

On the Application of A-Transform in the Fractional Calculus

Arkan Shareef Hasan^{1*}, Hela Shawkat Ahmed²

¹Department of Nursing, Duhok Polytechnic University, Iraq

²Department of Mathematics, University of Duhok, Iraq

Abstract. This paper investigates the application of the A-Transform within the framework of fractional calculus. The main objective is to establish analytical formulas for the A-Transform of key fractional operators and to demonstrate its capability in solving fractional differential and integrodifferential equations. The study derives simplified expressions for the Mittag–Leffler function, the Caputo derivative, and the mixed Caputo–Volterra equation using the A-Transform. In addition, a numerical scheme based on integral reformulation and Gauss–Jacobi quadrature is developed to handle non-homogeneous fractional models involving multiple Caputo derivatives and non-polynomial forcing terms. The results show that the proposed analytical and numerical approaches provide accurate, stable, and efficient solutions, confirming the A-Transform as a useful tool for the analysis of fractional-order systems.

Key words and Phrases: A-transform, Mittag Leffler Function, Caputo operator, Caputo-Volterra equation, Gauss-Jacobi

1. INTRODUCTION

Fractional calculus extends classical differentiation and integration to non-integer orders, providing powerful tools for modelling memory-dependent and hereditary processes that arise in physics, engineering, and applied sciences. Integral equations, on the other hand, offer a natural framework for describing systems governed by accumulation, convolution, and nonlocal interactions. The close relationship between these two areas—where fractional operators often appear as integral kernels—has led to significant advances in analysing complex dynamical systems and solving a wide range of boundary value and applied problems[1, 2, 3, 4].

*Corresponding author: arkan.hasan@dpu.edu.krd

2020 Mathematics Subject Classification: 34A08, 44Axx, 45D05, 65R10.

Received: 11-05-2025, accepted: 05-04-2026.

D. Das et al. [5] developed the framework of dislocated quasi-modular and non-Archimedean dislocated quasi-modular metric spaces, established several fixed-point theorems via C-class and A-class functions, and applied their results to the stability of Jungck-type iterations and an associated integral equation; M. Sharma et al. [6] developed a green fractional transportation model using dual hesitant Fermatean fuzzy sets and solved it through fuzzy programming and the global criterion method; L. Rathour et al. [7] studied application of the function in basic mathematics; G. Farid et al. [8] derived fractional Grüss type integral inequalities for generalized k -fractional integral operators with Mittag-Leffler function in the kernel; S. B. Akbar et al. [9] established refinements of the classical Hadamard inequality for Riemann–Liouville fractional integrals via strongly (α, m) -convex functions. Fractional differential equations (FDEs) and fractional integral equations (FIEs) have been extensively studied, with numerous analytical and numerical methods developed for their solution; Murad and Rasheed [10] applied the Adomian Decomposition Method to solve a class of FDEs. Volterra-type fractional integral equations have also been investigated using hybrid contraction techniques by B. Alqahtani et al. [11]. In addition, a wide variety of numerical methods for FDEs have been proposed in the literature. These include approaches based on Haar wavelets and other computational techniques [12, 13, 14, 15]. Among the tools commonly employed to tackle fractional equations, integral transforms have proven particularly effective. Recent developments feature Laplace-based transforms, including the Mellin transform, the α -transform, and the Yang–Laplace transform, as well as other generalized integral transforms, in solving various FDEs [16, 17, 18, 19].

In this paper, we focus on the application of the A-Transform that introduced by A. S. HASAN [20] in the context of fractional calculus. We revisit fundamental functions and operators frequently encountered in this field, including the Mittag-Leffler function, the Caputo fractional derivative, and the mixed Caputo-Volterra equation, and explore how the A-Transform facilitates their analysis. To further illustrate the utility of this transform, we present both analytical and numerical solutions to representative fractional differential equations involving Caputo operators and the Caputo-Volterra equation. Additionally, we develop and apply a robust numerical framework based on integral reformulation and Gauss-Jacobi quadrature to solve non-homogeneous fractional differential equations with multiple Caputo derivatives and the Caputo-Volterra integrodifferential equation, ensuring accurate approximations and preserving physical and mathematical consistency.

Next, we present some basic definitions and theories in fractional calculus and the A-transform.

1.1. Preliminaries.

Definition 1.1. [21] *The Gamma function, symbolized by $\Gamma(\alpha)$, provides a continuous extension of the factorial function to the domain of positive real numbers. It*

is defined for all $\alpha > 0$ by the improper integral:

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \quad (1)$$

Definition 1.2. [22] The Mittag-Leffler function is expressed through the infinite series below:

$$E_{\alpha,\beta}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j + \beta)}, \alpha, \beta > 0 \quad (2)$$

Definition 1.3. [23] [17] Let u be a function defined on the interval (a, ∞) , and let $\alpha > 0$. The Riemann–Liouville fractional integral of order α of the function u , symbolized by ${}^{RL}_a I_t^\alpha u(t)$, is given by:

$${}^{RL}_a I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} u(\tau) d\tau, \alpha > 0 \quad (3)$$

Definition 1.4. [24] Let $a \in \mathbb{R}$ and let $n \in \mathbb{Z}$, then the Caputo fractional derivative of order α is given by:

$${}_a^C D_t^\alpha u(t) = {}_a I_t^\alpha \frac{d^n}{dt^n} u(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - \tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} u(\tau) d\tau, \alpha > 0 \quad (4)$$

Definition 1.5. [25] Let $T > 0$, and let $\alpha, \beta \in (0, T]$, $n_\alpha = [\alpha]$, $n_\beta = [\beta]$. The function $u : [0, T] \rightarrow \mathbb{R}$ is said to be a classical solution if:

- (1) $u \in C^{n_*}((0, T])$, $n_* = \max\{[\alpha], [\beta]\}$;
- (2) All derivatives ${}_a^C D_t^\alpha u(t)$ and, when applicable ${}_a^C D_t^\beta u(t)$ exist and continuous on $(0, T]$;
- (3) the Volterra integral

$$\int_0^t (t - \tau)^{\alpha-2} u(\tau) d\tau$$

is well defined and continuous on $(0, T]$ whenever it appears;

- (4) The corresponding ordinary or fractional differential equation is satisfied point-wise for all $t \in (0, T]$,
- (5) The initial conditions

$$u^{(k)}(0) = c_k, k = 0, 1, \dots, n_* - 1$$

are satisfied in the classical sense.

1.2. A-Transform and Basic Properties.

Definition 1.6. [20] Let the function $u(t) \in \mathbb{R}_0^+$. The A-transform of $u(t)$ is a function of s , say, $U(s)$ and is denoted by $A\{u(t)\}$, thus,

$$A\{u(t)\} = U(s) = \int_0^\infty u(t)e^{-\frac{1}{s^2}t}dt, \quad (5)$$

for $s > 0$. And the inverse of A-transform takes the form:

$$u(t) = A^{-1}\{U(s)\} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} U(s)e^{\frac{1}{s^2}t}ds \quad (6)$$

Theorem 1.7. [20] (Linear property of A - Transform)

Let $U(s)$ and $V(s)$ be A-transforms of $u(t)$ and $v(t)$, respectively, for any constants c_1 and c_2 , then

$$A\{c_1u(t) + c_2v(t)\} = c_1A\{u(t)\} + c_2A\{v(t)\} = c_1U(s) + c_2V(s) \quad (7)$$

Theorem 1.8. [20] (A-transform of Derivatives)

Let $u(t)$ be differentiable and $s > 0$, then

$$A\{u^{(n)}(t)\} = \frac{1}{s^{2n}}U(s) - \sum_{j=0}^{n-1} \frac{1}{s^{2(n-1-j)}}u^{(j)}(0) \quad (8)$$

Theorem 1.9. [20] (The A-transform of convolution type)

Let $U(s)$ and $V(s)$ be the A-transforms of $u(t)$ and $v(t)$, respectively, then A transform of the Convolution of u and v is:

$$A\{u(t) * v(t)\} = U(s)V(s) \quad (9)$$

2. APPLICATION OF THE A-TRANSFORM TO THE FRACTIONAL CALCULUS

2.1. A-Transform of Mittag-Leffler Function.

Consider the Mittag-Leffler function:

$$t^{\beta-1}E_{\alpha,\beta}(t) = t^{\beta-1} \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j + \beta)}. \quad (10)$$

Taking the A-transform of Eq. (10) gives:

$$\begin{aligned}
A \{ t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha) \} &= A \left\{ t^{\beta-1} \sum_{j=0}^{\infty} \frac{(\lambda t^\alpha)^j}{\Gamma(\alpha j + \beta)} \right\} \\
&= \int_0^\infty e^{-\frac{1}{s^2} t} t^{\beta-1} \sum_{j=0}^{\infty} \frac{\lambda^j t^{\alpha j}}{\Gamma(\alpha j + \beta)} dt \\
&= \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(\alpha j + \beta)} \int_0^\infty e^{-\frac{1}{s^2} t} t^{\alpha j + \beta - 1} dt \\
&= \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(\alpha j + \beta)} A \{ t^{\alpha j + \beta - 1} \} \\
&= s^{2\beta} \sum_{j=0}^{\infty} (\lambda s^{2\alpha})^j = s^{2\beta} (1 + \lambda s^{2\alpha} + (\lambda s^{2\alpha})^2 + (\lambda s^{2\alpha})^3 + \dots) \\
&= \frac{s^{2\beta}}{1 - \lambda s^{2\alpha}},
\end{aligned} \tag{11}$$

so

$$A \{ t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha) \} = \frac{s^{2\beta}}{1 - \lambda s^{2\alpha}}. \tag{12}$$

2.2. A-Transform of Caputo Fractional Derivative.

Using the simple fact without proof that the A-transform of Riemann-Liouville fractional integral Eq. (3) is:

$$A \{ {}^{RL}I_t^\alpha u(t) \} = s^{2\alpha} U(s). \tag{13}$$

Using equations (13) and (8), the A-transform of the Caputo Fractional Derivative (4) is:

$$\begin{aligned}
A \{ {}^C D_t^\alpha u(t) \} &= A \left\{ {}_a I_t^{n-\alpha} \frac{d^n}{dt^n} u(t) \right\}; \\
&= s^{2(n-\alpha)} \left[\frac{1}{s^{2n}} U(s) - \sum_{j=0}^{n-1} \frac{1}{s^{2(n-1-j)}} u^{(j)}(0) \right]; \\
&= \frac{1}{s^{2\alpha}} U(s) - \sum_{j=0}^{n-1} \frac{1}{s^{2(\alpha-1-j)}} u^{(j)}(0),
\end{aligned} \tag{14}$$

so

$$A \{ {}^C D_t^\alpha u(t) \} = \frac{1}{s^{2\alpha}} U(s) - \sum_{j=0}^{n-1} \frac{1}{s^{2(\alpha-1-j)}} u^{(j)}(0), n-1 < \alpha < n. \tag{15}$$

Now, we apply the A-transform to the non-homogeneous Caputo fractional differential equation of multiple order $1 < \alpha < \beta < 2$, giving

$${}_0^C D_t^\alpha u(t) + {}_0^C D_t^\beta u(t) = f(t), t > 0, u(0) = c_1, u'(0) = c_2. \quad (16)$$

Since $1 < \alpha < \beta < 2$, then $n = 2$, therefore, by applying the A-transform and using Eq. (15), we obtain the following:

$$\begin{aligned} \frac{1}{s^{2\alpha}} U(s) - \sum_{j=0}^1 \frac{1}{s^{2(\alpha-1-j)}} u^{(j)}(0) + \frac{1}{s^{2\beta}} U(s) - \sum_{j=0}^1 \frac{1}{s^{2(\beta-1-j)}} u^{(j)}(0) &= F(s); \\ \frac{1}{s^{2\alpha}} U(s) - \frac{1}{s^{2(\alpha-1)}} u(0) - \frac{1}{s^{2(\alpha-2)}} u'(0) + \frac{1}{s^{2\beta}} U(s) - \frac{1}{s^{2(\beta-1)}} u(0) - \frac{1}{s^{2(\beta-2)}} u'(0) &= F(s); \end{aligned} \quad (17)$$

Factoring $U(s)$, $u(0)$ and $u'(0)$, and using the initial conditions, we get:

$$U(s) = \frac{s^{2\beta}}{1 + s^{2(\beta-\alpha)}} F(s) + c_1 s^2 + c_2 s^4, \quad (18)$$

let

$$Q(s) = \frac{s^{2\beta}}{1 + s^{2(\beta-\alpha)}}, \quad (19)$$

then:

$$q(t) = t^{\beta-1} E_{\beta-\alpha, \beta}(-t^{\beta-\alpha}). \quad (20)$$

In the A-transform theory, when we have $U_1(s) = Q(s)F(s)$, then by the convolution theorem, we get:

$$u_1(t) = (q * f)(t) = \int_0^t q(t-\tau) f(\tau) d\tau, \quad (21)$$

so, combining Eqs. (20) and (21), we get:

$$u_1(t) = \int_0^t (t-\tau)^{\beta-1} E_{\beta-\alpha, \beta}(-(t-\tau)^{\beta-\alpha}) f(\tau) d\tau, \quad (22)$$

and let

$$u_2(t) = A^{-1}\{c_1 s^2 + c_2 s^4\} = c_1 + c_2 t. \quad (23)$$

Multiplication in the transform domain correspond to convolution in the time domain. Therefore, by summing equations (22) and (23), we obtain the following general solution:

$$u(t) = \int_0^t (t-\tau)^{\beta-1} E_{\beta-\alpha, \beta}(-(t-\tau)^{\beta-\alpha}) f(\tau) d\tau + c_1 + c_2 t. \quad (24)$$

2.3. A-Transform of Caputo-Volterra Equation.

Define the following Caputo-Volterra equation:

$${}_0^C D_t^\alpha u(t) = f(t) + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} u(\tau) d\tau, \quad (25)$$

where the function $u(t)$, which will be evaluated, occurs on the left with a fractional derivative and on the right inside the integral sign. The kernel $K(t, \tau) = (t-\tau)^{\alpha-2}$ is of fractional order $\alpha-2$, and $1 < \alpha < 2$, and $f(t)$ is a given function, and let the initial conditions be

$$u(0) = c_1, u'(0) = c_2. \quad (26)$$

Applying the A-transform, we get:

$$\begin{aligned} A\{{}_0^C D_t^\alpha u(t)\} &= A\{f(t)\} + \frac{1}{\Gamma(\alpha-1)} A\{t^{\alpha-2}\} A\{u(t)\}; \\ \frac{1}{s^{2\alpha}} U(s) - \sum_{j=0}^{n-1} \frac{1}{s^{2(\alpha-1-j)}} u^{(j)}(0) &= F(s) + s^{2\alpha-2} U(s), \end{aligned} \quad (27)$$

since $1 < \alpha < 2$, then $n = 2$, so the above equation reduces to:

$$\begin{aligned} \frac{1}{s^{2\alpha}} U(s) - \sum_{j=0}^1 \frac{1}{s^{2(\alpha-1-j)}} u^{(j)}(0) &= F(s) + s^{2\alpha-2} U(s); \\ \frac{1 - s^{2(2\alpha-1)}}{s^{2\alpha}} U(s) &= F(s) + s^{2(1-\alpha)} u(0) + s^{2(2-\alpha)} u'(0); \\ U(s) &= \frac{s^{2\alpha}}{1 - s^{2(2\alpha-1)}} F(s) + \frac{s^2}{1 - s^{2(2\alpha-1)}} u(0) + \frac{s^4}{1 - s^{2(2\alpha-1)}} u'(0), \end{aligned} \quad (28)$$

let

$$Q(s) = \frac{s^{2\alpha}}{1 - s^{2(2\alpha-1)}}, \quad (29)$$

then:

$$q(t) = t^{\alpha-1} E_{2\alpha-1, \alpha}(t^{2\alpha-1}). \quad (30)$$

In the A-transform theory, when we have $U_1(s) = Q(s)F(s)$, then by Convolution Theorem, we obtain:

$$u_1(t) = (q * f)(t) = \int_0^t q(t - \tau)f(\tau)d\tau, \quad (31)$$

so,

$$u_1(t) = \int_0^t ((t - \tau)^{\alpha-1} E_{2\alpha-1, \alpha}((t - \tau)^{2\alpha-1})f(\tau)d\tau, \quad (32)$$

and let

$$\begin{aligned} u_2(t) &= A^{-1} \left\{ \frac{s^2}{1 - s^{2(2\alpha-1)}} u(0) + \frac{s^4}{1 - s^{2(2\alpha-1)}} u'(0) \right\} \\ &= c_1 E_{2\alpha-1, 1}(t^{2\alpha-1}) + c_2 t E_{2\alpha-1, 2}(t^{2\alpha-1}). \end{aligned} \quad (33)$$

Multiplication in the transform domain corresponds to convolution in the time domain. Therefore, by summing equations (32) and (33), we get the general form of the solution:

$$u(t) = \int_0^t ((t - \tau)^{\alpha-1} E_{2\alpha-1, \alpha}((t - \tau)^{2\alpha-1})f(\tau)d\tau + c_1 E_{2\alpha-1, 1}(t^{2\alpha-1}) + c_2 t E_{2\alpha-1, 2}(t^{2\alpha-1})) \quad (34)$$

3. THE A-TRANSFORM GAUSS-JACOBI HYBRID (ATGJ) METHOD

To accurately evaluate the integral term in the solution involving weakly singular kernels of the form $(t - \tau)^{\alpha-1}$, we derived the ATGJ Method. This quadrature technique is well-suited for integrals with endpoint singularities, as it uses weight functions tailored to the singular behavior, resulting in high accuracy with relatively few nodes. Specifically, the Gauss-Jacobi quadrature approximates integrals of the form

$$\int_0^t (t - \tau)^{\alpha-1} g(\tau) d\tau, \quad (35)$$

By transforming the integration interval and applying appropriately chosen orthogonal polynomials with weights. The method efficiently handles the fractional kernel's singularity without requiring excessive mesh refinement. For details on the Gauss-Jacobi quadrature and its theoretical properties, classical references such as Szegő's *Orthogonal Polynomials* [26, 27], respectively, provide comprehensive treatment. Incorporating this numerical integration within the computational scheme

allows reliable approximation of the solution $u(t)$ to the fractional differential equation.

4. NUMERICAL EXAMPLES

Example 4.1. Given the Caputo differential equation with fractional order $1 < \alpha < \beta < 2$:

$${}_0^C D_t^\alpha u(t) + {}_0^C D_t^\beta u(t) = 1 + \frac{(\alpha + 1)^2}{\alpha(\beta + 2)} t^2, t > 0, \quad (36)$$

and

$$u(0) = 1, u'(0) = -1. \quad (37)$$

Applying the A-transform and using Eq. (15) and the initial conditions, we obtain the following general solution:

$$u(t) = t^\beta E_{\beta-\alpha, \beta+1}(-t^{\beta-\alpha}) + \frac{2(\alpha + 1)^2}{\alpha(\beta + 2)} t^{\beta+2} E_{\beta-\alpha, \beta+3}(-t^{\beta-\alpha}) + 1 - t. \quad (38)$$

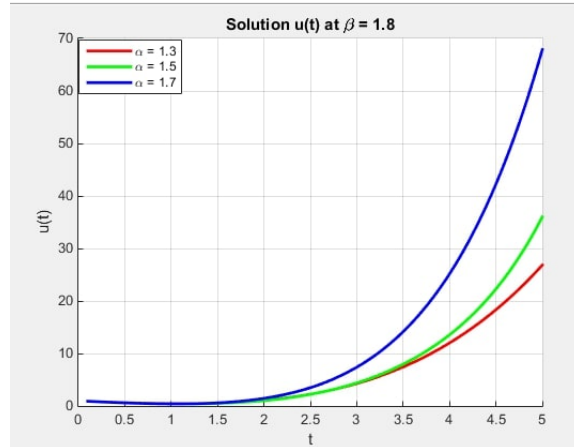


FIGURE 1. Plot of $u(t)$ of Example (4.1) at $\beta = 1.8$ and various values of α

Figure 1 illustrates the numerical behavior of the solution $u(t)$ for Example 5.1 of fractional order $\beta = 1.8$ and several values of α . The curves show a smooth transition of the solution as t increases, while different values of α shift the amplitude and growth rate of the solution. Smaller values of α produce a slower response with stronger memory effects, whereas larger values lead to a faster rise in the solution. This demonstrates the sensitivity of the Caputo-type model to the fractional order and highlights the accuracy of the A-Transform in capturing these behaviors.

Remark 4.2. (Classical Solution). In Example 4.1, since α and β cannot be equal, therefore, for $0 < \alpha < 1 < \beta < 2$, the solution of Eq. (36) becomes:

$$u(t) = t^\beta E_{\beta-\alpha, \beta+1}(-t^{\beta-\alpha}) + \frac{2(\alpha+1)^2}{\alpha(\beta+2)} t^{\beta+2} E_{\beta-\alpha, \beta+3}(-t^{\beta-\alpha}) + t E_{\beta-\alpha, 2}(-t^{\beta-\alpha}) + 1. \quad (39)$$

For integer orders $\alpha = 1$ and $\beta = 2$, Eq.(36) reduces to:

$$u''(t) + u'(t) = 1 + t^2, \quad (40)$$

whose solution is:

$$u(t) = \frac{1}{3}t^3 - t^2 - 3t - 3 + 4e^{-t}. \quad (41)$$

From Eq. (39), for the same values of α and β , and using the expansion of the following series:

$$t^2 E_{1,3}(-t) = t - 1 + e^{-t}, \quad (42)$$

$$2t^4 E_{1,5}(-t) = \frac{1}{3}t^3 - t^2 + 3t - 3 + 2e^{-t}, \quad (43)$$

$$t E_{1,2}(-t) = 1 - e^{-t}, \quad (44)$$

we get exactly the solution in Eq. (41).

Example 4.3. Giving a Caputo-Volterra Equation of order $1 < \alpha < 2$

$${}_0^C D_t^\alpha u(t) = t + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} u(\tau) d\tau, \quad (45)$$

with

$$u(0) = 0, u'(0) = 1. \quad (46)$$

Applying the A-transform and using Eqs. (15) and (46), we obtain the following solution:

$$u(t) = t^{\alpha+1} E_{2\alpha-1, \alpha+2}(t^{2\alpha-1}) + t E_{2\alpha-1, 2}(t^{2\alpha-1}). \quad (47)$$

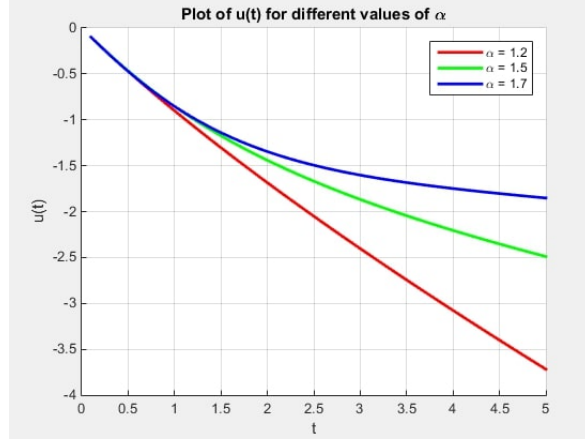


FIGURE 2. Plot of $u(t)$ of Example (4.3) at various values of α

Figure 2 shows the solution of Example 5.2, representing a Caputo–Volterra fractional equation for several values of α . As α increases, the solution becomes steeper and grows more rapidly, reflecting the weakening of long-term memory effects. For smaller α , the solution exhibits a more gradual growth, consistent with stronger hereditary behavior. The figure confirms that the A-Transform yields stable numerical solutions that correctly reflect the fractional dynamics of the Caputo–Volterra model.

Remark 4.4. (Classical Solution). In Example 4.3, for $\alpha = 2$, the Eq. (45) reduces to the Volterra integro-differential equation:

$$u''(t) = t + \int_0^t u(\tau) d\tau, \quad (48)$$

whose solution is:

$$u(t) = -1 + \frac{2}{3}e^t - e^{-t/2} \left(\frac{1}{3}\cos\left(\frac{\sqrt{3}t}{2}\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right) \right). \quad (49)$$

From Eq. (47), for the same value of α , we get:

$$t^3 E_{3,4}(t^3) = -1 + \frac{1}{3}e^t + \frac{2}{3}e^{-t/2}\cos\left(\frac{\sqrt{3}t}{2}\right), \quad (50)$$

$$tE_{3,2}(t^3) = \frac{1}{3}e^t - \frac{1}{3}e^{-t/2}\cos\left(\frac{\sqrt{3}t}{2}\right) + \frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}t}{2}\right); \quad (51)$$

By summing the two terms together from equations (50) and (51), we obtain exactly the solution in Eq. (49).

Example 4.5. Consider the Caputo differential equation of order $1 < \alpha < \beta < 2$:

$${}_0^C D_t^\alpha u(t) + {}_0^C D_t^\beta u(t) = \sin t, t > 0, \quad (52)$$

with the initial conditions:

$$u(0) = 1, u'(0) = \frac{1}{2}. \quad (53)$$

Using Eq. (24), and the initial conditions above, the solution of Eq. (52) is:

$$u(t) = \int_0^t (t - \tau)^{\beta-1} E_{\beta-\alpha, \beta}(-(t - \tau)^{\beta-\alpha}) \sin(\tau) d\tau + 1 + \frac{t}{2}, \quad (54)$$

that we have non-polynomial $f(t) = \sin(t)$. Using Gauss-Jacobi, let $\tau = t\xi$, so $d\tau = t d\xi$, then:

$$I(t) = t^\beta \int_0^1 (1 - \xi)^{\beta-1} E_{\beta-\alpha, \beta}(t^{\beta-\alpha}(1 - \xi)^{\beta-\alpha}) \sin(t\xi) d\xi, \quad (55)$$

Gauss-Jacobi on $[0, 1]$ with weight $w(\xi) = (1 - \xi)^{\beta-1}$:

$$I(t) \approx t^\beta \sum_{j=0}^{N_q} w_j E_{\beta-\alpha, \beta}(t^{\beta-\alpha}(1 - x_j)^{\beta-\alpha}) \sin(tx_j); \quad (56)$$

Hence, the final approximation is:

$$u(t) \approx I(t) + 1 + \frac{t}{2}. \quad (57)$$

Table 1 provides a comparison between the numerical solution and the high-resolution reference solution for Example 5.3 at $\beta = 1.9$ and different values of α . The table lists values of $u(t)$, reference values, and the absolute errors. The errors remain on the order of $10^{-12} - 10^{-14}$, indicating excellent accuracy of the proposed method. The consistency across all values of t and α confirms the reliability of the A-Transform combined with Gauss-Jacobi quadrature.

TABLE 1. Comparison of numerical and reference values for $\beta = 1.9$, and different values of α , and the corresponding error.

t	$u_{\alpha=1.2}(t)$	$u_{ref,\alpha=1.2}(t)$	$Error_{\alpha=1.2}$	$u_{\alpha=1.4}(t)$	$u_{ref,\alpha=1.4}(t)$	$Error_{\alpha=1.4}$
0.10101	1.0507	1.0507	1.5143e-12	1.0507	1.0507	1.3438e-12
0.30303	1.1565	1.1565	1.7979e-12	1.1561	1.1561	1.4222e-12
0.50505	1.273	1.273	4.4409e-14	1.2711	1.2711	3.7081e-14
0.70707	1.4044	1.4044	1.072e-12	1.3999	1.3999	8.9595e-13
0.90909	1.5533	1.5533	7.9248e-13	1.545	1.545	6.9322e-13
t	$u_{\alpha=1.6}(t)$	$u_{ref,\alpha=1.6}(t)$	$Error_{\alpha=1.6}$	$u_{\alpha=1.8}(t)$	$u_{ref,\alpha=1.8}(t)$	$Error_{\alpha=1.8}$
0.10101	1.0507	1.0507	1.0045e-12	1.0506	1.0506	5.6155e-13
0.30303	1.1555	1.1555	1.0492e-12	1.1548	1.1548	4.5741e-13
0.50505	1.269	1.269	3.153e-14	1.2667	1.2667	1.5321e-14
0.70707	1.395	1.395	7.5495e-13	1.3906	1.3906	1.8385e-13
0.90909	1.5364	1.5364	6.1684e-13	1.5303	1.5303	1.33e-13

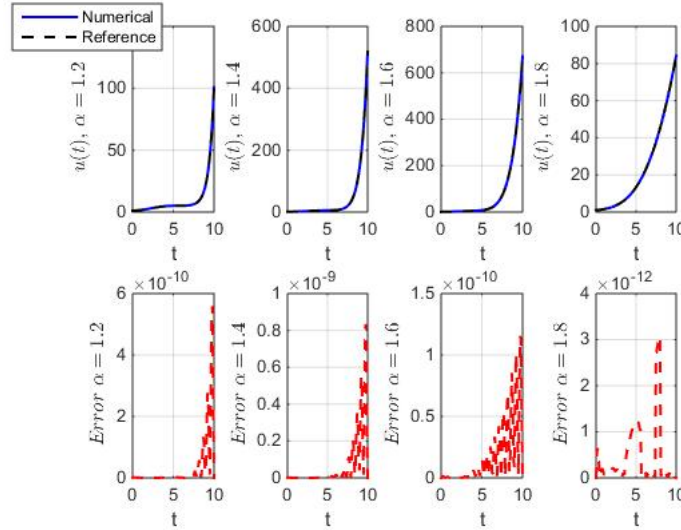


FIGURE 3. Shows the numerical solution and the numerical errors of $u(t)$

Figure 3 presents both the numerical solution and the corresponding numerical errors for Example 5.3 using the Gauss–Jacobi quadrature. The solution curves demonstrate the oscillatory nature of the forcing term $\sin(t)$, while maintaining stability for several values of α . The error plot shows that the absolute error remains extremely small across the domain, confirming the high accuracy and robustness of the combined A-Transform and Gauss–Jacobi approach, even for non-polynomial forcing functions.

Remark 4.6. (Classical Solution). In Example 4.5, for $0 < \alpha < 1 < \beta < 2$, the general solution of Eq. (52) is:

$$u(t) = \int_0^t (t-\tau)^{\beta-1} E_{\beta-\alpha, \beta}(-(t-\tau)^{\beta-\alpha}) \sin(\tau) d\tau + \frac{t}{2} E_{\beta-\alpha, 2}(-t^{\beta-\alpha}) + 1, \quad (58)$$

for integer orders $\alpha = 1$ and $\beta = 2$, Eq. (52) reduces to:

$$u''(t) + u'(t) = \sin(t), \quad (59)$$

whose solution is:

$$u(t) = \frac{5}{2} - e^{-t} - \frac{1}{2}(\sin(t) + \cos(t)). \quad (60)$$

From Eq.(58), for the same values of α and β , and using:

$$E_{1,2}(-t) = 1 - e^{-t}, \quad (61)$$

we get exactly the solution in Eq. (60).

Example 4.7. Consider the Caputo-Volterra equation of order $1 < \alpha < 2$:

$${}_0^C D_t^\alpha u(t) = e^t + 2\cos(t) + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} u(\tau) d\tau, \quad (62)$$

with the following initial conditions:

$$u(0) = \frac{-1}{2}, u'(0) = 2. \quad (63)$$

Using Eq. (34), and the initial conditions above, the solution of this equation will be:

$$u(t) = \int_0^t ((t-\tau)^{\alpha-1} E_{2\alpha-1, \alpha}((t-\tau)^{2\alpha-1})(e^\tau + 2\cos(\tau)) d\tau - \frac{1}{2} E_{2\alpha-1, 1}(t^{2\alpha-1}) + 2t E_{2\alpha-1, 2}(t^{2\alpha-1})), \quad (64)$$

that we have non-polynomial $f(t) = e^t + 2\cos(t)$. Using Gauss-Jacobi, let $\tau = t\xi$, so $d\tau = t d\xi$, then

$$I(t) = t^\alpha \int_0^1 (1-\xi)^{\alpha-1} E_{2\alpha-1, \alpha}(t^{2\alpha-1}(1-\xi)^{2\alpha-1}) f(t\xi) d\xi, \quad (65)$$

Gauss-Jacobi on $[0, 1]$ with weight $w(\xi) = (1 - \xi)^{\alpha-1}$:

$$I(t) \approx t^\alpha \sum_{j=0}^{N_q} w_j E_{2\alpha-1, \alpha}(t^{2\alpha-1}(1-x_j)^{2\alpha-1}) f(tx_j) d\xi; \quad (66)$$

Hence, the final approximation will be:

$$u(t) \approx I(t) - \frac{1}{2} E_{2\alpha-1, 1}(t^{2\alpha-1}) + 2t E_{2\alpha-1, 2}(t^{2\alpha-1}). \quad (67)$$

Table 2 shows a detailed comparison for Example 5.4, presenting numerical and reference solutions for several fractional orders α . The errors are extremely small—typically within $10^{-10} - 10^{-13}$ —which verifies the strong numerical stability of the method even in the presence of highly nonlinear and non-polynomial forcing terms. The results demonstrate that the A-Transform-based framework is capable of producing highly accurate solutions for complex fractional models.

TABLE 2. Comparison of Numerical and reference values for different α values and the corresponding errors

t	$u_{\alpha=1.3}(t)$	$u_{ref, \alpha=1.3}(t)$	$Error_{\alpha=1.3}$	$u_{\alpha=1.5}(t)$	$u_{ref, \alpha=1.5}(t)$	$Error_{\alpha=1.5}$
0.10101	0.023129	0.023129	7.4359e-10	-0.091149	-0.091149	6.0445e-10
0.30303	1.5197	1.5197	1.0808e-10	1.2214	1.2214	1.1152e-10
0.50505	3.4428	3.4428	1.0978e-12	3.0141	3.0141	1.2039e-12
0.70707	5.8002	5.8002	3.1965e-11	5.2624	5.2624	3.4946e-11
0.90909	8.656	8.656	2.874e-11	8.0019	8.0019	3.1433e-11
t	$u_{\alpha=1.3}(t)$	$u_{ref, \alpha=1.3}(t)$	$Error_{\alpha=1.3}$	$u_{\alpha=1.5}(t)$	$u_{ref, \alpha=1.5}(t)$	$Error_{\alpha=1.5}$
0.10101	-0.16788	-0.16788	3.1229e-10	-0.21784	-0.21784	7.3511e-11
0.30303	0.97742	0.97742	7.1054e-11	0.77533	0.77533	1.9813e-11
0.50505	2.6588	2.6588	9.4236e-13	2.3341	2.3341	4.9383e-13
0.70707	4.8592	4.8592	3.4062e-11	4.4806	4.4806	2.9439e-11
0.90909	7.6078	7.6078	3.4113e-11	7.2616	7.2616	3.5519e-11

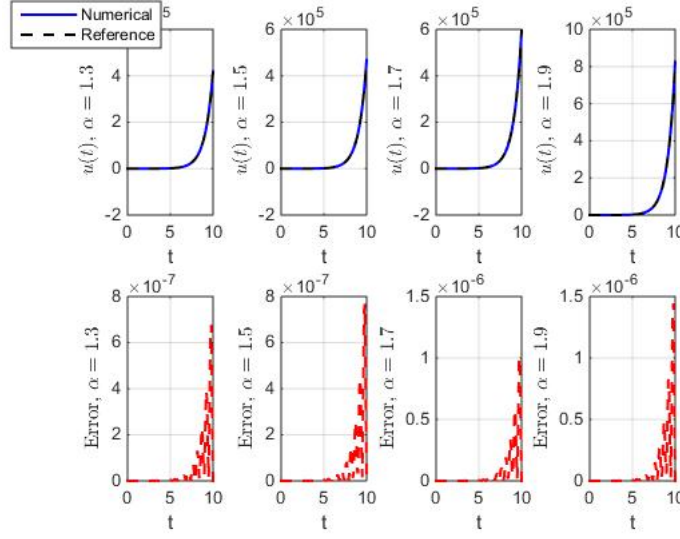


FIGURE 4. Shows the numerical solution and the numerical errors of $u(t)$

Figure 4 depicts the numerical solution and the associated numerical errors for Example 5.4, which involves a Caputo–Volterra equation with the non-polynomial forcing term $e^t + 2\cos(t)$. The solution curves display the characteristic growth induced by the exponential term, with variations depending on the fractional order α . The accompanying error plot shows consistently small error magnitudes across all tested values, demonstrating that the proposed method accurately handles both the integral kernel and the non-polynomial terms.

Remark 4.8. (Classical Solution). In Example 4.7, for $\alpha = 2$, Eq. (62) reduces to the following Volterra integro-differential equation:

$$u''(t) = e^t + 2\cos(t) + \int_0^t u(\tau)d\tau, \quad (68)$$

whose solution is:

$$u(t) = \left(\frac{5}{6} + \frac{t}{3}\right)e^t + \sin(t) - \cos(t) - e^{-t/2} \left(\frac{1}{3}\cos\left(\frac{\sqrt{3}t}{2}\right) + \frac{1}{2\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right) \right). \quad (69)$$

From Eq. (64), for $\alpha = 2$, and using Eq. (51), and

$$E_{3,1}(t^3) = \frac{1}{3}e^t + e^{-t/2} \left(\frac{1}{3}\cos\left(\frac{\sqrt{3}t}{2}\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right) \right), \quad (70)$$

we obtain exactly the solution in Eq. (69).

In the above examples, we also considered the corresponding integer-order cases, such as $\alpha = 1$ and $\beta = 2$, to verify the obtained solutions. For integer orders, the Caputo derivatives reduce to the corresponding ordinary derivatives, and the resulting solutions, expressed in terms of exponential and trigonometric functions, satisfy the differential equations pointwise together with the prescribed initial conditions. Hence, they are classical solutions. For the fractional-order cases considered above, the solutions are expressed in terms of Mittag–Leffler functions and convolution integrals, and their regularity is consistent with the requirements of the classical-solution definition given in Section 1.1.

5. COMPARISON WITH EXISTING METHODS

Recent studies have employed wavelet-based operational matrix methods including Hermite wavelets, Bernoulli wavelets, Haar wavelets, and Legendre wavelets [28, 29, 30, 31], respectively to solve fractional differential equations with Caputo-type operators. Although none of these works address the specific fractional models considered in this paper, they provide an appropriate methodological benchmark. Compared to these approaches, the proposed A-Transform framework offers simpler analytical formulas and avoids large operational matrices, while Gauss–Jacobi quadrature allows efficient treatment of non-polynomial forcing terms. The numerical results in this work exhibit very small errors (typically $10^{-12} - 10^{-14}$ and $10^{-10} - 10^{-13}$), which are comparable to or better than the accuracy reported in the above methods. Therefore, the proposed approach is competitive and effective even when applied to new fractional models not previously examined in the literature.

6. CONCLUSION

This work establishes the A-Transform as an effective analytical tool for fractional calculus, providing simplified expressions for key operators such as the Mittag–Leffler function, the Caputo derivative, and the mixed Caputo–Volterra equation. The proposed A-Transform Gauss–Jacobi hybrid framework efficiently solves non-homogeneous fractional differential and integro-differential equations, including cases with non-polynomial forcing terms. Numerical experiments confirm the accuracy and stability of the method, with smooth solution behavior and error levels consistent with high-resolution reference computations. The approach handles singular kernels efficiently and preserves physical consistency through direct treatment of initial conditions. These analytical and numerical results demonstrate the potential of the A-Transform-based methodology as a reliable and flexible tool for future studies involving complex fractional models and applications. MATLAB R2014b was used with moderate discretizations ($N = 100$ points). Since exact solutions of the two systems ((52)-(53), (62)-(63)) are unknown or too complicated

for non-polynomial $f(t)$, a high-resolution reference ($N_{ref} = 2000$) was computed for error estimation:

$$Error = |U_{Numerical} - U_{Reference}| \quad (71)$$

7. FUTURE WORKS

- (1) **Extension to Other Fractional Operators:** Apply the A-Transform framework to generalized fractional derivatives such as Atangana–Caputo, Caputo–Fabrizio, and Prabhakar operators, and examine whether similar analytical simplifications can be obtained.
- (2) **Integration with Wavelet Operational Matrices:** Develop a hybrid A-Transform/Hermite-wavelet scheme to handle multi-dimensional or strongly nonlinear fractional models, benefiting from both analytical and wavelet-based numerical efficiencies.
- (3) **Higher-Dimensional Fractional Equations:** Extend the method to fractional partial differential equations (FPDEs) in space–time domains, where singular kernels and memory effects play a stronger role.
- (4) **Adaptive Numerical Strategies:** Incorporate adaptive Gauss–Jacobi quadrature or error-controlled mesh refinement to improve accuracy near singularities or regions with rapid solution changes.

Data Availability Statement. No datasets were generated or analyzed during the current study. All results presented in this manuscript are derived analytically and numerically from the mathematical models considered.

Declarations. The author declares that there are no conflicts of interest related to this work.

Authors Contributions. Arkan Shareef Hasan: Conceptualization, methodology, analytical derivations, numerical analysis, and writing of the original manuscript. Hela Shawkat Ahmed: Review and editing of the manuscript. Both authors read and approved the final version of the manuscript.

Funding Statement. This research received no external funding.

Acknowledgement. We would like to thank the reviewers for their comments and suggestions.

REFERENCES

- [1] R. M. Pandey, A. Chandola, L. Rathour, and V. N. Mishra, "Fractional calculus operators applied on m-parameter mittag-leffler function and bessel-maitland function with applications," in *AIP Conference Proceedings*, vol. 3005, p. 020020, AIP Publishing LLC, 2024. <https://doi.org/10.1063/5.0210377>.
- [2] K. Diethelm and A. D. Freed, "On the solution of nonlinear fractional-order differential equations used in the modeling of viscoplasticity," in *Scientific computing in chemical engineering II: computational fluid dynamics, reaction engineering, and molecular properties*, pp. 217–224, Springer, 1999. https://link.springer.com/chapter/10.1007/978-3-642-60185-9_24.
- [3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, vol. 204. elsevier, 2006. https://sutlib2.sut.ac.th/sut_contents/H103746.pdf.
- [4] W. Lin, "Global existence theory and chaos control of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 1, pp. 709–726, 2007. <https://doi.org/10.1016/j.jmaa.2006.10.040>.
- [5] D. Das, B. C. Ghosh, L. Rathour, S. K. Sahani, L. N. Mishra, and V. N. Mishra, "Dislocated quasi modular metric spaces and its applications in fixed point theory via a-class and c-class function," in *AIP Conference Proceedings*, vol. 3005, p. 020033, AIP Publishing LLC, 2024. <https://doi.org/10.1063/5.0210617>.
- [6] M. Sharma, S. Chaudhary, L. Rathour, and V. N. Mishra, "A green fractional transportation system under dual hesitant fermatean fuzzy configuration with safety factor," in *Optimization and Computing using Intelligent Data-Driven Approaches for Decision-Making*, pp. 270–282, CRC Press. <https://www.taylorfrancis.com/chapters/edit/10.1201/9781003536796-12>.
- [7] L. Rathour, D. Obradovic, L. N. Mishra, K. Khatri, and V. N. Mishra, "Application of the function in basic mathematics," *International Journal of Advanced Science and Engineering*, vol. 9, no. 4, pp. 3144–3150, 2023. <https://doi.org/10.29294/IJASE.9.4.2023.3144-3150>.
- [8] G. Farid, S. Mehmood, L. Rathour, M. Elamin, H. U. M. Ahamd, and N. Yahia, "Grüss type k-fractional integral operator inequalities and allied results," *International Journal of Analysis and Applications*, vol. 21, pp. 133–133, 2023. <https://doi.org/10.28924/2291-8639-21-2023-133>.
- [9] G. Farid, S. B. Akbar, L. Rathour, and L. N. Mishra, "Riemann-liouville fractional versions of hadamard inequality for strongly (α, m) -convex functions," *Korean Journal of Mathematics*, vol. 29, no. 4, pp. 687–704, 2021. <https://doi.org/10.11568/kjm.2021.29.4.687>.
- [10] S. A. Murad and S. M. Rasheed, "Application of a domian decomposition method for solving fractional differential equation," *Journal of Education and Science*, vol. 22, no. 1, pp. 93–0, 2009. <https://ojs.uomosul.edu.iq/index.php/edusj/article/view/42082>.
- [11] B. Alqahtani, H. Aydi, E. Karapinar, and V. Rakočević, "A solution for volterra fractional integral equations by hybrid contractions," *Mathematics*, vol. 7, no. 8, p. 694, 2019. <https://doi.org/10.3390/math7080694>.
- [12] U. Lepik, "Solving fractional integral equations by the haar wavelet method," *Applied Mathematics and Computation*, vol. 214, no. 2, pp. 468–478, 2009. <https://doi.org/10.1016/j.amc.2009.04.015>.
- [13] R. Garrappa, "Numerical solution of fractional differential equations: A survey and a software tutorial," *Mathematics*, vol. 6, no. 2, p. 16, 2018. <https://doi.org/10.3390/math6020016>.
- [14] A. Jannelli, "Numerical solutions of fractional differential equations arising in engineering sciences," *Mathematics*, vol. 8, no. 2, p. 215, 2020. <https://doi.org/10.3390/math8020215>.
- [15] R. T. Matoog, A. M. Mahdy, M. A. Abdou, and D. S. Mohamed, "A computational method for solving nonlinear fractional integral equations," *Fractal and Fractional*, vol. 8, no. 11, p. 663, 2024. <https://doi.org/10.3390/fractalfract8110663>.
- [16] H. Jafari, "A new general integral transform for solving integral equations," *Journal of Advanced Research*, vol. 32, pp. 133–138, 2021. <https://doi.org/10.1016/j.jare.2020.08.016>.

- [17] Y. Luchko and V. Kiryakova, "The mellin integral transform in fractional calculus," *Fractional Calculus and Applied Analysis*, vol. 16, no. 2, pp. 405–430, 2013. <https://doi.org/10.2478/s13540-013-0025-8>.
- [18] S. Sattaso, K. Nonlaopon, H. Kim, and S. Al-Omari, "Certain solutions of abel's integral equations on distribution spaces via distributional g α -transform," *Symmetry*, vol. 15, no. 1, p. 53, 2022. <https://doi.org/10.3390/sym15010053>.
- [19] H. Liu and G. Cao, "Effectiveness of the young-laplace equation at nanoscale," *Scientific reports*, vol. 6, no. 1, p. 23936, 2016. <https://www.nature.com/articles/srep23936>.
- [20] A. S. Hasan, "The a-variational iteration transform," *Journal of Applied Computer Science and Mathematics*, vol. 19, no. 38, pp. 22–27, 2025. <https://doi.org/10.4316/JACSM.202501004>.
- [21] A. Munir, A. Qayyum, L. Rathour, G. Atta, S. S. Supadi, and U. Ali, "A study on milne-type inequalities for a specific fractional integral operator with applications," *Korean Journal of Mathematics*, vol. 32, no. 2, pp. 297–314, 2024. <https://doi.org/10.11568/kjm.2024.32.2.297>.
- [22] F. Mainardi, "Why the mittag-leffler function can be considered the queen function of the fractional calculus?," *Entropy*, vol. 22, no. 12, 2020. <https://www.mdpi.com/1099-4300/22/12/1359>.
- [23] G. Farid, S. B. Akbar, L. Rathour, L. N. Mishra, and V. N. Mishra, "Riemann-liouville fractional versions of hadamard inequality for strongly m-convex functions," *International Journal of Analysis and Applications*, vol. 20, pp. 5–5, 2022. <https://doi.org/10.28924/2291-8639-20-2022-5>.
- [24] S. S. Redhwan, S. L. Shaikh, and M. S. Abdo, "Some properties of sadik transform and its applications of fractional-order dynamical systems in control theory," *Advances in the Theory of Nonlinear Analysis and its Application*, vol. 4, p. 51–66, 2020. <https://doi.org/10.31197/atnaa.647503>.
- [25] J. Sousa, T. Abdeljawad, and D. Oliveira, "Mild and classical solutions for fractional evolution differential equation," *arXiv preprint arXiv:1908.04948*, 2019. <https://doi.org/10.48550/arXiv.1908.04948>.
- [26] S. Jahanshahi, E. Babolian, D. F. Torres, and A. Vahidi, "A fractional gauss-jacobi quadrature rule for approximating fractional integrals and derivatives," *Chaos, Solitons & Fractals*, vol. 102, pp. 295–304, 2017. <https://doi.org/10.1016/j.chaos.2017.04.034>.
- [27] D. S. Dimitrov, "Gauss-jacobi quadrature method for approximating fractional integrals and derivatives," *Applied Mathematics and Computation*, vol. 274, no. 1, p. 395–405, 2016. <https://doi.org/10.1016/j.amc.2015.12.034>.
- [28] A. Turan Dincel, S. N. Tural Polat, and P. Sahin, "Hermite wavelet method for nonlinear fractional differential equations," *Fractal and Fractional*, vol. 7, no. 5, p. 346, 2023. <https://doi.org/10.3390/fractalfract7050346>.
- [29] H. M. Arafa, M. A. Ramadan, and N. Althobaiti, "A numerical technique based on bernoulli wavelet operational matrices for solving a class of fractional order differential equations," *Fractal and Fractional*, vol. 7, no. 8, p. 604, 2023. <https://doi.org/10.3390/fractalfract7080604>.
- [30] N. A. Khan, M. Ali, A. Ara, I. Alraddadi, and H. Ahmad, "A haar wavelet operational matrix method for fractional derivatives with non-singular kernel," in *Operations Research Forum*, vol. 6, p. 118, Springer, 2025. <https://link.springer.com/article/10.1007/s43069-025-00502-4>.
- [31] E. A. Al-Taai, "A new operational matrix approach based on legendre wavelets for fractional differential equations," *International Journal of Applied Mathematics*, vol. 38, no. 8s, pp. 1167–1184, 2025. <https://doi.org/10.12732/ijam.v38i8s.633>.