lambda_t N_{si} I_t + l_2 (\sigma_m N_{se} - \theta_m N_{si}),
\end{aligned} \tag{30}$$

$$\begin{aligned} L_7\psi_7 - \theta_m u_2\psi_6 + (\mu_h + \omega_m)\psi_7 - \epsilon_t\psi_8 + \lambda_t(N_{sr}\psi_{11} + I_t\psi_7)(1 - u_1) \\ = l_1\lambda_t N_{sr}I_t + l_2\theta_m N_{is}, \end{aligned} \quad (31)$$

$$\begin{aligned} L_8\psi_8 - \rho_m\lambda_m(N_{es}\psi_{14} + I_m\psi_2)(1 - u_1) - \rho_m\lambda_m(N_{is}\psi_{14} + I_m\psi_3)(1 - u_1) \\ - \rho_t\lambda_t(N_{se}\psi_{11} + I_t\psi_5)(1 - u_1) - \rho_t\lambda_t(N_{si}\psi_{11} + I_t\psi_6)(1 - u_1) \\ + (\mu_h + \epsilon_t + \epsilon_m)\psi_8 - \lambda_t(N_{sr}\psi_{11} + I_t\psi_7)(1 - u_1) \\ - \lambda_m(N_{rs}\psi_{14} + I_m\psi_4)(1 - u_1) = -l_1[\rho_m\lambda_m I_m(N_{es} + N_{is}) \\ + \rho_t\lambda_t I_t(N_{se} + N_{si}) + \lambda_m N_{rs}I_m + \lambda_t N_{sr}I_t], \end{aligned} \quad (32)$$

$$L_9\psi_9 + c_t\phi_t(N_{is}\psi_9 + S_t\psi_3)(1 - u_1) + (\mu_t + u_3)\psi_9 = l_3c_t\phi_t N_{is}S_t, \quad (33)$$

$$L_{10}\psi_{10} - c_t\phi_t(N_{is}\psi_9 + S_t\psi_3)(1 - u_1) + (\alpha_t + \mu_t + u_3)\psi_{10} = -l_3c_t\phi_h N_{is}S_t, \quad (34)$$

$$L_{11}\psi_{11} - \alpha_t\psi_{10} + (\delta_t + \mu_t + u_3)\psi_{11} = 0, \quad (35)$$

$$L_{12}\psi_{12} + c_m\beta_m(N_{si}\psi_{12} + S_m\psi_6)(1 - u_1) + (\mu_m + u_3)\psi_{12} = l_3c_m\beta_m N_{si}S_m, \quad (36)$$

$$\begin{aligned} L_{13}\psi_{13} - c_m\beta_m(N_{si}\psi_{12} + S_m\psi_6)(1 - u_1) + (\alpha_m + \mu_m + u_3)\psi_{13} \\ = -l_3c_m\beta_m N_{si}S_m, \end{aligned} \quad (37)$$

$$L_{14}\psi_{14} - \alpha_m\psi_{13} + (\delta_m + \mu_m + u_3)\psi_{14} = 0. \quad (38)$$

where $L_j\psi_j = \frac{\partial\psi_j}{\partial t} - \Delta(D_j\psi_j)$.

The quotients are also used to derive the initial and boundary condition.

The initial condition is obtained as $\left(\frac{N_{ss}^\epsilon - N_{ss}}{\epsilon}\right)(0, x) = \frac{N_{ss}^0 - N_{ss}^0}{\epsilon} = 0$ which gives $\psi_1(0, x) = 0$ as $\epsilon \rightarrow 0$. Similar procedure gives the initial condition as below:

$$\psi_1(0, x) = 0, \psi_2(0, x) = 0, \dots, \psi_{14}(0, x) = 0, \in \Omega.$$

The boundary conditions is obtained as $\frac{\partial\left(\frac{N_{ss}^\epsilon - N_{ss}}{\epsilon}\right)}{\partial n} = 0$ which gives $\frac{\partial\psi_1}{\partial n} = 0$ as $\epsilon \rightarrow 0$. Similar procedure gives the boundary condition as below:

$$\frac{\partial\psi_1}{\partial n} = 0, \frac{\partial\psi_2}{\partial n} = 0, \dots, \frac{\partial\psi_{14}}{\partial n} = 0 \in \partial\Omega \times (0, T).$$

The sensitivity system can thus be expressed as

$$\begin{pmatrix} \mathcal{L}\psi_1 \\ \mathcal{L}\psi_2 \\ \mathcal{L}\psi_3 \\ \mathcal{L}\psi_4 \\ \mathcal{L}\psi_5 \\ \mathcal{L}\psi_6 \\ \mathcal{L}\psi_7 \\ \mathcal{L}\psi_8 \\ \mathcal{L}\psi_9 \\ \mathcal{L}\psi_{10} \\ \mathcal{L}\psi_{11} \\ \mathcal{L}\psi_{12} \\ \mathcal{L}\psi_{13} \\ \mathcal{L}\psi_{14} \end{pmatrix} = \begin{pmatrix} l_1 N_{ss}(c_t \phi_h I_t + c_m \beta_h I_m) \\ -l_1(c_t \phi_h N_{ss} I_t - \rho_m \lambda_m N_{es} I_m) - l_2 \sigma_t N_{es} \\ l_1 \rho_m \lambda_m N_{is} I_m + l_2(\sigma_t N_{es} - \theta_t N_{is}) \\ l_1 \lambda_m N_{rs} I_m + l_2 \theta_t N_{is} \\ -l_1(c_m \beta_h N_{ss} I_m - \rho_t \lambda_t N_{se} I_t) - l_2 \sigma_m N_{se} \\ l_1 \rho_t \lambda_t N_{si} I_t + l_2(\sigma_m N_{se} - \theta_m N_{si}) \\ l_1 \lambda_t N_{sr} I_t + l_2 \theta_m N_{is} \\ -l_1[\rho_m \lambda_m I_m(N_{es} + N_{is}) + \rho_t \lambda_t I_t(N_{se} + N_{si}) + \lambda_m N_{rs} I_m + \lambda_t N_{sr} I_t] \\ l_3 c_t \phi_t N_{is} S_t \\ -l_3 c_t \phi_h N_{is} S_t \\ 0 \\ l_3 c_m \beta_m N_{si} S_m \\ -l_3 c_m \beta_m N_{si} S_m \\ 0 \end{pmatrix},$$

where

$$\mathcal{L} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{14} \end{pmatrix} = \begin{pmatrix} L_1 \psi_1 \\ L_2 \psi_2 \\ \vdots \\ L_{14} \psi_{14} \end{pmatrix} + M \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{14} \end{pmatrix}.$$

$$\begin{pmatrix} L_1 \psi_1 \\ L_2 \psi_2 \\ \vdots \\ L_{14} \psi_{14} \end{pmatrix} = \begin{pmatrix} \frac{\partial \psi_1}{\partial t} - \Delta(D_1 \psi_1) \\ \frac{\partial \psi_2}{\partial t} - \Delta(D_2 \psi_2) \\ \vdots \\ \frac{\partial \psi_{14}}{\partial t} - \Delta(D_{14} \psi_{14}) \end{pmatrix}.$$

and M is

$$\begin{pmatrix} e_1 & 0 & 0 & -\omega_t & 0 & 0 & -\omega_m & 0 & 0 & 0 & e_2 & 0 & 0 & e_3 \\ e_4 & e_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_6 & 0 & 0 & e_7 \\ 0 & -\sigma_t u_2 & e_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_9 \\ 0 & 0 & -\theta_t u_2 & e_{10} & 0 & 0 & 0 & -\epsilon_m & 0 & 0 & 0 & 0 & 0 & e_{11} \\ e_{12} & 0 & 0 & 0 & e_{13} & 0 & 0 & 0 & 0 & 0 & e_{14} & 0 & 0 & e_{15} \\ 0 & 0 & 0 & 0 & -\sigma_m u_2 & e_{16} & 0 & 0 & 0 & 0 & e_{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\theta_m u_2 & e_{18} & -\epsilon_t & 0 & 0 & e_{19} & 0 & 0 & 0 \\ 0 & e_{20} & e_{20} & e_{21} & e_{22} & e_{22} & e_{23} & e_{24} & 0 & 0 & e_{25} & 0 & 0 & e_{26} \\ 0 & 0 & e_{27} & 0 & 0 & 0 & 0 & e_{28} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_{29} & 0 & 0 & 0 & 0 & 0 & e_{30} & e_{31} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_t & e_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{33} & 0 & 0 & 0 & 0 & 0 & e_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{35} & 0 & 0 & 0 & 0 & 0 & e_{36} & e_{37} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_m & e_{38} \end{pmatrix}$$

where $e_1 = (c_t \phi_h I_t + c_m \beta_h I_m)(1 - u_1) + \mu_h$, $e_2 = c_t \phi_h N_{ss}(1 - u_1)$, $e_3 = c_m \beta_h N_{ss}(1 - u_1)$, $e_4 = -c_t \phi_h I_t(1 - u_1)$, $e_5 = (\mu_h + \sigma_t u_2) + \rho_m \lambda_m I_m(1 - u_1)$, $e_6 = -c_t \phi_h N_{ss}(1 - u_1)$, $e_7 = \rho_m \lambda_m N_{es}(1 - u_1)$, $e_8 = (\mu_h + \delta_a + \theta_t u_2) + \rho_m \lambda_m I_m(1 - u_1)$, $e_9 = \rho_m \lambda_m N_{is}(1 - u_1)$, $e_{10} = (\mu_h + \omega_t) + \lambda_m I_m(1 - u_1)$, $e_{11} = \lambda_m N_{rs}(1 - u_1)$, $e_{12} = -c_m \beta_h I_m(1 -$

$$\begin{aligned}
u_1), e_{13} &= (\mu_h + \sigma_m u_2) + \rho_t \lambda_t I_t (1 - u_1), e_{14} = \rho_t \lambda_t N_{se} (1 - u_1), e_{15} = -c_m \beta_h N_{ss} (1 - \\
u_1), e_{16} &= (\mu_h + \delta_b + \theta_m u_2) + \rho_t \lambda_t I_t (1 - u_1), e_{17} = \rho_t \lambda_t N_{si} (1 - u_1), e_{18} = (\mu_h + \\
\omega_m) + \lambda_t I_t (1 - u_1), e_{19} &= \lambda_t N_{sr} (1 - u_1), e_{20} = -\rho_m \lambda_m I_m (1 - u_1), e_{21} = -\lambda_m I_m (1 - \\
u_1), e_{22} &= -\rho_t \lambda_t I_t (1 - u_1), e_{23} = -\lambda_t I_t (1 - u_1) e_{24} = (\mu_h + \epsilon_t + \epsilon_m), e_{25} = -(\rho_t \lambda_t N_{se} + \\
\rho_t \lambda_t N_{si} + \lambda_t N_{sr}) &(1 - u_1), e_{26} = -(\rho_m \lambda_m N_{es} + \rho_m \lambda_m N_{is} + \lambda_m N_{rs}) (1 - u_1), e_{27} = \\
c_t \phi_t S_t (1 - u_1), e_{28} &= c_t \phi_t N_{is} (1 - u_1) + \mu_t + u_3, e_{29} = -c_t \phi_t S_t (1 - u_1), e_{30} = \\
-c_t \phi_t N_{is} (1 - u_1), e_{31} &= (\alpha_t + \mu_t + u_3), e_{32} = (\delta_t + \mu_t + u_3), e_{33} = c_m \beta_m S_m (1 - \\
u_1), e_{34} &= c_m \beta_m N_{si} (1 - u_1) + \mu_m + u_3, e_{35} = -c_m \beta_m S_m (1 - u_1), e_{36} = -c_m \beta_m N_{si} (1 - \\
u_1), e_{37} &= (\alpha_m + \mu_m + u_3), e_{38} = (\delta_m + \mu_m + u_3).
\end{aligned}$$

Given adjoint variable, $\lambda_1, \lambda_2, \dots, \lambda_{14}$, the adjoint operator can be derived using (39):

$$\langle \lambda, \mathcal{L}\psi \rangle = \langle \mathcal{L}^* \lambda, \psi \rangle. \quad (39)$$

The derivative with respect to the states of the objective functional gives the R.H.S of the adjoint system.

The Adjoint system is expressed as

$$\begin{pmatrix} \mathcal{L}^* \lambda_1 \\ \mathcal{L}^* \lambda_2 \\ \mathcal{L}^* \lambda_3 \\ \mathcal{L}^* \lambda_4 \\ \mathcal{L}^* \lambda_5 \\ \mathcal{L}^* \lambda_6 \\ \mathcal{L}^* \lambda_7 \\ \mathcal{L}^* \lambda_8 \\ \mathcal{L}^* \lambda_9 \\ \mathcal{L}^* \lambda_{10} \\ \mathcal{L}^* \lambda_{11} \\ \mathcal{L}^* \lambda_{12} \\ \mathcal{L}^* \lambda_{13} \\ \mathcal{L}^* \lambda_{14} \end{pmatrix} = \begin{pmatrix} 0 \\ p \\ q \\ 0 \\ p \\ q \\ 0 \\ r \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The adjoint operator is

$$\mathcal{L}^* \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{14} \end{pmatrix} = \begin{pmatrix} L_1^* \lambda_1 \\ L_2^* \lambda_2 \\ \vdots \\ L_{14}^* \lambda_{14} \end{pmatrix} + M^T \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{14} \end{pmatrix},$$

where M^T represents the transpose of the matrix and

$$\begin{pmatrix} L_1^* \lambda_1 \\ L_2^* \lambda_2 \\ \vdots \\ L_{14}^* \lambda_{14} \end{pmatrix} = \begin{pmatrix} -\frac{\partial \lambda_1}{\partial t} - \Delta(D_1 \lambda_1) \\ -\frac{\partial \lambda_2}{\partial t} - \Delta(D_2 \lambda_2) \\ \vdots \\ -\frac{\partial \lambda_{14}}{\partial t} - \Delta(D_{14} \lambda_{14}) \end{pmatrix}.$$

The Adjoint system in expanded form becomes

$$\begin{aligned} & L_1^* \lambda_1 + [(c_t \phi_h I_t + c_m \beta_h I_m)(1 - u_1) + \mu_h] \lambda_1 - (c_t \phi_h I_t)(1 - u_1) \lambda_2 \\ & - (c_m \beta_h I_m)(1 - u_1) \lambda_5 = \mathcal{L}^* \lambda_1, \\ & L_2^* \lambda_2 + [(\mu_h + \sigma_t u_2) + \rho_m \lambda_m I_m(1 - u_1)] \lambda_2 - \sigma_t u_2 \lambda_3 - \rho_m \lambda_m I_m(1 - u_1) \lambda_8 = \mathcal{L}^* \lambda_2, \\ & L_3^* \lambda_3 + [(\mu_h + \delta_a + \theta_t u_2) + \rho_m \lambda_m I_m(1 - u_1)] \lambda_3 - \theta_t u_2 \lambda_4 - \rho_m \lambda_m I_m(1 - u_1) \lambda_8 \\ & + c_t \phi_t S_t(1 - u_1) \lambda_9 - c_t \phi_t S_t(1 - u_1) \lambda_{10} = \mathcal{L}^* \lambda_3, \\ & L_4^* \lambda_4 - \omega_t \lambda_1 + [(\mu_h + \omega_t) + \lambda_m I_m(1 - u_1)] \lambda_4 - \lambda_m I_m(1 - u_1) \lambda_8 = \mathcal{L}^* \lambda_4, \\ & L_5^* \lambda_5 + [(\mu_h + \sigma_m u_2) + \rho_t \lambda_t I_t(1 - u_1)] \lambda_5 - \sigma_m u_2 \lambda_6 - \rho_t \lambda_t I_t(1 - u_1) \lambda_8 = \mathcal{L}^* \lambda_5, \\ & L_6^* \lambda_6 + [(\mu_h + \delta_b + \theta_m u_2) + \rho_t \lambda_t I_t(1 - u_1)] \lambda_6 - \theta_m u_2 \lambda_7 \\ & - \rho_t \lambda_t I_t(1 - u_1) \lambda_8 + c_m \beta_m S_m(1 - u_1) \lambda_{12} - c_m \beta_m S_m(1 - u_1) \lambda_{13} = \mathcal{L}^* \lambda_6, \\ & L_7^* \lambda_7 - \omega_m \lambda_1 + [(\mu_h + \omega_m) + \lambda_t I_t(1 - u_1)] \lambda_7 - \lambda_t I_t(1 - u_1) \lambda_8 = \mathcal{L}^* \lambda_7, \\ & L_8^* \lambda_8 - \epsilon_m \lambda_4 - \epsilon_t \lambda_7 + (\mu_h + \epsilon_t + \epsilon_m) \lambda_8 = \mathcal{L}^* \lambda_8, \\ & L_9^* \lambda_1 + [c_t \phi_t N_{is}(1 - u_1) + \mu_t + u_3] \lambda_9 - c_t \phi_t N_{is}(1 - u_1) \lambda_{10} = \mathcal{L}^* \lambda_9, \\ & L_{10}^* \lambda_{10} + (\alpha_t + \mu_t + u_3) \lambda_{10} - \alpha_t \lambda_{11} = \mathcal{L}^* \lambda_{10}, \\ & L_{11}^* \lambda_{11} + c_t \phi_h N_{ss}(1 - u_1)(\lambda_1 - \lambda_2) + \rho_t \lambda_t N_{se}(1 - u_1) \lambda_5 + \rho_t \lambda_t N_{si}(1 - u_1) \lambda_6 \\ & + \lambda_t N_{sr}(1 - u_1) \lambda_7 - [\rho_t \lambda_t (N_{se} + N_{si})(1 - u_1) + \lambda_t (N_{sr}(1 - u_1))] \lambda_8 \\ & + (\delta_t + \mu_t + u_3) \lambda_{11} = \mathcal{L}^* \lambda_{11}, \\ & L_{12}^* \lambda_{12} + [c_m \beta_m N_{si}(1 - u_1) + \mu_m + u_3] \lambda_{12} - c_m \beta_m N_{si}(1 - u_1) \lambda_{13} = \mathcal{L}^* \lambda_{12}, \\ & L_{13}^* \lambda_{13} + (\alpha_m + \mu_m + u_3) \lambda_{13} - \alpha_m \lambda_{14} = \mathcal{L}^* \lambda_{13}, \\ & L_{14}^* \lambda_{14} + c_m \beta_h N_{ss}(1 - u_1)(\lambda_1 - \lambda_5) + \rho_m \lambda_m N_{es}(1 - u_1) \lambda_2 \\ & + \rho_m \lambda_m N_{is}(1 - u_1) \lambda_3 + \lambda_m N_{rs}(1 - u_1) \lambda_4 \\ & - [\rho_m \lambda_m (N_{es} + N_{is})(1 - u_1) + \lambda_m (N_{rs}(1 - u_1))] \lambda_8 + (\delta_m + \mu_m + u_3) \lambda_{14} = \mathcal{L}^* \lambda_{14}. \end{aligned}$$

where $L_j^* \lambda_j = -\frac{\partial \lambda_j}{\partial t} - \Delta(D_j \lambda_j)$.

The transversality condition are represented by

$$\lambda_1(T, x) = 0, \lambda_2(T, x) = 0, \dots, \lambda_{14}(T, x) = 0.$$

The boundary conditions are given by

$$\frac{\partial \lambda_1}{\partial n} = 0, \frac{\partial \lambda_2}{\partial n} = 0, \dots, \frac{\partial \lambda_{14}}{\partial n} = 0.$$

□

3.3. Characterization of Optimal Control.

It can be demonstrated, in the suitable weak formulation, that system (4) admits a unique solution, and likewise, the optimal control problem has a solution for sufficiently short time intervals [33]. The solution to the optimal control problem is derived by solving an optimality system that includes the state equations, the corresponding adjoint equations, and the condition defining the optimal control. The sensitivity function, adjoint systems together with the control bounds will be applied to characterize the optimal control.

Theorem 3.3. Characterization of Optimal Control. *Given Optimal control u_1^*, u_2^*, u_3^* and solutions of the corresponding state system that minimize $J(u_1, u_2, u_3)$, we obtain a characterization of our optimal control as*

$$\begin{aligned} u_1^* &= \min \left(u_{1\max}, \max \left(0, \frac{1}{2n_1} [c_t \phi_h N_{ss} I_t (\lambda_2 - \lambda_1) + c_m \beta_h N_{ss} I_m (\lambda_5 - \lambda_1) + \right. \right. \\ &\quad \left. \left. \rho_m \lambda_m N_{es} I_m (\lambda_8 - \lambda_2) + \rho_m \lambda_m N_{is} I_m (\lambda_8 - \lambda_3) + \lambda_m N_{rs} I_m (\lambda_8 - \lambda_4) + \right. \right. \\ &\quad \left. \left. \rho_t \lambda_t N_{se} I_t (\lambda_8 - \lambda_5) + \rho_t \lambda_t N_{si} I_t (\lambda_8 - \lambda_6) + \lambda_t N_{sr} I_t (\lambda_8 - \lambda_7)] \right) \right), \\ u_2^* &= \min \left(u_{2\max}, \max \left(0, \frac{1}{2n_2} [\sigma_t N_{es} (\lambda_2 - \lambda_3) + \theta_t N_{is} (\lambda_3 - \lambda_4) \right. \right. \\ &\quad \left. \left. + \sigma_m N_{se} (\lambda_5 - \lambda_6) + \theta_m N_{si} (\lambda_6 - \lambda_7)] \right) \right), \\ u_3^* &= \min \left(u_{3\max}, \max \left(0, \frac{1}{2n_3} [c_t \phi_t N_{is} S_t (\lambda_{10} - \lambda_9) + c_m \beta_m N_{si} S_m (\lambda_{13} - \lambda_{12})] \right) \right). \end{aligned}$$

where $u_{1\max}, u_{2\max}$ and $u_{3\max}$ are upper bounds on the controls.

Proof. A characterization for u_1^*, u_2^*, u_3^* by using the adjoint and sensitivity system is obtained.

The map $u_1 \rightarrow J(u_1, u_2, u_3)$ at u_1^* has a derivative in the direction of l_1 as shown below:

$$\lim_{\epsilon \rightarrow 0^+} \frac{J(u_1^* + \epsilon l_1, u_2, u_3) - J(u_1^*, u_2, u_3)}{\epsilon} \geq 0. \quad (40)$$

$$\begin{aligned} J(u_1^* + \epsilon l_1, u_2, u_3) &= \int_Q \left[p(N_{es}^\epsilon + N_{se}^\epsilon) + q(N_{is}^\epsilon + N_{si}^\epsilon) + r(I_t^\epsilon + I_m^\epsilon) + n_1(u_1^* + \epsilon l_1)^2 \right. \\ &\quad \left. + n_2 u_2^2 + n_3 u_3^2 \right] d\mathbf{x} dt. \end{aligned}$$

$$\begin{aligned} J(u_1^* + \epsilon l_1, u_2, u_3) &= \int_Q \left[p(N_{es}^\epsilon + N_{se}^\epsilon) + q(N_{is}^\epsilon + N_{si}^\epsilon) + r(I_t^\epsilon + I_m^\epsilon) + n_1(u_1^*)^2 \right. \\ &\quad \left. + 2\epsilon n_1 l_1 u_1^* + n_1 \epsilon^2 l_1^2 + n_2 u_2^2 + n_3 u_3^2 \right] d\mathbf{x} dt. \end{aligned} \quad (41)$$

$$J(u_1^*, u_2, u_3) = \int_Q \left[p(N_{es} + N_{se}) + q(N_{is} + N_{si}) + r(I_t + I_m) + n_1(u_1^*)^2 + n_2 u_2^2 + n_3 u_3^2 \right] d\mathbf{x} dt. \quad (42)$$

The characterization for u_1^* is obtained below using (40), (41) and (42).

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0^+} \frac{J(u_1^* + \epsilon l_1, u_2, u_3) - J(u_1^*, u_2, u_3)}{\epsilon} \geq 0, \\ & \lim_{\epsilon \rightarrow 0^+} \int_Q \left[p \left(\frac{N_{es}^\epsilon - N_{es}}{\epsilon} + \frac{N_{se}^\epsilon - N_{se}}{\epsilon} \right) + q \left(\frac{N_{is}^\epsilon - N_{is}}{\epsilon} + \frac{N_{si}^\epsilon - N_{si}}{\epsilon} \right) \right. \\ & \quad \left. + r \left(\frac{I_t^\epsilon - I_t}{\epsilon} + \frac{I_m^\epsilon - I_m}{\epsilon} \right) + 2n_1 l_1 u_1^* + n_1 \epsilon l_1^2 \right] d\mathbf{x} dt \geq 0, \\ & \int_Q \left[p(\psi_2 + \psi_5) + q(\psi_3 + \psi_7) + r(\psi_{11} + \psi_{14}) + 2n_1 l_1 u_1^* \right] d\mathbf{x} dt \geq 0, \\ & \int_Q \left[(\psi_1, \psi_2, \dots, \psi_{14})(0, p, q, 0, p, q, 0, 0, 0, 0, 0, 0, 0, 0)^T + 2l_1 n_1 u_1^* \right] d\mathbf{x} dt \geq 0, \\ & \int_Q \left[(\psi_1, \psi_2, \dots, \psi_{14}) \mathcal{L}^*(\lambda_1, \lambda_2, \dots, \lambda_{14})^T + 2l_1 n_1 u_1^* \right] d\mathbf{x} dt \geq 0, \\ & \int_Q \left[(\lambda_1, \lambda_2, \dots, \lambda_{14}) \mathcal{L}(\psi_1, \psi_2, \dots, \psi_{14})^T + 2l_1 n_1 u_1^* \right] d\mathbf{x} dt \geq 0, \\ & \int_Q l_1 \left[c_t \phi_h N_{ss} I_t (\lambda_1 - \lambda_2) + c_m \beta_h N_{ss} I_m (\lambda_1 - \lambda_5) + \rho_m \lambda_m N_{es} I_m (\lambda_2 - \lambda_8) \right. \\ & \quad \left. + \rho_m \lambda_m N_{is} I_m (\lambda_3 - \lambda_8) + \rho_t \lambda_t N_{si} I_t (\lambda_6 - \lambda_8) + \lambda_t N_{sr} I_t (\lambda_7 - \lambda_8) + 2n_1 u_1^* \right] d\mathbf{x} dt \geq 0. \end{aligned}$$

Within the region $\{(\mathbf{x}, t) : 0 < u_1^*(t, x) < u_{1\max}\}$, the variation l_1 can have any sign since the optimal control can be adjusted a little down or up and still remain within the bounds. Hence,

$$\begin{aligned} u_1^* = & \min \left(u_{1\max}, \max \left(0, \frac{1}{2n_1} [c_t \phi_h N_{ss} I_t (\lambda_2 - \lambda_1) + c_m \beta_h N_{ss} I_m (\lambda_5 - \lambda_1) + \right. \right. \\ & \rho_m \lambda_m N_{es} I_m (\lambda_8 - \lambda_2) + \rho_m \lambda_m N_{is} I_m (\lambda_8 - \lambda_3) + \lambda_m N_{rs} I_m (\lambda_8 - \lambda_4) + \\ & \left. \left. \rho_t \lambda_t N_{se} I_t (\lambda_8 - \lambda_5) + \rho_t \lambda_t N_{si} I_t (\lambda_8 - \lambda_6) + \lambda_t N_{sr} I_t (\lambda_8 - \lambda_7)] \right) \right). \end{aligned}$$

Similar procedure gives

$$\begin{aligned} u_2^* = & \min \left(u_{2\max}, \max \left(0, \frac{1}{2n_2} [\sigma_t N_{es} (\lambda_2 - \lambda_3) + \theta_t N_{is} (\lambda_3 - \lambda_4) + \sigma_m N_{se} (\lambda_5 - \lambda_6) \right. \right. \\ & \left. \left. + \theta_m N_{si} (\lambda_6 - \lambda_7)] \right) \right), \end{aligned}$$

and

$$u_3^* = \min \left(u_{3 \max}, \max \left(0, \frac{1}{2n_3} [c_t \phi_t N_{is} S_t (\lambda_{10} - \lambda_9) + c_m \beta_m N_{si} S_m (\lambda_{13} - \lambda_{12})] \right) \right).$$

□

4. NUMERICAL SIMULATION AND DISCUSSION

The spatio-temporal dynamics of the malaria and sleeping sickness co-infection model are simulated using a finite difference method applied to a system of fourteen coupled reaction-diffusion partial differential equations. The computational domain spanned 100 spatial units discretized into 50 grid points with spacing $\Delta x = 2$, while the temporal domain extended over 50 days using a time step of $\Delta t = 0.05$ days. Spatial derivatives were computed via second-order central differences with zero-flux Neumann boundary conditions. The optimal control problem is simulated using parameter values outlined in Table 1 subject to the following initial conditions: N_{ss} population was initialized uniformly at density 1.0 across the domain, N_{is} population were initialized with a Gaussian distribution centered at $x = 70$ with peak density 0.02, representing a localized malaria outbreak. Similarly, N_{si} were initialized with a Gaussian distribution centered at $x = 30$ with the same peak density. All other human compartments began at zero density. Susceptible mosquitoes and tsetse flies were uniformly distributed at density 0.6. Infected mosquitoes I_m were concentrated near the malaria outbreak location $x = 30$, while infected tsetse flies I_t were concentrated near the sleeping sickness outbreak location $x = 70$, both following Gaussian distributions with peak density 0.4. A uniform diffusion coefficient of $D = 0.02$ was used for all population compartments, representing moderate spatial mobility of both human and vector populations. The following weight parameters were used: $p = 1.0$, $q = 2.0$, $r = 0.5$, and $n_1 = n_2 = n_3 = 0.1$.

An iterative forward-backward scheme using MATLAB is used to solve the optimality system. The numerical simulations show significant epidemiological differences between the controlled and uncontrolled cases. In the absence of intervention, both malaria and trypanosomiasis persist at high levels within the human population, indicating sustained transmission and a high risk of co-infection spread, as illustrated in Figures 1 and 2. However, under the implementation of control strategies, there is a marked reduction in the densities of infected individuals, showing the effectiveness of the interventions in disrupting transmission pathways. Biologically, this reduction suggests that preventive and treatment-based controls significantly limit contact between susceptible humans and infectious vectors, thereby reducing the force of infection. As a direct consequence, more individuals transition into the recovered class, as shown in Figures 3 and 4. This increase reflects improved treatment outcomes, which in turn reduces the reservoir of infection and further suppresses disease transmission.

The impact of the control measures is also evident in the vector population dynamics. Figures 5 and 6 show a substantial decline in infected vector densities under control conditions. This indicates that interventions such as insecticide application and vector management strategies effectively reduce vector infection rates

and disrupt the transmission cycle between humans and vectors. From a biological standpoint, lowering the infected vector population is critical, as it directly reduces the probability of new infections in humans.

From a spatial perspective, the results further demonstrate that control strategies not only reduce infection intensity but also limit geographical spread. The uncontrolled case exhibits a pronounced peak around the center of the spatial domain, with a maximum density exceeding 0.06 as shown in Figure 7. This indicates a strong spatial concentration of infections, suggesting intense local transmission in the absence of intervention measures. In contrast, the controlled scenario shows a significantly reduced peak, with the maximum density dropping to below 0.02. This substantial decline demonstrates the effectiveness of the implemented control strategies in suppressing malaria transmission. Additionally, the controlled curve is slightly more spread out and flattened compared to the uncontrolled case, indicating a reduction in spatial clustering and a more uniform, lower-level distribution of infection. Similarly in Figure 8, the introduction of control measures leads to a noticeable reduction in the peak infection density, confirming the effectiveness of the implemented interventions in lowering trypanosomiasis burden. However, the persistence of the peak location despite the application of controls implies that the interventions are not sufficient to completely disrupt transmission dynamics in high-risk areas. This highlights the need for more targeted and possibly intensified strategies focused on identified hotspots.

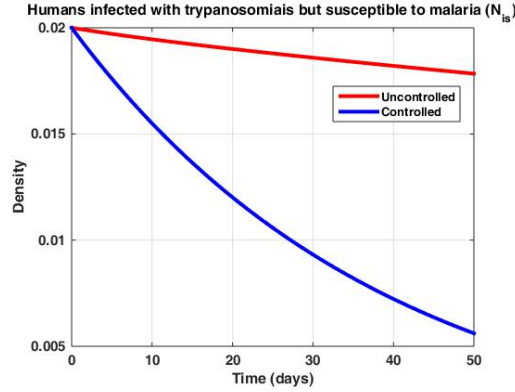


FIGURE 1. Control dynamics on infected humans with trypanosomiasis.

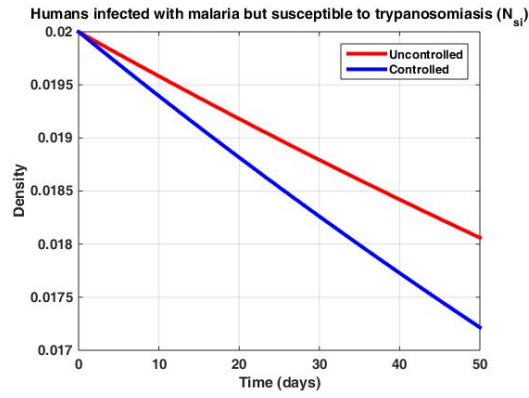


FIGURE 2. Control dynamics on infected humans with malaria.

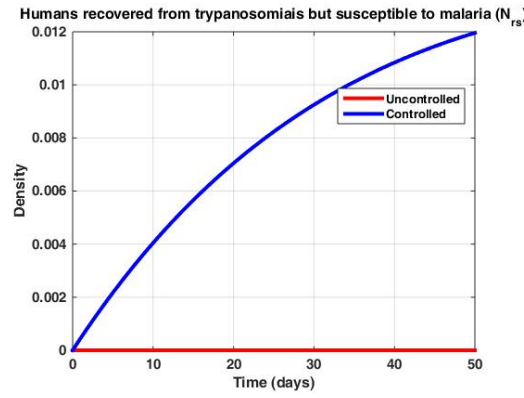


FIGURE 3. Control dynamics on recovered humans from trypanosomiasis

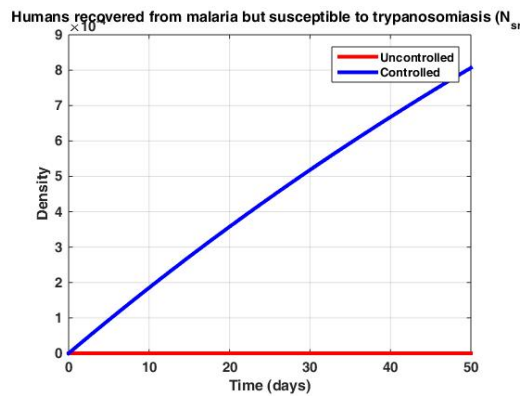


FIGURE 4. Control dynamics on recovered humans from malaria

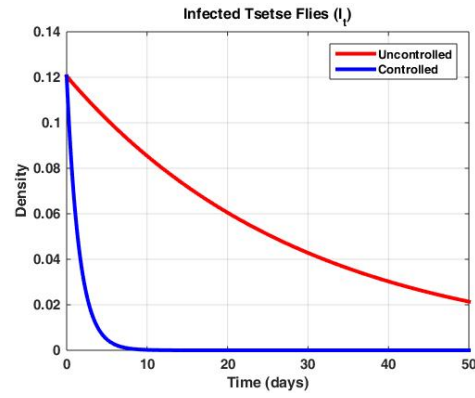


FIGURE 5. Control dynamics on infected tsetse flies

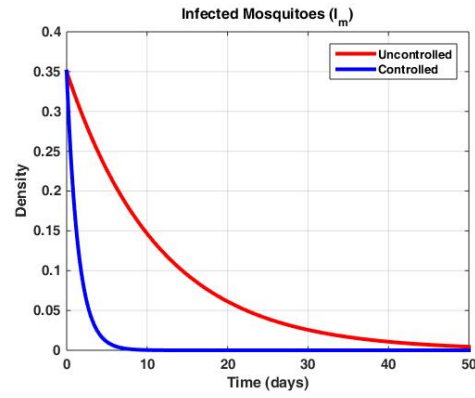


FIGURE 6. Control dynamics on infected mosquitoes

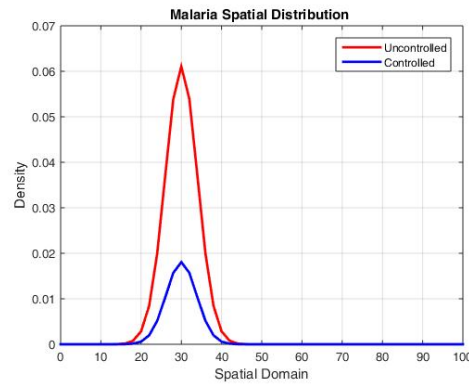


FIGURE 7. Control intervention on malaria spatial distribution

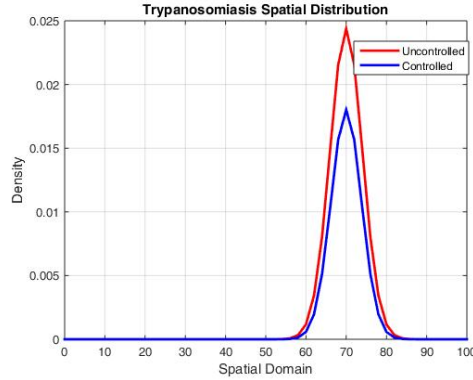


FIGURE 8. Control intervention on trypanosomiasis spatial distribution

5. CONCLUSION

Optimal control strategy for a spatial model describing the transmission dynamics of trypanosomiasis-malaria co-infection, governed by a system of partial differential equations is developed and analyzed. The model incorporates two preventive and a treatment control. These controls are optimized through the formulation of a cost functional by minimizing the costs related to prevention and treatment application. By applying Pontryagin's Maximum Principle to the spatial framework, the necessary conditions for optimality and constructed the optimality system is derived. Simulation results indicate that the treatment-focused intervention significantly increases the number of individuals who recover from the co-infection compared to scenarios without any control. On the other hand, prevention efforts, which primarily target vector reduction, are shown to remarkably decrease the overall vector population. Also, the differential spatial behavior has crucial implications for targeted intervention strategies, suggesting that control measures may need to be spatially adapted based on disease-specific propagation characteristics rather than implemented uniformly across the domain.

This study demonstrates that the applied interventions can effectively suppress co-infection dynamics, and highlights the complementary roles of prevention and treatment in disease management. However, there are some limitations. The model uses the same parameter values across all location and times, which may not reflect in climate or environment. In future studies, the model can be improved by adding spatial between areas, using real data for validation.

Data Availability Statement

No new data were created or analyzed in this study.

Declarations. The authors declare no conflicts of interest

Author Contributions. All authors contributed equally to the conception, design, analysis, and writing of this manuscript. All authors have read, reviewed, and approved the final version of the manuscript.

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