

Existence and Uniqueness of Solution for an Infection-Age Structured Mathematical Model of Ebola Virus Disease Dynamics

Emmanuel Azuaba^{1*}, Ninuola Ifeoluwa Akinwande², Abdulrahman Sirajo³, Faruk Adamu Kuta⁴

¹Department of Mathematical Sciences, Bingham University, Nigeria,

²Department of Mathematics, Federal University of Technology, Nigeria,

³Department of Mathematics, Federal University Birnin Kebbi, Nigeria,

⁴Department of Microbiology, Federal University of Technology Minna, Nigeria.

Abstract. This study examines an infection-age-structured epidemic model for Ebola Virus Disease (EVD) dynamics. We incorporate the effects of multiple quarantine stages and dead bodies of Ebola victims into a hybrid system comprising ordinary differential equations (ODEs), partial differential equations (PDEs), and integro-differential equations to investigate the transmission dynamics of EVD. We prove the existence and uniqueness of solutions using Volterra integral equations and classical Gronwall's inequality $B(t) \leq |\beta|_L e^{t|\beta|_L} |\psi|_L$. We also show that the solutions of partial differential equations and integro-differential equations depend continuously on the initial infection age of EVD distributions. We show that the solutions of ordinary differential equations exist and are unique. The effective reproduction number (R_{eff}) was obtained with a threshold value of $R_P + R_Q + R_G$ and is used to analyze the local and global stability of disease-free equilibrium. The local stability of the disease-free equilibrium was linearized to obtain $|G_1(a_1)| \leq G_1(0) < 1$, where $G_1(a_1)$ is a function of R_{eff} . A suitable Lyapunov function was constructed and used to analyze the global stability of the disease-free equilibrium. The result obtained was $q(t, \tau) \geq 0$, where $q(t, \tau)$ is the density function of infection-age for all $t > \tau$. This implies that disease-free equilibrium is locally and globally asymptotically stable if $R_{eff} < 1$ and $R_{eff} \leq 1$ respectively. The local and global stability of endemic equilibrium were analyzed using LaSalle's Invariance theory and Lyapunov function.

Key words and Phrases: Infection-age, Ebola Virus Disease, Burial, Existence and uniqueness, Quarantine, Classical Gronwall's inequality.

*Corresponding author: emmanuel.azuaba@binghamuni.edu.ng

2020 Mathematics Subject Classification: 34A12, 26D10, 92B05

Received: 15-04-2025, accepted: 05-04-2026.

1. INTRODUCTION

Ebola Virus Disease (EVD) is a *haemorrhagic* fever transmitted to humans by Ebola virus genes. The virus belongs to one of the three genera of *Filoviridae*, along with the *Marburgvirus* and *Cuevavirus* genera. In 2014, WHO described Ebola as a fatal disease caused by infection from the family of *Ribonucleic Acid* (RNA) viruses known as *Filoviruses*. The Ebola virus comprises five genomes, which are *Zaire ebolavirus*, *Sudan ebolavirus*, *Bundibugyo ebolavirus*, *Tai Forest ebolavirus*, and *Reston ebolavirus*. The first four viruses have caused diseases in the human populations, and the fifth virus spawned disease in nonhuman primates like monkeys and chimpanzees as well as fruit bats [1]. Ebola *virions* are filamentous particles that appear in the shape of a shepherd's crook and may also be coiled like a toroid.



FIGURE 1. Ebola Virus Genus [2]

In this work, we study the *Zaire Ebolavirus*. The virus was first identified in 1976 near the Ebola River in the Democratic Republic of Congo [3]; [4]. However, in March 2014, the epidemic outbreak of *Zaire Ebolavirus* occurred in Guinea, and the virus has spread to other West African nations like Liberia, Sierra Leone, Senegal, and Nigeria [5]. In September 2025, an Ebola Zaire outbreak was confirmed at Bulape and Mweka in the Democratic Republic of Congo; 28 suspected cases and 15 deaths were reported, including four public health workers [6].

The reservoir of *Zaire Ebolavirus* remains unknown, but because of similarities in the viruses, many scientists and researchers believe that the virus is from forest animals, and bats are the most likely carriers. Thus, the virus was initially transmitted to people from wild animals and spread in the human population through human-to-human transmission via blood and body fluid contacts. Individuals infected with EVD can spread the virus to other people once they have developed symptoms. The symptoms usually start 2 to 14 days after contracting the virus with a fever, headache, muscle pain, and sore throat. The vomiting, diarrhea, and rashes could typically follow along with inflammation of the liver and

kidneys. Most people at that stage begin to bleed both internally and externally [5]. EVD is only transmitted by direct contact with the blood or body fluids of an infected person. An infection also occurs when there is direct contact with dead bodies of Ebola victims or materials like clothing that are soiled with infected bodily fluids.

The spread of EVD is attributed to many factors, such as lack of safe burial of Ebola victims and the absence of isolation centers to quarantine the infectious persons [7]. The overall risk to West African countries directly affected by EVD can be curbed if health workers are given a robust health infrastructure such as quarantine logistics, epidemiological surveillance, ventilators, the Ebola Contact Tracing application (ECT app) and appropriate Personal Protection Equipment (PPE) to use in treating infected Ebola patients [8]. Ebola virus disease claimed thousands of lives in the 2014 outbreak, and since then there are series of sporadic outbreaks across West African countries. Despite the fact that some countries are able to control the virus, there is still more to be done in order to understand the dynamics of the virus. The aim of this study is to develop and analyze a mathematical model of EVD dynamics by incorporating infection age and multiple quarantine stages with safe burial practices as effective control measures using ordinary differential equations, partial differential equations, and integro-differential equations. Thus, this study presents effective control measures of EVD. Significantly, the study encourages governments of the affected countries to use aggressive quarantine strategies and safe burial practices of Ebola dead bodies.

We present the mathematical model in Section 2 and analyze the model in Section 3. Section 4 presents results and discussions, while Section 5 concludes the study.

2. MODEL

Several authors have researched EVD including [9, 10, 11] without infection age. However, more advanced population models are being structured with infection age. This is because age is one of the most important characteristics in modelling population dynamics and infectious diseases. Individuals with different ages have different reproductive rates and survival capacities. Epidemic diseases also have different infection rates and different mortality rates for different individuals of different immune systems with different age groups. Infection age helps in modelling the impact of behaviour changes on the infected individuals and interventions over the course of an epidemic. By infection age we are referring to the time since the infection began, which is a transient state relative to lifespan of an infected individual. In the past five decades, the age-structured SIR epidemic model has been extensively studied by many authors, including [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

2.1. Assumptions of the model.

We assume the following vital dynamics in line with the nature and rate of transfer from one compartment to another as follows.

- (i) Susceptible individuals $S(t)$ are infected by having effective physical contact with either $I(t)$ or $G(t)$ at the rates α_1 and α_2 respectively;
- (ii) A suspected individual from $S(t)$ can be moved to $P(t)$ at a rate γ to be observed. A newly observed individual can be virus-free with probability ϕ , or shows Ebola disease symptoms with probability $1 - \phi$.
- (iii) The recruitment into the population is through $S(t)$ at a rate Λ , and the recruitment includes natural birth and migration;
- (iv) The compartment $G(t)$ contains both natural and disease-induced death from $I(t)$ and $Q(t)$ only;
- (v) The population natural death rate μ is applicable to all compartments of the living except $G(t)$;
- (vi) Recovered individuals are permanently immune;
- (vii) The two compartments $I(t)$ and $Q(t)$ are highly infectious among the living population and hence, the two compartments are considered to be the same and also equal, i.e. $I(t) = Q(t)$ in terms of effective EVD transmission dynamics;
- (viii) The initial infection-age $q(0, \tau) = \psi(\tau)$ is continuous, nonnegative and integrable function of $\tau \in [0, T]$.

TABLE 1. Parameters of the Model

Symbol	Interpretation
t	Time
Λ	Recruitment rate into $S(t)$ only
μ	Per capital natural death rate
δ_1	Disease-induced death rate of the infected class $I(t)$
δ_2	Disease-induced death rate of the quarantine with treatment class $Q(t)$
α_1	Effective virus transmission rate from $I(t)$ to $S(t)$
α_2	Effective virus transmission rate from $G(t)$ to $S(t)$
γ	The rate at which suspected individuals moved from $S(t)$ to $P(t)$
φ	Quarantine rate from $I(t)$ to $Q(t)$
ω	Effective treatment rate from $Q(t)$ to $R(t)$
θ	Safe rate of burial practices
ϕ	Probability of a newly observed individual is virus-free
$1 - \phi$	Probability of a newly observed individual is infected
$\alpha_2(1 - \theta)$	Effective contact rate between $G(t)$ and $S(t)$
$\sigma(\tau)$	Death rate due to infection age
$\psi(\tau)$	Initial age of infection
τ	Infection age
T	Maximum infection age
δ	Maximum death rate from infection
$q(t, \tau)$	Density function of infection-age
k	Control parameter

2.2. Model formulation.

In this section, the transmission dynamics of EVD are developed using ordinary differential equations, partial differential equations, and integro-differential equations. We partitioned the living population into five compartments of susceptible $S(t)$, quarantine under observation $P(t)$, quarantine with treatment $Q(t)$, infected individuals $I(t)$, recovered individuals $R(t)$ and a compartment of dead bodies $G(t)$. The transfer rates between the sub-classes are composed of several epidemiological parameters as follows.

Susceptible $S(t)$.

The susceptible population is increased by birth and migration at a rate Λ , also by a probability ϕ from the members of $P(t)$ who are virus-free after observation. $S(t)$ members are free of infection but open to infection if they interact with either $I(t)$ or $G(t)$ at the rates α_1 and $\alpha_2(1 - \theta)$ respectively. Where α_1 is a disease transmission rate between $S(t)$ and $I(t)$, while $\alpha_2(1 - \theta)$ is an effective contact rate between $S(t)$ and $G(t)$. Susceptible individuals are reduced at a rate γ as well as natural death at a rate μ . Thus, the rate change of susceptible per unit time is given by

$$\frac{dS}{dt} = \Lambda - \frac{\alpha_1 IS}{N} - \frac{\alpha_2(1 - \theta)GS}{N} + \phi P - (\mu + \gamma)S.$$

Quarantine under observation $P(t)$.

Quarantine under observation is increased at a rate γ due to the number of suspected individuals from $S(t)$. After observation, a newly observed individual can be virus-free with probability ϕ and moved back to $S(t)$, or shows Ebola disease symptoms with probability $1 - \phi$ to progress to $Q(t)$. The quarantine under observation can also be reduced by natural death at a rate μ . Thus, $0 \leq \phi \leq 1$. Hence, the rate change of quarantine under observation at a time t is given as

$$\frac{dP}{dt} = \gamma S - ((\mu + \phi)(1 - \phi))P.$$

Infected individuals $I(t)$.

The compartment is increased due to effective transmission contacts at the rates α_1 and $\alpha_2(1 - \theta)$. Infected individuals $I(t)$ are quarantined to $Q(t)$ at a rate φ and reduced by δ_1 due to disease-induced death as well as natural death at a rate μ . We assume that the infected individuals $I(t)$ could only be quarantined to receive an effective medical treatment and recovered or leave the study population $N(t)$ via death. So, the rate change of infected individuals at a time t is given by

$$\frac{dI}{dt} = \frac{\alpha_1 IS}{N} + \frac{\alpha_2(1 - \theta)GS}{N} - (\mu + \delta_1 + \varphi)I.$$

Dead bodies $G(t)$.

We incorporate $G(t)$ in the model because it is capable of spreading disease or can be reduced through a safe burial practices by θ . This is a compartment of dead

bodies for both natural and Ebola victims generated from $I(t)$ and $Q(t)$ only. We assume that even individuals who died naturally at a rate μ from $I(t)$ and $Q(t)$ automatically become members of $G(t)$ due to the fact that they are more likely to spread infection as Ebola disease is very severe. The compartment is also increased by disease-induced death from $I(t)$ and $Q(t)$ at the rates δ_1 and δ_2 respectively. Hence, the rate change of dead bodies at a time t is given as

$$\frac{dG}{dt} = (\mu + \delta_1)I + (\mu + \delta_2)Q - \theta G.$$

Recovered individuals $R(t)$.

The recovered individuals are increased due to effective medical treatment at a rate ω and reduced by natural death at a rate μ . Thus, the rate change of recovered individuals at a time t is given by

$$\frac{dR}{dt} = \omega Q - \mu R.$$

Quarantine with treatment $Q(t)$.

This is the second compartment of infected individuals that increased by $1 - \phi$ and at a rate φ . The quarantine with treatment can also be reduced by effective medical treatment, disease-induced death, and natural death at the rates ω , δ_2 , and μ respectively. The quarantine with treatment is structured by the infection-age with the density function $q(t, \tau)$, where t is the time parameter and τ is the infection age ([24]). The choice of age structured in $Q(t)$ is based on the fact that the compartment alone contains two control measures which are quarantine and effective medical treatment. Thus, we are interested to analyze the disease dynamics of infected individuals who are quarantine with treatment. We assume that there is a maximum infection age at which a member of the quarantine with treatment must leave the compartment via death; and so $0 \leq \tau \leq T$.

The death rate due to infection is given by

$$\sigma(\tau) = \delta e^{-k(T-\tau)}$$

where δ and k are constants. δ is the maximum death rate from infection while k is a control parameter which could be a measure of slowing down the death of the infected, this is a measure of effectiveness of slowing down the death of infected individuals; for example, the effectiveness of ω and a high value of k means positive effectiveness of such control measures and vice versa ([25]). By biological implication, the expression of death rate in exponential form reflects the deadliest nature of EVD. The infection age provides more accurate modeling where infectivity or recovery rates depend on the duration of infection rather than just the passage of time. This is true because the traditional ordinary differential equations for the models like SIR assume that the probability of leaving a compartment is constant, implying an exponential distribution of time spent in that state. But infection age allows non-exponential, realistic recovery and infectiousness periods. Infectivity often varies over time, for example, peak infectivity occurring days after infection.

Infection age makes infectivity a function by providing a more realistic transmission mechanism. Therefore, infection age provides a tool for modeling of changed behaviours or treatment effectiveness, for example, antiviral treatment that depends on how long an individual has been sick.

Thus, the infection age density for this model is governed by a partial differential equation of the type

$$\frac{\partial q(t, \tau)}{\partial t} + \frac{\partial q(t, \tau)}{\partial \tau} = -(\mu + \sigma(\tau))q(t, \tau).$$

The partial differential equation is coupled with ordinary differential equations and integro-differential equations for the remaining compartments of susceptible $S(t)$, quarantine under observation $P(t)$, infected individuals $I(t)$, recovered individuals $R(t)$ and a compartment of dead bodies $G(t)$ to overcome the limitations of classical compartmental ordinary differential equation models like SIR or SEIR. The core justification is to explicitly incorporate disease memory and the fact that an individual's contagiousness and recovery likelihood depend heavily on the time elapsed since they were infected, rather than just the time they spent in an infectious compartment. This hybrid approach allows the model to capture complexities and time dependent dynamics via the partial differential equation while still using manageable equations for the remaining compartments via the ordinary differential equations.

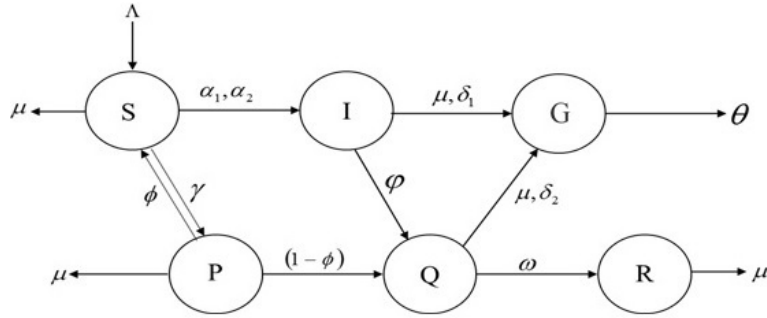


FIGURE 2. Schematic Diagram of Ebola Virus Disease [1]

The model equations are

$$\frac{dS}{dt} = \Lambda - \frac{\alpha_1 IS}{N} - \frac{\alpha_2(1 - \theta)GS}{N} + \phi P - (\mu + \gamma)S \quad (1)$$

$$\frac{dP}{dt} = \gamma S - ((\mu + \phi)(1 - \phi)) P \quad (2)$$

$$\frac{dI}{dt} = \frac{\alpha_1 IS}{N} + \frac{\alpha_2(1 - \theta)GS}{N} - (\mu + \delta_1 + \varphi)I \quad (3)$$

$$\frac{dG}{dt} = (\mu + \delta_1)I + (\mu + \delta_2)Q - \theta G \quad (4)$$

$$\frac{dR}{dt} = \omega Q - \mu R \quad (5)$$

$$\frac{\partial q(t, \tau)}{\partial t} + \frac{\partial q(t, \tau)}{\partial \tau} + (\mu + \sigma(\tau))q(t, \tau) = 0. \quad (6)$$

We assume the following conditions hold.

- (1) $0 \leq \theta \leq 1, (1 - \theta) \geq 0, \theta \geq 0,$
- (2) $0 \leq \phi \leq 1, (1 - \phi) \geq 0, \phi \geq 0.$

Equation (6) describes Ebola Virus disease prognosis whereby the virus infects individuals of any age and is highly dependent on infection-age progression, which also correlates with significant differences in the immune system of humans. The virus destroys the immune cells and suppresses the immune system of the host. The ability of the host cells to respond varies across age groups, which directly impacts survival rates. In most cases, a balanced and effective immune response is crucial for survival.

The boundary conditions are

$$q(t, 0) = B(t) = \frac{\alpha_1 IS}{N} + \frac{\alpha_2(1 - \theta)GS}{N} + \varphi I + (1 - \phi)P - \omega Q \quad (7)$$

$$q(t, T) = 0 \quad (8)$$

$$N(t) = S(t) + P(t) + I(t) + R(t) + Q(t). \quad (9)$$

The initial age and maximum infection age involved integro-differential equations; For example, from equation (8). T is the maximum infection age, i.e. when $\tau = T$ the infected member dies of EVD. The total population of infected individuals from quarantine with treatment is given as

$$Q(t) = \int_0^T q(t, \tau) d\tau \quad (10)$$

According to [25], death rate via infection is given as

$$\sigma(\tau) = \delta e^{-k(T-\tau)}. \quad (11)$$

The initial conditions are

$$\begin{aligned} S(0) &= S_0, P(0) = P_0, I(0) = I_0, G(0) = G_0, \\ R(0) &= R_0, Q(0) = Q_0, N(0) = N_0 \end{aligned} \quad (12)$$

and

$$q(0, \tau) = \psi(\tau). \quad (13)$$

We integrate equation (6) over τ to obtain

$$\frac{dQ}{dt} + q(t, T) - q(t, 0) = -\mu \int_0^T q(t, \tau) d\tau - \int_0^T \sigma(\tau)q(t, \tau) d\tau. \quad (14)$$

From (14), we have

$$\int_0^T \sigma(\tau)q(t, \tau) d\tau = \delta \int_0^T e^{-k(T-\tau)}q(t, \tau) d\tau = 0 \quad (15)$$

where, $\int_0^T q(t, \tau) d\tau$ and $\sigma(\tau)$ are as defined in (10) and (11) respectively. Thus, applying integration by parts on RHS of (15) equals to zero i.e $\int_0^T \sigma(\tau)q(t, \tau) d\tau = 0$. We substitute equations (7),(8), (10) and (15) into (14) to obtain

$$\frac{dQ}{dt} = \frac{\alpha_1 IS}{N} + \frac{\alpha_2(1-\theta)GS}{N} + \varphi I + (1-\phi)P - (\mu + \omega)Q. \quad (16)$$

Equation (16) follows from (6) and (10). Hence, equation (16) gives us the total population of infected individuals from quarantine with treatment in the form of ordinary differential equation.

3. EXISTENCE AND UNIQUENESS OF SOLUTION AND CONTINUOUS DEPENDENCE ON INITIAL INFECTION-AGE DISTRIBUTIONS

In this section, we prove the existence and uniqueness of solutions for equations (6), (7), and (13). We show that solutions of problems (6), (7), and (13) depend continuously on the initial infection-age of the infected individuals, for example, see [26, 27, 28, 29, 30], which defined the partial derivative part of equation (6) as

$$\frac{\partial q(t, \tau)}{\partial t} + \frac{\partial q(t, \tau)}{\partial \tau} = \lim_{h \rightarrow 0} \frac{q(t+h, \tau+h) - q(t, \tau)}{h}. \quad (17)$$

Let

$$f(h) = q(t_0 + h, \tau_0 + h). \quad (18)$$

Then, the ordinary differential equation is

$$\frac{df(h)}{dh} + (\mu + \sigma(h))f(h) = 0. \quad (19)$$

Equation (19) provides the tool for a unique solution as

$$f(h) = f(0)e^{-\int_0^h (\mu + \sigma(s))ds}. \quad (20)$$

Substituting equation (18) into equation (20), we have

$$g(t_0 + h, \tau_0 + h) = g(t_0, \tau_0)e^{-\int_0^h (\mu + \sigma(s))ds}. \quad (21)$$

This gives the value of q at all points on the characteristics passing through the point $q(t_0, \tau_0)$. When $q(t_0, \tau_0) = (0, \tau - t)$; $h = t$ we have that

$$q(t, \tau) = \psi(\tau - t)e^{-\int_0^t (\mu + \sigma(s))ds}, \quad t \leq \tau. \quad (22)$$

Also, when $q(t_0, \tau_0) = (t - \tau, 0)$; $h = t$ we have

$$q(t, \tau) = B(t - \tau)e^{-\int_0^t (\mu + \sigma(s))ds}, \quad t > \tau. \quad (23)$$

Then equations (22) and (23) position the solution in the positive quadrant $t \geq 0, \tau \geq 0$. Thus, the problem (6) to (13) has the following unique solution that exists for all time.

$$q(t, \tau) = \begin{cases} B(t - \tau)\pi(\tau), & t > \tau, \\ \psi(\tau - t)\pi(\tau), & t < \tau. \end{cases} \quad (24)$$

Equation(24) is written in the following form

$$q(t, \tau) = \begin{cases} B(t - \tau)\pi(\tau), & t > \tau \\ \psi(\tau - t)\frac{\pi(\tau)}{\pi(\tau - t)}, & t < \tau \end{cases} \quad (25)$$

where

$$\pi(\tau) = e^{-\mu\tau} e^{-\int_0^h \sigma(s) ds}. \quad (26)$$

Equation (26) denotes the survival probability; i.e., the probability for an individual to survive to age τ ; Thus it must be $\pi(\tau) \geq 0$; and moreover, the function

$$K(\tau) = \psi(\tau)\pi(\tau), \quad \tau \in [0, T] \quad (27)$$

is an infectious function of newly infected individuals that synthesizes the dynamics of quarantine with the treatment population [31]. It is related to the following parameter.

$$\Gamma = \int_0^T \psi(\tau)\pi(\tau) d\tau. \quad (28)$$

Equation (28) represents the total infection rate, which is the number of newly infected people that an infected individual is expected to infect during the infection period. In essence, the total infection rate for Ebola virus disease translates the biological process of viral transmission within a population into a measurable public health metric of accumulated risk over time. Next, we consider the expected life span of an infected individual given by the following equation.

$$\bar{\pi} = \int_0^T \pi(\tau) d\tau. \quad (29)$$

Equation (29) provides the mean value of the life of an infected individual. Biologically, the expected life span of an infected individual describes the duration of time for the symptomatic phase of the virus within a host rather than the virus survival time outside the host's overall life expectancy. Empirical evidence shows that Ebola outbreaks impact population life expectancy, and for policy making, the statistical indicator helps in public health analysis and planning rather than clinical guidelines. Perhaps, equation (29) is better understood that $(\mu + \sigma(\tau))\pi(\tau)d\tau$ is the probability for an infected individual to survive to age τ and then die in $[\tau, \tau + d\tau]$; thus, we have

$$\bar{\pi} = \int_0^T \tau(\mu + \sigma(\tau))\pi(\tau) d\tau = - \int_0^T \tau \frac{d}{d\tau} \pi(\tau) d\tau - \tau\pi(\tau) \Big|_0^T + \int_0^T \pi(\tau) d\tau = \int_0^T \pi(\tau) d\tau. \quad (30)$$

We list some of the assumptions that equations (6), (7), and (13) are supposed to fulfill in order to show that (6), (7), and (13) depend continuously on the initial age of infection and are biologically significant or mathematically well-posed.

$$\psi(\cdot) \text{ is non-negative and belongs to } L^T(0, \tau) \quad (31)$$

$$\sigma(\cdot) \text{ is non-negative and belongs to } L^1_{\text{loc}}([0, \tau]) \quad (32)$$

$$\int_0^\tau (\mu + \sigma(s)) ds = T, \quad (33)$$

$$\psi \in L^1(0, \tau), \quad \psi(\tau) \geq 0 \quad \text{a.e in } [0, \tau] \quad (34)$$

equation (7) is now written in the form

$$B(t) = \int_0^T (\beta(\tau)q(t, \tau) d\tau + C(t)). \quad (35)$$

Thus, from the model assumptions, we assume that

$$Q(t) = I(t) \quad (36)$$

where as,

$$\beta(\tau) = \left(\frac{\alpha_1 S(t)}{N} + \varphi - \omega \right) \quad (37)$$

$$C(t) = \frac{\alpha_1(1-\theta)GS}{N} + (1-\phi)P \quad (38)$$

and equation (37) is also non-negative and belongs to $L^T(0, \tau)$, and since (38) is an arbitrary equation, we set

$$C(t) = \frac{\alpha_2(1-\theta)GS}{N} + (1-\phi)P = 0 \quad (39)$$

and the condition is sufficient. Suppose now that

$$B(t) = \int_0^T (\beta(\tau)q(t, \tau) d\tau). \quad (40)$$

Substituting (25) into (40) we have for $t \leq \tau$ that is

$$B(t) = \int_0^T \beta(\tau)B(t-\tau)\pi(\tau)d\tau + \int_0^T \beta(\tau)\frac{\pi(\tau)}{\pi(t-\tau)}\psi(t-\tau)d\tau \quad (41)$$

and for $t > \tau$ we have

$$B(t) = \int_0^T \beta(\tau)B(t-\tau)\pi(\tau) d\tau. \quad (42)$$

Thus, $B(t)$ satisfies the following Volterra integral equation of the second kind i.e.

$$B(t) = F(t) + \int_0^t K(t-s)B(s) ds \quad (43)$$

and

$$F(t) = \int_t^T \beta(\tau)\frac{\pi(\tau)}{\pi(\tau-t)}\psi(\tau-t) d\tau = \int_0^T \beta(\tau+t)\frac{\pi(\tau+t)}{\pi(\tau)}\psi(\tau) d\tau \quad (44)$$

$$K(t) = \beta(t)\pi(t) \quad (45)$$

where $t \geq 0$, and the functions β, π and ψ are extended by zero outside the interval $[0, T]$. Thus, equation (43) is known as the renewal equation and also as the Lotka equation; we observe that the Kernel $K(t)$ is an infectious function of newly infected individuals defined in (27) ([32]). Now, equation (43) is equivalent to (6), (7), and (13); actually, (43) is the main tool to investigate the existence of (6), (7), and (13). The following Lemma states some properties of (43) on the basis of the assumptions of (31) to (34).

Lemma 3.1. *Let (31) to (34) be satisfied, then*

$$K(t) \geq 0 \quad \text{a.e.,} \quad K(t) = 0 \quad \text{for } t > \tau, \quad K \in L^1(\mathfrak{R}_+) \cap L^T(\mathfrak{R}_+), \quad (46)$$

$$F(t) \geq 0, \quad K(t) = 0 \quad \text{for } t > \tau, \quad F \in C(\mathbb{R}_+), \quad (47)$$

Moreover, if

$$\psi \in W^{1,1}(0, \tau), \quad \text{and } (\mu + \sigma(\cdot))\psi(\cdot) \in L^1(0, \tau), \quad \text{then } F \in W^{1,T}(\mathfrak{R}_+) \quad [33] \quad (48)$$

Proof. Equation (46) and the first part of (47) are obvious. To prove that, $F \in C(\mathfrak{R}_+)$, we take $t_0 \geq 0$ then

$$\begin{aligned} F(t) &= \int_t^T \beta(\tau) \frac{\pi(\tau)}{\pi(\tau-t)} (\psi(\tau-t) - \psi(\tau-t_0)) d\tau \\ &\quad + \int_t^T \beta(\tau) \frac{\pi(\tau)}{\pi(\tau-t)} \psi(\tau-t_0) d\tau, \quad \psi \in L^1(\mathfrak{R}) \end{aligned} \quad (49)$$

$$\left| \int_t^T \beta(\tau) \frac{\pi(\tau)}{\pi(\tau-t)} (\psi(\tau-t) - \psi(\tau-t_0)) d\tau \right| \leq |\beta|_{L^T} \int_0^T |\psi(\tau-t) - \psi(\tau-t_0)| d\tau \rightarrow 0. \quad (50)$$

as $t \rightarrow t_0$, and equation (44) yields

$$\lim_{t \rightarrow t_0} F(t) = \int_{t_0}^T \beta(\tau) \frac{\pi(\tau)}{\pi(\tau-t_0)} \psi(\tau-t_0) d\tau = F(t_0). \quad (51)$$

Similarly, equation (48) implies $F \in W^{1,T}(\mathfrak{R}_+)$. This completes the proof of Lemma 3.1. $\square$

In the next theorem, we establish the existence and uniqueness of solutions to equations (6), (7), and (13) through (43). To proceed, we analyze equations (6), (7), and (13) by considering (43) to (45). Since the function F is continuous, for the purpose of this section it is sufficient to study (43) within the class of continuous functions. Thus, we introduce the following definition of a solution.

Definition 3.2. *A solution to (43) is a function $B \in C(\mathfrak{R}_+, \mathfrak{R})$ satisfying (43). First, we have the following theorem, which is standard in the theory of Volterra equations. We provide a proof here for the sake of completeness.*

Theorem 3.3. *Let (31) to (34) be satisfied, then equations (43) to (45) has a unique solution $B \in C(\mathfrak{R}_+)$ such that $B(t) \geq 0$ for all t . In addition, if ψ satisfies (48), then $B \in W_{loc}^{1,T}(\mathbb{R}_+)$ and*

$$B'(t) = F'(t) + K(t)B(0) + \int_0^t K(t-s)B'(s) ds. \quad (52)$$

Proof. First, we assume that

$$|K|_{L^1(\mathbb{R}_+)} = \int_0^T K(s) ds < 1. \quad (53)$$

The solution of (43) is obtained by standard Picard iterations, for $t \geq 0$ therefore

$$B^{k+1}(t) = F(t) + \int_0^t K(s)B^k(t-s) ds \quad (54)$$

and initialized by $B^0(t) = 0$ on $\mathfrak{R}_+$. Thus

$$\begin{cases} B^0(t) = F(t), \\ B^{k+1}(t) = F(t) + \int_0^t K(t-s)B^k(s) ds. \end{cases} \quad (55)$$

If we take any $T > 0$, then by (46) and (47) we obtain $B^k \in ([0, T])$, $B^k(T) \geq 0$; however

$$|B^{k+1}(t) - B^k(t)| \leq \int_0^t K(t-s) |B^k(s) - B^{k-1}(s)| ds \quad (56)$$

and

$$|B^{k+1} - B^k|_{C([0, T])} \leq |K|_{L^1(\mathbb{R}_+)} |B^k - B^{k-1}|_{C([0, T])}. \quad (57)$$

This implies that by (53), the sequence $B^k(t)$ converges uniformly on $[0, T]$, to a solution $B(t)$ of (43), such that $B \in C([0, T])$ and $B(t) \geq 0$. Concerning the uniqueness of this solution, we set $B(t)$ and $\bar{B}(t)$ to be the two solutions of (43), so that from (56) we obtain the following

$$|B - \bar{B}|_{C([0, T])} \leq |K|_{L^1(\mathbb{R}_+)} |B - \bar{B}|_{C([0, T])} \quad (58)$$

which yields (53), $B(t) = \bar{B}(t)$. In addition, if ψ satisfies (48), then, by Lemma 3.1 and (55), we have $B^k \in W^{1,T}(\mathfrak{R}_+)$ and setting

$$V^k(t) = \frac{d}{dt} B^k(t) \quad \text{a.e., we have } V^k \in L^T(\mathbb{R}_+) \quad \text{and,} \quad (59)$$

$$V^{k+1}(t) = F'(t) + K(t)F(0) + \int_0^t K(t-s)V^k(s)ds \quad (60)$$

which yields

$$|V^{k+1} - V^k|_{L^T(\mathbb{R}_+)} \leq |K|_{L^T(\mathbb{R}_+)} |V^k - V^{k-1}|_{L^T(\mathbb{R}_+)}. \quad (61)$$

From equation (53), the sequence V^k converges in $L^T(\mathfrak{R}_+)$ to $V(t) = \frac{d}{dt}B(t)$, thus, (52) follows from (60). In addition to equation (53), and in order to make our argument valid, we take α such that

$$\int_0^T e^{-\alpha t} K(t) dt < 1. \quad (62)$$

Setting $\bar{B} = e^{-\alpha t} B(t)$, $\bar{F} = e^{-\alpha t} F(t)$, $\bar{K} = e^{-\alpha t} K(t)$, equation (43) is transformed into an equivalent equation (63) as follows

$$\bar{B}(t) = \bar{F}(t) + \int_0^t \bar{K}(t-s) \bar{B}(s) ds \quad (63)$$

and since (63) satisfies (53), this is similar to equation (61). The next theorem allows us to state results for equations (6), (7) and (13) through (25). $\square$

Theorem 3.4. *Let (31) to (34) and (48) be satisfied. We also assume that*

$$\psi(0) = \int_0^T \beta(\tau) \psi(\tau) d\tau \quad (64)$$

and let $q(t, \tau)$ define by (25) such that $B(t)$ is the solution of (43) to (45). Then

$$q \in C([0, \tau] \times \mathbb{R}_+), \quad q(t, \tau) \geq 0, \quad (\mu + \sigma(\cdot))q(t, \cdot) \in L^1(0, \tau), \quad \forall t > 0. \quad (65)$$

$$\frac{\partial q}{\partial t}(t, \tau) \quad \text{and} \quad \frac{\partial q}{\partial \tau}(t, \tau) \quad \text{exist a.e in} \quad [0, \tau] \times [0, T] \quad (66)$$

and (6), (7), and (13) is satisfied. [34] surmised that $q(t, \tau)$ is the only solution of (65) and (66).

Proof. The Proof of (65) to (66) is quite straightforward from the properties of $B(t)$ which is stated in Theorem (3.3). We only note that the following inequality concerns the last part of (65).

$$\begin{aligned} \int_0^t (\mu + \sigma(\tau)) q(t, \tau) d\tau &= \int_0^{t \wedge \tau} (\mu + \sigma(\tau)) B(t - \tau) \pi(\tau) d\tau \\ &\quad + \int_{t \wedge \tau}^t (\mu + \sigma(\tau)) \psi(t \wedge \tau) \frac{\pi(\tau)}{\pi(\tau - t)} d\tau \\ &\leq \max_{s \in [0, t]} |B(s)| \int_0^{t \wedge \tau} (\mu + \sigma(\tau)) \pi(\tau) d\tau \\ &\quad + e^{\int_0^t (\mu + \sigma(s)) ds} |\psi|_{C([0, t])} \int_{t \wedge \tau}^t (\mu + \sigma(\tau)) \pi(\tau) ds \\ &\leq \max_{s \in [0, t]} |B(s)| + e^{\int_0^t (\mu + \sigma(s)) ds} |\psi|_{C([0, \tau])}. \end{aligned} \quad (67)$$

Where $t \wedge \tau$ is the average number of secondary infections produced by a primary case at a specific age of infection. This implies that infectivity is not constant but depends on the time elapsed since an individual was infected, which is the relationship often described by a density function.

Equation (67) provides the fact that (64) implies the continuity of $(q(t, \tau))$ through the line $t = \tau$; so that

$$B(0) = \int_0^T \beta(\tau) \psi(\tau) d\tau = \psi(0). \quad (68)$$

□

We note that the solution of (6), (7), and (13) must be in the form of (25) with $B(t)$ satisfying (43) to (45) such that equations (25) and (48) are enough to provide a classical solution in the next following theorem.

Theorem 3.5. *Let (31) to (34) be satisfied, then $q(t, \tau)$, defined by (25), has the following properties*

$$q(t, \cdot) \in C([0, T]; L^1(0, \tau)), \quad q(t, \tau) \geq 0, \quad a.e \text{ in } [0, \tau] \times \mathfrak{R}_+. \quad (69)$$

$$|q(t, \cdot)|_{L^1} \leq e^{t|\beta|_{L^T}} |\psi|_{L^1}, \quad (70)$$

$$q(t, \tau) \text{ is continuous for } \tau < t \text{ and satisfies (7) for } t > 0 \quad (71)$$

$$\lim_{h \rightarrow 0} \frac{1}{h} [q(t+h, \tau+h) - q(t, \tau)] = -(\mu + \sigma(\tau))q(t, \tau) \quad a.e \text{ in } [0, \tau] \times \mathfrak{R}_+. \quad (72)$$

Proof. Let us prove (70), first from (44) we obtain

$$F(t) \leq |\beta|_{L^T} |\psi|_{L^1}, \quad K(t) \leq |\beta|_{L^T}; \quad (73)$$

then, from (43)

$$B(t) \leq |\beta|_{L^T} |\psi|_{L^1} + |\beta|_{L^T} \int_0^t B(s) ds. \quad (74)$$

Thus, by classical Gronwall's inequality ([35]),

$$B(t) \leq |\beta|_{L^T} e^{t|\beta|_{L^T}} |\psi|_{L^1} \quad (75)$$

from this estimate, equation (25) yields

$$\begin{aligned} |q(t, \cdot)|_{L^1} &= \int_0^t B(t-\tau) \pi(\tau) d\tau + \int_0^t \frac{\pi(\tau+t)}{\pi(\tau)} \psi(\tau) d\tau \\ &\leq \left(|\beta|_{L^T} \int_0^t e^{(t-\tau)|\beta|_{L^T}} d\tau + 1 \right) |\psi|_{L^1} = e^{t|\beta|_{L^T}} |\psi|_{L^1}. \end{aligned} \quad (76)$$

Equation (71) follows easily from (69). Now, for a given $\psi \in L^1(0, \tau)$, let ψ^n be a sequence such that ψ^n satisfies (48) and (64),

$$\lim_{n \rightarrow T} |\psi^n - \psi|_{L^1} = 0 \quad (77)$$

and let q^n be the solution of (22), (23), and (29) corresponding to ψ^n . Thus $q^n \in C([0, T]; L^1(0, \tau))$ and, by (68) and linearity, we have

$$|q^n(t, \cdot) - q(t, \cdot)|_{L^1} \leq e^{t|\beta|_{L^T}} |\psi^n - \psi|_{L^1}. \quad (78)$$

So that q is the limit of the sequence q^n in the space $C([0, T]; L^1(0, \tau))$, i.e., (67) is true. Finally, equations (69) and (70) are straightforward, and that completes the proof.

In the next theorem, we prove the existence and uniqueness of a solution for equations (1) to (5), as in [36].

Theorem 3.6. *The solution of (1) to (5) exists and is unique for*

$$|t-t_0| \leq a, \quad |S-S_0| \leq b, \quad |P-P_0| \leq c, \quad |I-I_0| \leq d, \quad |G-G_0| \leq e, \quad \& \quad |R-R_0| \leq f.$$

Proof.

$$f_1(S, P, I, G, R, Q) = \Lambda - \frac{\alpha_1 IS}{N} - \frac{\alpha_2(1-\theta)GS}{N} + \phi P - (\mu + \gamma)S, \quad S(0) = S_0 \quad (79)$$

$$f_2(S, P, I, G, R, Q) = \gamma S - ((\mu + \phi)(1 - \phi))P, \quad P(0) = P_0 \quad (80)$$

$$f_3(S, P, I, G, R, Q) = \frac{\alpha_1 IS}{N} + \frac{\alpha_2(1-\theta)GS}{N} - (\mu + \varphi + \delta_1)I, \quad I(0) = I_0 \quad (81)$$

$$f_4(S, P, I, G, R, Q) = (\mu + \delta_1)I + (\mu + \delta_2)Q - \theta G, \quad G(0) = G_0 \quad (82)$$

$$f_5(S, P, I, G, R, Q) = \omega Q - \mu R, \quad R(0) = R_0 \quad (83)$$

Then,

$$\begin{aligned} \frac{\partial f_1}{\partial S} &= - \left(\frac{\alpha_1 I + \alpha_2(1-\theta)G}{N} \right) - (\mu + \gamma), \\ \frac{\partial f_1}{\partial P} &= \phi, \quad \frac{\partial f_1}{\partial I} = - \frac{\alpha_1 S}{N}, \end{aligned} \quad (84)$$

$$\frac{\partial f_1}{\partial G} = - \frac{\alpha_2(1-\theta)S}{N}, \quad \frac{\partial f_1}{\partial R} = 0$$

$$\frac{\partial f_2}{\partial S} = \gamma, \quad \frac{\partial f_2}{\partial P} = -(\mu + \phi)(1 - \phi), \quad \frac{\partial f_2}{\partial I} = 0, \quad \frac{\partial f_2}{\partial G} = 0, \quad \frac{\partial f_2}{\partial R} = 0 \quad (85)$$

$$\frac{\partial f_3}{\partial S} = \frac{\alpha_1 I + \alpha_2(1-\theta)G}{N}, \quad \frac{\partial f_3}{\partial P} = 0, \quad \frac{\partial f_3}{\partial I} = \left(\frac{\alpha_1 S}{N} \right) - (\mu + \varphi + \delta_1), \quad (86)$$

$$\frac{\partial f_3}{\partial G} = \frac{\alpha_2(1-\theta)S}{N}, \quad \frac{\partial f_3}{\partial R} = 0$$

$$\frac{\partial f_4}{\partial S} = 0, \quad \frac{\partial f_4}{\partial P} = 0, \quad \frac{\partial f_4}{\partial I} = \mu + \delta_1, \quad \frac{\partial f_4}{\partial G} = -\theta, \quad \frac{\partial f_4}{\partial R} = 0 \quad (87)$$

$$\frac{\partial f_5}{\partial S} = 0, \quad \frac{\partial f_5}{\partial P} = 0, \quad \frac{\partial f_5}{\partial I} = 0, \quad \frac{\partial f_5}{\partial G} = 0, \quad \frac{\partial f_5}{\partial R} = -\mu \quad (88)$$

Taking the absolute values of equations (84) to (88) as in [37], we obtain

$$\left| \frac{\partial f_1}{\partial S} \right| = \left| - \left(\frac{\alpha_1 I + \alpha_2(1-\theta)G}{N} \right) - (\mu + \gamma) \right|, \quad \left| \frac{\partial f_1}{\partial P} \right| = |\phi|, \quad \left| \frac{\partial f_1}{\partial I} \right| = \left| - \frac{\alpha_1 S}{N} \right| \quad (89)$$

$$\left| \frac{\partial f_1}{\partial G} \right| = \left| - \frac{\alpha_2(1-\theta)S}{N} \right|, \quad \left| \frac{\partial f_1}{\partial R} \right| = |0|$$

$$\left| \frac{\partial f_2}{\partial S} \right| = |\gamma|, \quad \left| \frac{\partial f_2}{\partial P} \right| = |-(\mu + \phi)(1 - \phi)|, \quad \left| \frac{\partial f_2}{\partial I} \right| = |0|, \quad \left| \frac{\partial f_2}{\partial G} \right| = |0|, \quad \left| \frac{\partial f_2}{\partial R} \right| = |0| \quad (90)$$

$$\begin{aligned} \left| \frac{\partial f_3}{\partial S} \right| &= \left| \left(\frac{\alpha_1 I + \alpha_2(1-\theta)G}{N} \right) \right|, \quad \left| \frac{\partial f_3}{\partial P} \right| = |0|, \quad \left| \frac{\partial f_3}{\partial I} \right| = \left| \frac{\alpha_1 S}{N} - (\mu + \varphi + \delta_1) \right| \\ \left| \frac{\partial f_3}{\partial G} \right| &= \left| \frac{\alpha_2(1-\theta)S}{N} \right|, \quad \left| \frac{\partial f_3}{\partial R} \right| = |0| \end{aligned} \quad (91)$$

$$\left| \frac{\partial f_4}{\partial S} \right| = |0|, \quad \left| \frac{\partial f_4}{\partial P} \right| = |0|, \quad \left| \frac{\partial f_4}{\partial I} \right| = |\mu + \delta_I|, \quad \left| \frac{\partial f_4}{\partial G} \right| = |-\theta|, \quad \left| \frac{\partial f_4}{\partial R} \right| = |0| \quad (92)$$

$$\left| \frac{\partial f_5}{\partial S} \right| = |0|, \quad \left| \frac{\partial f_5}{\partial P} \right| = |0|, \quad \left| \frac{\partial f_5}{\partial I} \right| = |0|, \quad \left| \frac{\partial f_5}{\partial G} \right| = |0|, \quad \left| \frac{\partial f_5}{\partial R} \right| = |-\mu| \quad (93)$$

This implies that $\left| \frac{\partial f_i}{\partial x_j} \right|$, $i = 1, 2, 3, 4, 5$ and $j = S, P, I, G, R$. are bounded. Hence, there exists a unique solution of the model equations (1) to (5). $\square$

3.1. Existence of Equilibria.

Equations (1) to (13) are re-written as follows

$$\frac{dS}{dt} = \Lambda - \frac{\alpha_1 S}{N} \int_0^T q(t, \tau) d\tau - \frac{\alpha_2 k_1 GS}{N} + \phi P - k_2 S \quad (94)$$

$$\frac{dP}{dt} = \gamma S - k_3 P \quad (95)$$

$$\frac{dI}{dt} = \frac{\alpha_1 S}{N} \int_0^T q(t, \tau) d\tau + \frac{\alpha_2 k_1 GS}{N} - k_4 \int_0^T q(t, \tau) d\tau \quad (96)$$

$$\frac{dG}{dt} = (k_5 + k_6) \int_0^T q(t, \tau) d\tau - \theta G \quad (97)$$

$$\frac{dR}{dt} = \omega \int_0^T q(t, \tau) d\tau - \mu R \quad (98)$$

$$\frac{\partial q(t, \tau)}{\partial t} + \frac{\partial q(t, \tau)}{\partial \tau} + (\mu + \sigma(\tau)) q(t, \tau) = 0. \quad (99)$$

The boundary conditions are given as

$$q(t, 0) = B(t) = \frac{\alpha_1 S}{N} \int_0^T q(t, \tau) d\tau + \frac{\alpha_2 k_1 GS}{N} + k_7 P + \varphi \int_0^T q(t, \tau) d\tau \quad (100)$$

where

$$k_1 = (1 - \theta), \quad k_2 = (\mu + \gamma), \quad k_3 = (\mu + \phi)(1 - \phi), \quad k_4 = (\mu + \varphi + \delta_1), \quad (101)$$

$$k_5 = (\mu + \delta_1), \quad k_6 = (\mu + \delta_2), \quad k_7 = (1 - \phi). \quad (102)$$

We assume that $Q(t) = I(t)$, but from equations (10) and (36), we have $Q(t) = \int_0^T q(t, \tau) d\tau$

$$\implies Q(t) = I(t) = \int_0^T q(t, \tau) d\tau. \quad (103)$$

The initial conditions are

$$\begin{aligned} N(0) &= N_0, & S(0) &= S_0, & P(0) &= P_0, \\ I(0) &= I_0, & G(0) &= G_0, \\ R(0) &= R_0, & q(0, \tau) &= \psi(\tau). \end{aligned} \quad (104)$$

□

3.2. Disease-free equilibrium state (E^0).

The disease-free equilibrium state describes the total absence of EVD in the studied population. At this state all the infected classes will be zero and the population will be made up of susceptible and quarantine under observation only.

Lemma 3.7. *A disease-free equilibrium state of the model equations (94) to (100) exists as follows.*

$$\begin{pmatrix} S \\ P \\ I \\ G \\ R \\ Q \end{pmatrix} = \begin{pmatrix} \frac{(\mu + \phi)(1 - \phi)\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \\ \frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = E^0. \quad (105)$$

Proof. : According to [38], equilibria are the points where the variables do not change with time. Thus

$$\frac{dS}{dt} = \frac{dP}{dt} = \frac{dI}{dt} = \frac{dG}{dt} = \frac{dR}{dt} = 0. \quad (106)$$

At the disease-free equilibrium state, we set

$$S(t) = x^0; \quad P(t) = u^0; \quad I(t) = y^0; \quad G(t) = g^0; \quad R(t) = r^0; \quad (107)$$

$$Q(t) = z^0; \quad N(t) = n^0; \quad q(t, \tau) = q(\tau);$$

$$n^0 = x^0 + u^0 + y^0 + g^0 + r^0 + z^0, \quad n = x + u + y + g + r + z; \quad (108)$$

If $I(t) = y^0$, $Q(t) = z^0$, and $I(t) = Q(t)$, from equations (103) and (107), this implies that

$$z^0 = y^0 = \int_0^T q(\tau) d\tau. \quad (109)$$

Equations (94) to (100) yields

$$\Lambda - \frac{\alpha_1 x^0 z^0}{n^0} - \frac{\alpha_1 k_1 g^0 x^0}{n^0} + \phi u^0 - k_2 x^0 = 0. \quad (110)$$

$$\gamma x^0 - k_3 u^0 = 0 \quad (111)$$

$$\frac{\alpha_1 x^0 z^0}{n^0} + \frac{\alpha_2 k_1 g^0 x^0}{n^0} - k_4 z^0 = 0 \quad (112)$$

$$(k_5 + k_6)z^0 - \theta g^0 = 0 \quad (113)$$

$$\omega z^0 - \mu r^0 = 0 \quad (114)$$

$$\frac{dq(\tau)}{d\tau} + (\mu + \sigma(\tau))q(\tau) = 0 \quad (115)$$

$$q(0) = B_0 = \frac{\alpha_1 x^0 z^0}{n^0} + \frac{\alpha_2 k_1 x^0 g^0}{n^0} + \varphi z^0 + k_7 u^0. \quad (116)$$

Solving (115) yields

$$q(\tau) = q(0) e^{-\int_0^\tau (\mu + \sigma(s)) ds}. \quad (117)$$

Let

$$\pi(\tau) = e^{-\int_0^\tau (\mu + \sigma(s)) ds}. \quad (118)$$

Using (118), we re-write (117) in the following form.

$$q(\tau) = q(0) \pi(\tau). \quad (119)$$

Integrating both sides of (119) gives

$$\int_0^T q(\tau) d\tau = q(0) \int_0^T \pi(\tau) d\tau. \quad (120)$$

Equation (120) is re-written in the form

$$z^0 = q(0) \bar{\pi} \quad (121)$$

where

$$\bar{\pi} = \int_0^T \pi(\tau) d\tau. \quad (122)$$

Using (116) in (121) gives

$$\frac{\alpha_1 x^0 z^0}{n^0} \bar{\pi} + \frac{\alpha_2 k_1 x^0 g^0}{n^0} \bar{\pi} + \phi z^0 \bar{\pi} + k_7 u^0 \bar{\pi} = z^0. \quad (123)$$

Factorizing the LHS of (123) yields

$$\left[\frac{\alpha_1 x^0 z^0}{n^0} + \frac{\alpha_2 k_1 x^0 g^0}{n^0} + \phi z^0 + k_7 u^0 \right] \bar{\pi} = z^0. \quad (124)$$

From (124), either

$$\bar{\pi} = z^0 \quad (125)$$

or

$$\left[\frac{\alpha_1 x^0 z^0}{n^0} + \frac{\alpha_2 k_1 x^0 g^0}{n^0} + \phi z^0 + k_7 u^0 \right] = z^0, \quad (126)$$

if $\bar{\pi} = z^0$, then equation (121) yields

$$z^0 = 0, \quad (127)$$

$$q(0) = 1. \quad (128)$$

And so (122) to (126) equal zero. Substituting (127) into (114) yields

$$r^0 = 0. \quad (129)$$

Substituting (107) into (36) yields

$$z^0 = y^0. \quad (130)$$

This implies

$$y^0 = 0. \quad (131)$$

Solving equation (113) for g^0 yields

$$g^0 = \frac{(k_5 + k_6)z^0}{\theta}. \quad (132)$$

Substituting (127) into (132), we obtain

$$g^0 = 0. \quad (133)$$

Solving (111) for u^0 yields

$$u^0 = \frac{\gamma x^0}{k_3}. \quad (134)$$

Substituting (131) and (133) into (110) yields

$$\Lambda + \phi u^0 - k_2 x^0 = 0. \quad (135)$$

Substituting (134) into (135) and isolating x^0 yields

$$x^0 = \left(\frac{(\mu + \phi)(1 - \phi)\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \right). \quad (136)$$

Substituting (136) into (134), we obtain

$$u^0 = \left(\frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \right). \quad (137)$$

Substituting (127), (129), (131), (133), (136), and (137) into (109) and simplifying yields

$$n^0 = \left(\frac{\Lambda((\mu + \phi)(1 - \phi) + \gamma)}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \right). \quad (138)$$

This completes the proof of Lemma 3.7. $\square$

3.3. Effective Reproduction Number ($\mathcal{R}_{eff}$).

The effective reproduction number, denoted by $\mathcal{R}_{eff}$, represents the expected number of secondary cases produced in a completely susceptible population by a typical infective individual after intervention. For infection-age and age-structured population models, the effective reproduction number or basic reproduction number is often expressed as the sum of the infectivity contributions from each infected compartment [36], [39]. If $\mathcal{R}_{eff} < 1$, according to [26], this indicates that, on average, an infected individual produces less than one new infection during the infectious period, and hence the infection can be eradicated. However, if $\mathcal{R}_{eff} > 1$, then each infected individual produces, on average, more than one new infection, then the disease spreads in the population. Thus, $\mathcal{R}_{eff}$ of the model is as given

$$\mathcal{R}_{eff} = \left[\frac{(1-\phi)\gamma\Lambda}{(\mu+\gamma)(\mu+\phi)(1-\phi)-\phi\gamma} + \left(\frac{\alpha_1(\mu+\phi)(1-\phi)}{(\mu+\phi)(1-\phi)+\gamma} + \varphi \right) \int_0^T \pi(\tau) d\tau \right. \\ \left. + \left(\frac{\alpha_2(1-\theta)(\mu+\phi)(1-\phi)}{(\mu+\phi)(1-\phi)+\gamma} \right) \left(\frac{(\mu+\delta_1)+(\mu+\delta_2)}{\theta} \right) \int_0^T \pi(\tau) d\tau \right]. \quad (139)$$

The threshold value of (139) is of the form

$$\mathcal{R}_P + \mathcal{R}_Q + \mathcal{R}_G \quad (140)$$

where

$$R_P = (1-\phi)u^0. \quad (141)$$

Equation (141) describes the confirmed number of infected individuals after observation. The term $1-\phi$ is the probability of quarantine under observation members who are infected and capable of causing secondary infections. Term $u^0 = \left(\frac{\gamma\Lambda}{(\mu+\gamma)(\mu+\phi)(1-\phi)-\phi\gamma} \right)$ represents the total number of quarantine under observation.

$$\mathcal{R}_Q = \left(\frac{\alpha_1 x^0}{n^0} + \varphi \right) \int_0^T \pi(\tau) d\tau. \quad (142)$$

Equation (142) represents the number of infected individuals who are quarantine with treatment. Term $x^0 = \left(\frac{(\mu+\phi)(1-\phi)\Lambda}{(\mu+\gamma)(\mu+\phi)(1-\phi)-\phi\gamma} \right)$ represents the total number of susceptible individuals in the absence of EVD, and $n^0 = \left(\frac{\Lambda((\mu+\phi)(1-\phi)+\gamma)}{(\mu+\gamma)(\mu+\phi)(1-\phi)-\phi\gamma} \right)$ represents the total population in the absence of infection. The $\pi(\tau) = e^{-\int_0^\tau (\mu+\sigma(s)) ds}$ is the survival probability as a function of infection age τ in the compartment of quarantine with treatment.

$$\mathcal{R}_G = \frac{\alpha_2(1-\theta)x^0 g^0}{n^0} \quad (143)$$

Equation (143) represents the number of deceased individuals caused by EVD who are generated from $I(t)$ and $Q(t)$, respectively. The parameter $\alpha_2(1-\theta)$ represents effective contact rate between the susceptible individuals and dead bodies which is capable of spreading EVD through contaminated burial practices. Part $g^0 = \left(\frac{(\mu+\delta_1)+(\mu+\delta_2)}{\theta} \right) \int_0^T \pi(\tau) d\tau$ represents the total number of dead bodies both from natural and disease induced deaths of $I(t)$ and $Q(t)$ only. The effective reproduction number is very sensitive for assessing the effectiveness of control measures in real time and is influenced by factors of safe burial and multiple quarantine stages. Safe burial of Ebola virus victims is a central component of the EVD response because corpses remain highly contagious, and traditional funeral practices that involved direct contact with the dead bodies can lead to major reemergence of infection; implementing safe burial practices (PBP) is essential to break chains of transmission and control the epidemic. The multiple quarantine stages approach

provides flexibility and robustness to the public health responses, making it one of the effective control strategies for curtailing Ebola virus outbreaks, especially in challenging environments.

3.4. Local Stability of the Disease-Free Equilibrium (DFE) State.

Theorem 3.8. *The disease-free equilibrium state is locally asymptotically stable if $\mathcal{R}_{eff} < 1$, and unstable if $\mathcal{R}_{eff} > 1$.*

Proof. The local stability of the disease-free equilibrium state given by (105) is considered. Let

$$S(t) = \left(\frac{(\mu + \phi)(1 - \phi)\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \right) + x(t) \quad (144)$$

$$P(t) = \left(\frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \right) + u(t) \quad (145)$$

$$I(t) = y(t) \quad (146)$$

$$G(t) = g(t) \quad (147)$$

$$R(t) = r(t) \quad (148)$$

$$q(t, \tau) = z(t, \tau) \quad (149)$$

and from (107) and (138), we have

$$N(t) = n^0 = \left(\frac{\Lambda((\mu + \phi)(1 - \phi) + \gamma)}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \right). \quad (150)$$

We linearized (94) to (100) using (144) to (150) about E^0 to obtain

$$\frac{dx(t)}{dt} = \left\{ \begin{array}{l} \Lambda - (\mu + \gamma) \left(\frac{(\mu + \phi)(1 - \phi)\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + x(t) \right) \\ -\alpha_2(1 - \theta)g(t) \left[\left(\frac{(\mu + \phi)(1 - \phi)\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + x(t) \right) \div \left(\frac{\Lambda((\mu + \phi)(1 - \phi) + \gamma)}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \right) \right] \\ +\phi \left(\frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + u(t) \right) \\ -\alpha_1 \left[\left(\frac{(\mu + \phi)(1 - \phi)\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \right) \div \left(\frac{\Lambda((\mu + \phi)(1 - \phi) + \gamma)}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \right) \right] \int_0^T q(t, \tau) d\tau \end{array} \right\}. \quad (151)$$

We simplified (151) to obtain

$$\frac{dx(t)}{dt} = \left\{ \begin{aligned} & [((\mu + \gamma) + g(t)\alpha_2(1 - \theta)(\mu + \gamma)(\mu + \phi)(1 - \phi))\Lambda(\mu + \phi)(1 - \phi) + \gamma]^{-1} x(t) \\ & + \left(\frac{\Lambda\phi\gamma}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + \frac{\alpha_2(1 - \theta)(\mu + \phi)(1 - \phi)g(t)}{(\mu + \phi)(1 - \phi) + \gamma} \right) \\ & - \phi \left(\frac{\Lambda\gamma}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + u(t) \right) \\ & + \alpha_1 \left(\frac{(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) \int_0^T z(t, \tau) d\tau \end{aligned} \right\}. \quad (152)$$

Substituting (144) and (145) into (95) yields

$$\begin{aligned} \frac{du(t)}{dt} = & -(\mu + \phi)(1 - \phi) \left(\frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + u(t) \right) \\ & + \gamma \left(\frac{(\mu + \phi)(1 - \phi)\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + x(t) \right). \end{aligned} \quad (153)$$

Expanding and simplifying (153) yields

$$\frac{du(t)}{dt} = -(\mu + \phi)(1 - \phi)u(t) + \gamma x(t). \quad (154)$$

Substituting (144), (147), and (150) into (96) yields

$$\frac{dy(t)}{dt} = - \left\{ \begin{aligned} & (\mu + \delta_1 + \varphi) \int_0^T q(t, \tau) d\tau \\ & - \alpha_1 \left[\left(\frac{(\mu + \phi)(1 - \phi)\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \right) \div \left(\frac{\Lambda((\mu + \phi)(1 - \phi) + \gamma)}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \right) \int_0^T q(t, \tau) d\tau \right] \\ & - \alpha_2(1 - \theta)g(t) \left[\left(\frac{\Lambda(\mu + \phi)(1 - \phi)}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + x(t) \right) \div \left(\frac{\Lambda((\mu + \phi)(1 - \phi) + \gamma)}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} \right) \right] \end{aligned} \right\}. \quad (155)$$

Expanding and simplifying (155) yields

$$\frac{dy(t)}{dt} = \left\{ \begin{aligned} & (\mu + \delta_1 + \phi) - \alpha_1 ((\mu + \phi)(1 - \phi)((\mu + \phi)(1 - \phi) + \gamma)^{-1}) y(t) \\ & - \alpha_2(1 - \theta)g(t) \left[\left(\frac{(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) + x(t) \left(\frac{(\mu + \gamma)(\mu + \phi)(1 - \phi) + \gamma\phi}{\Lambda((\mu + \phi)(1 - \phi) + \gamma)} \right) \right] \end{aligned} \right\}. \quad (156)$$

where

$$y(t) = \int_0^T q(t, \tau) d\tau. \quad (157)$$

From (97), (98), (99), and (100), we have the following equations,

$$\frac{dg(t)}{dt} = -\theta g(t) + (\mu + \delta_1) \int_0^T z(t, \tau) d\tau + (\mu + \delta_2) \int_0^T z(t, \tau) d\tau, \quad (158)$$

$$\frac{dr(t)}{dt} = -\mu r(t) + \omega \int_0^T z(t, \tau) d\tau, \quad (159)$$

$$\frac{\partial z(t, \tau)}{\partial t} + \frac{\partial z(t, \tau)}{\partial \tau} + (\mu + \sigma(\tau)) z(t, \tau) = 0, \quad (160)$$

and

$$z(t, 0) = \left\{ \begin{aligned} & \left[\varphi + \left(\frac{\alpha_1(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) \right] \int_0^T z(t, \tau) d\tau \\ & + \alpha_2(1 - \theta)g(t) \left[\left(\frac{(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) + x(t) \left(\frac{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma}{\Lambda((\mu + \phi)(1 - \phi) + \gamma)} \right) \right] \\ & + (1 - \phi) \left(\frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + u(t) \right) \end{aligned} \right\}. \quad (161)$$

To analyze the asymptotic behaviour of E^0 , the solutions were considered in the form of (162) as shown

$$x(t) = \bar{x}e^{\lambda t}, \quad u(t) = \bar{u}e^{\lambda t}, \quad y(t) = \bar{y}e^{\lambda t}, \quad g(t) = \bar{g}e^{\lambda t}, \quad r(t) = \bar{r}e^{\lambda t}, \quad z(t, \tau) = \bar{z}(\tau)e^{\lambda t} \quad (162)$$

Substituting (162) into equations (152), (154), (156), (158), (159), and (161), and simplifying respectively gives the following eigen-values

$$\begin{aligned} & (\mu + \gamma) - \bar{g}(\alpha_2(1 - \theta)((\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma)(\Lambda(\mu + \phi)(1 - \phi) + \gamma)^{-1} + \lambda)\bar{x} \\ & = \phi \left(\frac{\Lambda\gamma}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + \bar{u} \right) \\ & - \left(\frac{\Lambda\phi\gamma}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + \frac{\alpha_2(1 - \theta)(\mu + \phi)(1 - \phi)\bar{g}}{(\mu + \phi)(1 - \phi) + \gamma} \right) \\ & - \left(\left(\frac{\alpha_1(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) \int_0^T \bar{z}(\tau) d\tau \right) \end{aligned} \quad (163)$$

$$((\mu + \phi)(1 - \phi) + \lambda)\bar{u} = \gamma\bar{x} \quad (164)$$

$$\begin{aligned} & ((\mu + \delta_1 + \varphi) - \alpha_1((\mu + \phi)(1 - \phi)((\mu + \phi)(1 - \phi) + \gamma)^{-1} + \lambda)\bar{y} \\ & = \alpha_2(1 - \theta)\bar{g} \left[\frac{(\mu + \varphi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} + \bar{x} \left(\frac{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma}{\Lambda(\mu + \phi)(1 - \phi) + \gamma} \right) \right] \end{aligned} \quad (165)$$

$$(\theta + \lambda)\bar{g} = (\mu + \delta_1) \int_0^T \bar{z}(\tau) d\tau + (\mu + \delta_2) \int_0^T \bar{z}(\tau) d\tau \quad (166)$$

$$(\mu + \lambda)\bar{r} = \omega \int_0^T \bar{z}(\tau) d\tau \quad (167)$$

$$\frac{d\bar{z}(\tau)}{d\tau} = -(\lambda + \mu + \sigma(\tau))\bar{z}(\tau) \quad (168)$$

$$\bar{z}(0) = \left\{ \begin{aligned} & \left(\varphi + \left(\frac{\alpha_2(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) \right) \int_0^T \bar{z}(\tau) d\tau \\ & + \alpha_2(1 - \theta) \bar{g} \left[\frac{(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} + \bar{x} \left(\frac{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma}{\Lambda((\mu + \phi)(1 - \phi) + \gamma)} \right) \right] \\ & + (1 - \phi) \left(\frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + \bar{u} \right) \end{aligned} \right\}. \quad (169)$$

Solving (168) for $\bar{z}(\tau)$ yields

$$\bar{z}(\tau) = \bar{z}(0)e^{-(\lambda + \mu + \sigma(\tau))\tau}. \quad (170)$$

Further simplification of (170) yields

$$\bar{z}(\tau) = \bar{z}(0)\pi(\tau) \quad (171)$$

where $\pi(\tau)$ is defined as in (117). From equation (164), we have

$$\bar{x} = \frac{((\mu + \phi)(1 - \phi) + \lambda)}{\gamma}, \quad (172)$$

$$\bar{u} = \frac{\gamma}{((\mu + \phi)(1 - \phi) + \lambda)}, \quad (173)$$

and from equation (166),

$$\bar{g} = \left(\frac{(\mu + \delta_1) + (\mu + \delta_2)}{\theta + \lambda} \right) \int_0^T z(\tau) d\tau. \quad (174)$$

Substituting equations (172), (173), and (174) into (169) gives

$$\bar{z}(0) = \left\{ \begin{aligned} & \left[\left(\varphi + \frac{\alpha_1(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) \int_0^T \bar{z}(\tau) d\tau + \alpha_2(1 - \theta) \left(\frac{(\mu + \delta_1) + (\mu + \delta_2)}{\theta + \lambda} \right) \right] \\ & \times \left[\left(\frac{(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} + \frac{(\mu + \phi)(1 - \phi) + \lambda}{\gamma} \cdot \frac{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma}{\Lambda((\mu + \phi)(1 - \phi) + \gamma)} \right) \int_0^T \bar{z}(\tau) d\tau \right] \\ & + (1 - \phi) \left(\frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + \frac{\gamma}{(\mu + \phi)(1 - \phi) + \lambda} \right) \end{aligned} \right\} \quad (175)$$

Equation (175) is rewritten as follows.

$$\begin{aligned} \bar{z}(0) - & \left[\left[\varphi + \left(\frac{\alpha_1(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) + \alpha_2(1 - \theta) \left(\frac{(\mu + \delta_1) + (\mu + \delta_2)}{\theta + \lambda} \right) \right] \right. \\ & \left[\frac{(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} + \frac{(\mu + \phi)(1 - \phi) + \lambda}{\gamma} \cdot \frac{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma}{\Lambda[(\mu + \phi)(1 - \phi) + \gamma]} \right] \int_0^T \bar{z}(\tau) d\tau \\ & = (1 - \phi) \left[\frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + \frac{\gamma}{(\mu + \phi)(1 - \phi) + \lambda} \right] \end{aligned} \quad (176)$$

Multiplying the RHS of (176) by $\int_0^T \bar{z}(d\tau)$ and solving for $\bar{z}(0)$ yields

$$\bar{z}(0) = \left\{ \begin{aligned} & \left[\left(\varphi + \frac{\alpha_1(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) \int_0^T \bar{z}(\tau) d\tau + \alpha_2(1 - \theta) \left(\frac{(\mu + \delta_1) + (\mu + \delta_2)}{\theta + \lambda} \right) \right] \\ & \left[\left(\frac{(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) + \left(\frac{(\mu + \phi)(1 - \phi) + \lambda}{\gamma} \right) \left(\frac{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma}{\Lambda((\mu + \phi)(1 - \phi) + \gamma)} \right) \right] \int_0^T \bar{z}(\tau) d\tau \\ & + (1 - \phi) \left(\frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + \frac{\gamma}{(\mu + \phi)(1 - \phi) + \lambda} \right) \int_0^T \bar{z}(\tau) d\tau \end{aligned} \right\} \quad (177)$$

Substituting (171) into (177) yields

$$\bar{z}(0) = \left\{ \begin{aligned} & \left[\left(\varphi + \frac{\alpha_1(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) \int_0^T \bar{z}(0)e^{-z\tau}\pi(\tau) d\tau + \alpha_2(1 - \theta) \left(\frac{(\mu + \delta_1) + (\mu + \delta_2)}{\theta + \lambda} \right) \right] \\ & \left[\left(\frac{(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) + \left(\frac{(\mu + \phi)(1 - \phi) + \lambda}{\gamma} \right) \left(\frac{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma}{\Lambda((\mu + \phi)(1 - \phi) + \gamma)} \right) \right] \\ & \int_0^T \bar{z}(0)e^{-z\tau}\pi(\tau) d\tau \\ & + (1 - \phi) \left(\frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + \frac{\gamma}{(\mu + \phi)(1 - \phi) + \lambda} \right) \int_0^T \bar{z}(0)e^{-z\tau}\pi(\tau) d\tau \end{aligned} \right\} \quad (178)$$

We dividing both sides of (178) by $\bar{z}(0)$ and obtain

$$1 = \left\{ \begin{aligned} & \left[\left(\varphi + \frac{\alpha_1(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) \int_0^T \bar{z}(0)e^{-z\tau}\pi(\tau) d\tau + \alpha_2(1 - \theta) \left(\frac{(\mu + \delta_1) + (\mu + \delta_2)}{\theta + \lambda} \right) \right] \\ & + \left[\left(\frac{(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) + \left(\frac{(\mu + \phi)(1 - \phi) + \lambda}{\gamma} \right) \left(\frac{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma}{\Lambda((\mu + \phi)(1 - \phi) + \gamma)} \right) \right] \\ & \int_0^T \bar{z}(0)e^{-z\tau}\pi(\tau) d\tau \\ & + (1 - \phi) \left(\frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + \frac{\gamma}{(\mu + \phi)(1 - \phi) + \lambda} \right) \int_0^T \bar{z}(0)e^{-z\tau}\pi(\tau) d\tau \end{aligned} \right\} \quad (179)$$

The RHS of (179) is defined by $G_1(\lambda)$. $G_1(\lambda)$ is continuously differentiable function with $\lim_{\lambda \rightarrow \infty} G_1(\lambda) = 0$. Direct computation shows that $G_1'(\lambda) < 0$ and therefore, $G_1(\lambda)$ is a decreasing function. Hence, any real solution of equation (179) is negative if $G_1(0) < 1$, and positive if $G_1(0) > 1$. Thus, if $G_1(0) > 1$, the disease-free equilibrium (DFE) state is unstable. It is shown next that equation (179) has no complex solution with non-negative real part if $G_1(0) < 1$. In fact, we set

$$H(\tau) = \left[\begin{array}{c} \left(\varphi + \frac{\alpha_1(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) \\ + (1 - \phi) \left(\frac{\gamma\Lambda}{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma} + \frac{\gamma}{(\mu + \phi)(1 - \phi) + \lambda} \right) \end{array} \right] \pi(\tau) \quad (180)$$

$$F(\tau) = \left[\begin{array}{c} \alpha_2(1 - \theta) ((\mu + \delta_1) + (\mu + \delta_2)) \left[\left(\frac{(\mu + \phi)(1 - \phi)}{(\mu + \phi)(1 - \phi) + \gamma} \right) \right. \\ \left. + \left(\frac{(\mu + \phi)(1 - \phi) + \lambda}{\gamma} \right) \left(\frac{(\mu + \gamma)(\mu + \phi)(1 - \phi) - \phi\gamma}{\Lambda((\mu + \phi)(1 - \phi) + \gamma)} \right) \right] \end{array} \right] \pi(\tau). \quad (181)$$

Thus

$$G_1(\lambda) = \int_0^T e^{-\lambda\tau} H(\tau) d\tau + \frac{1}{(\theta + \lambda)} \int_0^T e^{-\lambda\tau} F(\tau) d\tau. \quad (182)$$

Suppose $G_1(0) < 0$. Assume that $\lambda = a_1 + ib_1$ is a complex solution of equation (182) with $a_1 \geq 0$ ([39]). Then

$$\begin{aligned} |G_1(\lambda)| &= \left| \int_0^T e^{-\lambda\tau} H(\tau) d\tau + \frac{1}{\theta + \lambda} \int_0^T e^{-\lambda\tau} F(\tau) d\tau \right| \\ &\leq \left| \int_0^T e^{-(a_1 + ib_1)\tau} H(\tau) d\tau \right| + \left| \frac{1}{\theta + a_1 + ib_1} \right| \left| \int_0^T e^{-(a_1 + ib_1)\tau} F(\tau) d\tau \right| \\ &= \int_0^T \left| e^{-(a_1 + ib_1)\tau} \right| |H(\tau)| d\tau + \frac{1}{\sqrt{(a_1 + \theta)^2 + b_1^2}} \int_0^T \left| e^{-(a_1 + ib_1)\tau} \right| |F(\tau)| d\tau \\ &\leq \int_0^T e^{-a_1\tau} |H(\tau)| d\tau + \frac{1}{a_1 + \theta} \int_0^T e^{-a_1\tau} |F(\tau)| d\tau \end{aligned} \quad (183)$$

$$= |G_1(a_1)| \leq G_1(0) < 1 \quad (184)$$

where $G_1(a_1)$ is a function of effective reproduction number. Therefore, it follows from equation (179) that equation (182) has solutions $\lambda = a_1 + ib_1$ only if $a_1 < 0$. Thus, every solution of (182) must have a negative real part. We observe that $R_{eff} = G_1(0)$. Therefore, the disease-free equilibrium (DFE) state is locally asymptotically stable if $G_1(0) < 1$. This completes the proof of Theorem 3.8. $\square$

3.5. Global stability of the DFE state.

The global asymptotic stability condition demands that solutions approach equilibrium for all initial conditions and the restrictions on the initial conditions are removed. Therefore, we construct a similar Lyapunov function as outlined in the works of [29] and [40] to study the global stability of the disease-free equilibrium state in this work.

Theorem 3.9. *The DFE state E^0 of the model equations (94) to (100) is Globally Asymptotically Stable (GAS) in Ω if $\mathcal{R}_{eff} \leq 1$, while it is unstable if $\mathcal{R}_{eff} > 1$.*

Proof. In this section, the global stability of E^0 is studied by providing the proof of Theorem 3.9. Thus, the following approach by [36] and [37] were utilized by constructing a suitable Lyapunov function as follows.

$$L_1(t) = \left(x - x^0 - x^0 \ln \frac{x}{x^0}\right) + \left(u - u^0 - u^0 \ln \frac{u}{u^0}\right) + \int_0^T r(\tau) \frac{q(t, \tau)}{\pi(\tau)} d\tau. \quad (185)$$

Using (24) in (185) yields

$$L_1(t) = \left(x - x^0 - x^0 \ln \frac{x}{x^0}\right) + \left(u - u^0 - u^0 \ln \frac{u}{u^0}\right) + \int_0^T r(\tau) B(t - \tau) d\tau. \quad (186)$$

Let

$$r(\tau) = \left(\frac{\alpha_1 x^0}{n^0} + \varphi\right) \int_0^T \pi(\tau) d\tau + \frac{\alpha_2(1 - \theta)g^0 x^0}{n^0} + (1 - \phi)u^0 \quad (187)$$

and

$$r(0) = \left(\frac{a_1 x^0}{n^0} + \varphi\right) \int_0^T \pi(\tau) d\tau + \frac{a_2(1 - \theta)g^0 x^0}{n^0} + (1 - \phi)u^0 \quad (188)$$

where g^0 , x^0 , u^0 and n^0 , are as defined in (132), (136), (137), and (138) respectively. Direct differentiation of (186) yields

$$\frac{dL_1(t)}{dt} = \left(\frac{x - x^0}{x}\right) \frac{dS}{dt} + \left(\frac{u - u^0}{u}\right) \frac{dP}{dt} + \left(r(0)B(t) + \int_0^T [r'(\tau)B(t - \tau)] d\tau\right). \quad (189)$$

Substituting equations (94), (95), (96), and (188) into equation (189) yields

$$\frac{dL_1}{dt} = \left[\begin{aligned} &\left(\frac{x - x^0}{x}\right) \left(\Lambda - \frac{\alpha_1 y x}{n} - \frac{\alpha_2(1 - \theta)x}{n} + \phi u - (\mu + \gamma)x\right) \\ &+ \left(\frac{u - u^0}{u}\right) [\gamma x - (\mu + \phi)u] + B(t) \frac{\alpha_1 x^0}{n^0} \int_0^T \pi(s) ds + B(t) \frac{\alpha_2(1 - \theta)g^0 x^0}{n^0} \\ &+ B(t) \varphi \int_0^T \pi(s) ds + B(t)(1 - \phi)u^0 - \frac{\alpha_1 x^0}{n^0} \int_0^T B(t - \tau) \pi(\tau) d\tau \\ &- \frac{\alpha_2(1 - \theta)g^0 x^0}{n^0} B(t - \tau) - \varphi \int_0^T B(t - \tau) \pi(\tau) d\tau - (1 - \phi)u^0 B(t - \tau) \end{aligned} \right] \quad (190)$$

Using equations (100), (24), (136), (137), and (138) in (190) yields

$$\frac{dL_1}{dt} = \left[\begin{aligned} & \left(\frac{x - x^0}{x} \right) [(\mu + \gamma)x^0 - (\mu + \gamma)x] + \left(\frac{u - u^0}{u} \right) [(\mu + \phi)u^0 - (\mu + \phi)u] \\ & + q(t, 0)R_E \frac{\alpha_1 x^0}{n^0} \int_0^T B(t - \tau)\pi(\tau) d\tau - \frac{\alpha_2(1 - \Theta)g^0 x^0}{n^0} B(t - \tau) \\ & - \varphi \int_0^T B(t - \tau)\pi(\tau) d\tau - (1 - \phi)u^0 B(t - \tau) \end{aligned} \right] \quad (191)$$

The expansion and simplification of equation (191) yields

$$\frac{dL_1}{dt} = \left[\begin{aligned} & -\frac{(\mu + \gamma)}{x}(x + x^0)^2 - \frac{(\mu + \phi)(1 - \phi)}{u}(u - u^0)^2 + q(t, 0)R_{eff} \\ & - \left(\frac{\alpha_1 x^0}{n^0} + \varphi \right) \int_0^T B(t - \tau)\pi(\tau) d\tau - \left((1 - \phi)u^0 + \frac{\alpha_2(1 - \theta)g^0 x^0}{n^0} \right) B(t - \tau) \end{aligned} \right] \quad (192)$$

Further simplification of (192) yields

$$= -\frac{(\mu + \gamma)}{x}(x - x^0)^2 - \frac{(\mu + \phi)(1 - \phi)}{u}(u - u^0)^2 + q(t, 0)R_{eff} - q(t, 0). \quad (193)$$

Further simplification of (3.5) yields

$$= -\frac{(\mu + \gamma)}{x}(x - x^0)^2 - \frac{(\mu + \phi)(1 - \phi)}{u}(u - u^0)^2 - (1 - R_{eff})q(t, 0) \leq 0 \quad \text{For } R_{eff} \leq 1. \quad (194)$$

The equality $\frac{dL_1}{dt} = 0$ holds if and only if $x = x^0$, $u = u^0$, and $q(t, 0) = 0$. Thus, from solution (24), the model (42) to (43) along the characteristic lines, it can be seen that $q(t, 0) = 0$ for all $t > \tau$. Hence, $q(t, 0) \rightarrow 0$ as $t \rightarrow \infty$. Therefore, from the LaSalle Invariant Principle, the disease-free equilibrium is globally asymptotically stable if $R_{eff} \leq 1$. $\square$

3.6. Endemic Equilibrium (EE) State.

The EE state is referred to as the state in which EVD still persists in the population. Thus, at the EE state we set

$$\begin{aligned} S(t) &= x^*, & P(t) &= u^*, & I(t) &= y^*, & G(t) &= g^*, & R(t) &= r^*, \\ Q(t) &= z^*, & N(t) &= n^*, & q(t, \tau) &= q^*(\tau). \end{aligned} \quad (195)$$

Also, from equations (36), (103), (195), we assume that

$$z^* = y^* = \int_0^T q^*(\tau) d\tau. \quad (196)$$

Let $(x^*, u^*, y^*, g^*, r^*, q^*(\tau))$ represents any arbitrary endemic equilibrium (EE) of the model equations (94) to (100) This equilibrium satisfies the following equations.

$$\Lambda - \frac{\alpha_1 x^* z^*}{n} - \frac{\alpha_2(1 - \theta)x^* g^*}{n} + \phi u^* - (\mu + \gamma)x^* = 0 \quad (197)$$

$$\gamma x^* - (\mu + \phi)(1 - \phi)u^* = 0 \quad (198)$$

$$\frac{\alpha_1 x^* z^*}{n} + \frac{\alpha_2 (1 - \theta) x^* g^*}{n} - (\mu + \delta_1 + \varphi) z^* = 0 \quad (199)$$

$$(\mu + \delta_1) z^* + (\mu + \delta_2) z^* - \theta g^* = 0 \quad (200)$$

$$\omega z^* - \mu r^* = 0 \quad (201)$$

$$\frac{dq^*(\tau)}{d\tau} + (\mu + \sigma(\tau))q^*(\tau) = 0 \quad (202)$$

and

$$q^*(0) = B(0) = \frac{\alpha_1 x^* z^*}{n} + \frac{\alpha_2 (1 - \theta) x^* g^*}{n} + \varphi z^* + (1 - \phi)u^*. \quad (203)$$

Solve (202) yields

$$q^*(\tau) = C e^{-(\mu + \sigma(\tau))\tau}. \quad (204)$$

Equation (204) yields

$$q^*(0) = C. \quad (205)$$

Substituting (205) into (204) yields

$$q^*(\tau) = q^*(0) e^{-(\mu + \sigma(\tau))\tau}. \quad (206)$$

Equation is re-written as

$$q^*(\tau) = q^*(0) \pi(\tau). \quad (207)$$

where $\pi(\tau)$ is defined as in (118). Substituting (207) into (201), we have

$$\omega \int_0^T q^*(0) \pi(\tau) d\tau - \mu r^* = 0. \quad (208)$$

Isolating r^* in ((208) yields

$$r^* = \frac{\omega}{\mu} \int_0^T q^*(0) \pi(\tau) d\tau. \quad (209)$$

Substituting((207) into ((200), we have

$$(\mu + \delta_1) \int_0^T q^*(0) \pi(\tau) d\tau + (\mu + \delta_2) \int_0^T q^*(0) \pi(\tau) d\tau - \theta g^* = 0. \quad (210)$$

Isolating the desired variable in equation (210) we have

$$g^* = \left(\frac{(\mu + \delta_1) + (\mu + \delta_2)}{\theta} \right) \int_0^T q^*(0) \pi(\tau) d\tau. \quad (211)$$

Substituting equations (121) and (125) into (112) gives

$$\left[\begin{aligned} & \frac{\alpha_1 x^*}{n^*} \int_0^T q^*(0) \pi(\tau) d\tau + \frac{\alpha_2(1-\theta)}{n^*} \left(\frac{(\mu + \delta_1) + (\mu + \delta_2)}{\theta} \right) \int_0^T q^*(0) \pi(\tau) d\tau \\ & - (\mu + \delta_1 + \phi) \int_0^T q^*(0) \pi(\tau) d\tau \end{aligned} \right] = 0. \quad (212)$$

Equation (122) takes the form

$$\left[\left(\frac{\alpha_1 x^*}{n^*} + \frac{\alpha_2(1-\theta)}{n^*} \left(\frac{(\mu + \delta_1) + (\mu + \delta_2)}{\theta} \right) \right) - (\mu + \delta_1 + \phi) \right] \int_0^T q^*(0) \pi(\tau) d\tau = 0. \quad (213)$$

Either

$$\int_0^T q^*(0) \pi(\tau) d\tau = 0 \quad (214)$$

or

$$\left(\frac{\alpha_1 x^*}{n^*} + \frac{\alpha_2(1-\theta)}{n^*} \left(\frac{(\mu + \delta_1) + (\mu + \delta_2)}{\theta} \right) \right) - (\mu + \delta_1 + \phi) = 0. \quad (215)$$

Solving for x^* variable in (215) yields

$$x^* = \left(\frac{n^* \theta (\mu + \delta_1 + \phi) - \alpha_2(1-\theta) ((\mu + \delta_1) + (\mu + \delta_2))}{\alpha_1 \theta} \right). \quad (216)$$

Substituting equation (216) into (198) and isolating u^* yields

$$u^* = \left(\frac{\gamma (n^* \theta (\mu + \phi + \delta_1)) - \alpha_2(1-\theta) ((\mu + \delta_1) + (\mu + \delta_2))}{\alpha_1 \theta (\mu + \phi) (1 - \phi)} \right). \quad (217)$$

Substituting equation (214) into (209) and (211) gives

$$r^* = g^* = 0. \quad (218)$$

Substituting equations (218), (216), and (217) into (197) yields

$$\left[\begin{aligned} & \Lambda - \left(\frac{\alpha_1 z^*}{n^*} + (\mu + \gamma) \right) \left(\frac{n^* \theta (\mu + \delta_1 + \phi) - \alpha_2(1-\theta) ((\mu + \delta_1) + (\mu + \delta_2))}{\alpha_1 \theta} \right) \\ & + \left(\frac{\gamma (n^* \theta (\mu + \phi + \delta_1)) - \alpha_2(1-\theta) ((\mu + \delta_1) + (\mu + \delta_2))}{\alpha_1 \theta (\mu + \phi) (1 - \phi)} \right) \end{aligned} \right] = 0. \quad (219)$$

Isolating the z^* from equation (219) and simplifying, we obtain

$$z^* = \left[\frac{n^* ((\mu + \phi)(1 - \phi)(\mu + \gamma) - \gamma \phi)}{\alpha_1 (\mu + \phi) (1 - \phi)} + \frac{\Lambda}{\mu + \delta_1 + \phi} - \frac{\Lambda n^* \theta}{\alpha_2(1-\theta) ((\mu + \delta_1) + (\mu + \delta_2))} \right]. \quad (220)$$

From equations (196), (207), and (214), we obtain

$$z^* = 0. \quad (221)$$

Therefore, substituting equations (216), (217), (218), and (221) into (203) yields

$$q^*(0) = \left(\frac{\gamma(1 - \phi) [n^* \theta (\mu + \phi + \delta_1)] - \alpha_2(1 - \theta) [(\mu + \delta_1) + (\mu + \delta_2)]}{\alpha_1 \theta (\mu + \phi) (1 - \phi)} \right). \quad (222)$$

For the EVD to be controlled, it is sufficient that the effective reproduction number is less than one, that is

$$R_{\text{eff}} = \left(\frac{\alpha_1 x^0}{n^0} + \phi \right) \int_0^T \pi(\tau) d\tau + \frac{\alpha_2(1-\theta)x^0 g^0}{n^0} + (1-\phi)u^0 < 1. \quad (223)$$

Substituting equation (122) into (223) gives:

$$R_{\text{eff}} = \left(\frac{\alpha_1 x^0}{n^0} + \varphi \right) \bar{\pi} + \frac{\alpha_2(1-\theta)x^0 g^0}{n^0} + (1-\phi)u^0 < 1. \quad (224)$$

Equation (224) takes the form

$$\left(\frac{\alpha_1 x^0}{n^0} + \varphi \right) \bar{\pi} + \frac{\alpha_2(1-\theta)x^0 g^0}{n^0} + (1-\phi)u^0 < 1. \quad (225)$$

Solving for α_1 and α_2 in (225) gives

$$\alpha < \frac{n^0 (1 - (1-\phi)u^0 - \varphi \bar{\pi})}{x^0 (\bar{\pi} + (1-\theta)g^0)} \quad (226)$$

$$\alpha = \alpha_1 + \alpha_2 \quad (227)$$

where u^0 , g^0 , x^0 , n^0 , and $\bar{\pi}$ are defined as in equations (137), (132), (136), (138), and (122), respectively. From equation (226), an upper bound for the infection rate was obtained. This suggests that for EVD to be put under control in the population, the infection rate should not exceed the value given by the right-hand side of (226).

3.7. Local Stability of the Endemic Equilibrium (EE) State.

Lemma 3.10. *The unique EE $E^*(x^*, u^*, y^*, g^*, r^*, q^*(\tau))$ given by (216), (217), (196), (211) and (209), with $q^*(0)$ given by (222), is locally asymptotically stable (LAS) if $R_{\text{eff}} > 1$*

Proof. To show the local stability of EE, we linearize equation (94) to (100) around E^* . Therefore,

$$\begin{aligned} S(t) &= x^* + x(t), & P(t) &= u^* + u(t), & I(t) &= y^* + y(t), & G(t) &= g^* + g(t), \\ R(t) &= r^* + r(t), & q(t, \tau) &= q^*(\tau) + z(t, \tau). \end{aligned} \quad (228)$$

We let the solutions of the linearized system in exponential form to be as follows

$$\begin{aligned} x(t) &= x e^{\lambda t}, & u(t) &= u e^{\lambda t}, & y(t) &= y e^{\lambda t}, & g(t) &= g e^{\lambda t}, \\ r(t) &= r e^{\lambda t}, & z(t, \tau) &= z(\tau) e^{\lambda t}. \end{aligned} \quad (229)$$

Substituting equation (229) into (228) yields

$$\begin{aligned} S(t) &= x^* + x e^{\lambda t}, & P(t) &= u^* + u e^{\lambda t}, & I(t) &= y^* + y e^{\lambda t}, & G(t) &= g^* + g e^{\lambda t}, \\ R(t) &= r^* + r e^{\lambda t}, & q(t, \tau) &= q^*(\tau) + z(\tau) e^{\lambda t}, \end{aligned} \quad (230)$$

Substituting equations (195) and (230) into (94) yields

$$\frac{dx(t)}{dt} = \left[\Lambda - \left(\alpha_1 \int_0^T (q^*(\tau) + z(\tau)e^{\lambda t}) d\tau + \alpha_2(1-\theta)(g^* + ge^{\lambda t}) \right) \frac{x^* + xe^{\lambda t}}{n^*} \right. \\ \left. + \phi(u^* + ue^{\lambda t}) - (\mu + \gamma)(x^* + xe^{\lambda t}) \right] \quad (231)$$

We expand and simplify equation (231) using (196) to obtain

$$0 = \Lambda - \frac{\alpha_1 x^* z^*}{n^*} - \frac{\alpha_2(1-\theta)g^* x^*}{n^*} + \phi u^* - (\mu + \gamma)x^* - \frac{x^*}{n^*} \left(\alpha_1 \int_0^T z(\tau) d\tau + \alpha_2(1-\theta)g \right) \\ - \frac{\alpha_1 x}{n^*} \left(\int_0^T z(\tau) d\tau + z^* \right) - \frac{\alpha_2(1-\theta)x}{n^*} (g^* + g) + \phi u - \lambda x - \mu x - \gamma x. \quad (232)$$

Using equation (197) in (232) yields

$$(\lambda + \mu + \gamma)x = - \frac{x^*}{n^*} \left(\alpha_1 \int_0^T z(\tau) d\tau + \alpha_2(1-\theta)f \right) \\ - \frac{\alpha_1 x}{n^*} \left(\int_0^T z(\tau) d\tau + z^* \right) - \frac{\alpha_2(1-\theta)x}{n^*} (g^* + g) + \phi u. \quad (233)$$

Substituting (230) into (95) yields

$$\frac{du(t)}{dt} = \gamma (x^* + xe^{\lambda t}) - (\mu + \phi)(1 - \phi) (u^* + ue^{\lambda t}). \quad (234)$$

Expanding the right-hand side (RHS) of equation (234) yields

$$0 = \gamma x^* - (\mu + \phi)(1 - \phi)u^* + \gamma x - ((\mu + \phi)(1 - \phi) + \lambda) u. \quad (235)$$

We obtain (198) from (234)

$$((\mu + \phi)(1 - \phi) + \lambda) u = \gamma x. \quad (236)$$

Substituting (195) and (230) into (96) yields

$$\frac{dy(t)}{dt} = \left[\frac{\alpha_1(x^* + xe^{\lambda t})}{n^*} \int_0^T (q^*(\tau) + z(\tau)) d\tau + \frac{\alpha_2(1-\theta)(g^* + ge^{\lambda t})(x^* + xe^{\lambda t})}{n^*} \right. \\ \left. - (\mu + \delta_1 + \varphi) \int_0^T (q^*(\tau) + z(\tau)) d\tau. \right] \quad (237)$$

Expanding and simplifying equation (237) yields

$$y\lambda = \left[\begin{aligned} & \frac{\alpha_1 x^* z^*}{n^*} + \frac{\alpha_2(1-\theta)g^* x^*}{n^*} - (\mu + \delta_1 + \varphi)z^* \\ & + \frac{\alpha_1}{n^*} \left(x^* \int_0^T z(\tau) d\tau + x \int_0^T z(\tau) d\tau + xz^* \right) + \frac{\alpha_2(1-\theta)}{n^*} (g^* x + gx^* + gx) \\ & - (\mu + \delta_1 + \varphi) \int_0^T z(\tau) d\tau \end{aligned} \right]. \quad (238)$$

Using equation (199) in (236) yields

$$y\lambda = \left[\begin{aligned} & \frac{\alpha_1}{n^*} \left(x^* \int_0^T z(\tau) d\tau + x \int_0^T z(\tau) d\tau + xz^* \right) + \frac{\alpha_2(1-\theta)}{n^*} (g^* x + gx^* + gx) \\ & - (\mu + \delta_1 + \varphi) \int_0^T z(\tau) d\tau \end{aligned} \right]. \quad (239)$$

Substituting equation (230) into (97) yields

$$\frac{dg(t)}{dt} = ((\mu + \delta_1) + (\mu + \delta_2)) \int_0^T (q^*(\tau) + z(\tau)e^{\lambda t}) d\tau - \theta(g^* + ge^{\lambda t}) \quad (240)$$

Expanding the RHS of equation (240) and using equation (200) yields

$$(\theta + \lambda)g = 2\mu \int_0^T z(\tau) d\tau + \delta_1 \int_0^T z(\tau) d\tau + \delta_2 \int_0^T z(\tau) d\tau. \quad (241)$$

Substituting equation (230) into (98) yields

$$\frac{dr(t)}{dt} = \omega \int_0^T (q^*(\tau) + z(\tau)e^{\lambda t}) d\tau - \mu(r^* + re^{\lambda t}). \quad (242)$$

Expanding the RHS of equation (242) and using equation (201) gives

$$(\mu + \lambda)r = \omega \int_0^T z(\tau) d\tau. \quad (243)$$

Equation (244) was obtained from (99).

$$\frac{dz(\tau)}{dt} + (\lambda + \mu + \sigma(\tau))z(\tau) = 0. \quad (244)$$

Solving equation (244) yields

$$z(\tau) = Ce^{-(\lambda + \mu + \sigma(\tau))\tau}. \quad (245)$$

Substituting $\tau = 0$ into (245) gives

$$z(0) = C. \quad (246)$$

Substituting equation (245) into (244) gives

$$z(\tau) = z(0)e^{-(\lambda + \mu + \sigma(\tau))\tau}. \quad (247)$$

Simplifying equation (246) we obtain

$$z(\tau) = z(0) e^{-\lambda\tau} \pi(\tau). \quad (248)$$

Term $\pi(\tau)$ is defined as in equation (118). Substituting equations (195) and (230) into (100) yields

$$z(0) = \left[\frac{\alpha_1(x^* + xe^{\lambda t})}{n^*} \int_0^T (q^*(\tau) + z(\tau)e^{\lambda\tau}) d\tau + \frac{\alpha_2(1-\theta)(g^* + ge^{\lambda t})(x^* + xe^{\lambda t})}{n^*} \right. \\ \left. + (1-\phi)(u^* + ue^{\lambda t}) + \varphi \int_0^T (q^*(\tau) + z(\tau)e^{\lambda\tau}) d\tau \right]. \quad (249)$$

Expanding equation (249) and using equation (196) gives

$$z(0) = \left[\frac{\alpha_1 x^* z^*}{n^*} + \frac{\alpha_2(1-\theta)x^* g^*}{n^*} + \varphi z^* + (1-\phi)u^* \right. \\ \left. + \frac{\alpha_2 x^*}{n^*} \int_0^T z(\tau) d\tau + \frac{\alpha_1 x z^*}{n^*} + \frac{\alpha_1 x}{n^*} \int_0^T z(\tau) d\tau + \frac{\alpha_2(1-\theta)g^* x}{n^*} \right. \\ \left. + \frac{\alpha_2(1-\theta)gx^*}{n^*} + \frac{\alpha_2(1-\theta)gx}{n^*} + (1-\phi)u + \varphi \int_0^T z(\tau) d\tau \right]. \quad (250)$$

Using equation (203) in (250) gives

$$z(0) = \left[q^*(0) + \frac{\alpha_1 x^*}{n^*} \int_0^T z(\tau) d\tau + \frac{\alpha_1 x z^*}{n^*} + \frac{\alpha_1 x}{n^*} \int_0^T z(\tau) d\tau + \frac{\alpha_2(1-\theta)g^* x}{n^*} \right. \\ \left. + \frac{\alpha_2(1-\theta)gx^*}{n^*} + \frac{\alpha_2(1-\theta)gx}{n^*} + \frac{\alpha_2(1-\theta)gx}{n^*} + (1-\phi)u + \varphi \int_0^T z(\tau) d\tau \right]. \quad (251)$$

Equations (233), (236), (241), and (251) take the form

$$k_8 x = -\frac{\alpha_1 x^* m}{n^*} - \frac{\alpha_2 k_1 g^* x^*}{n^*} - \frac{\alpha_1 x^* x}{n^*} - \frac{\alpha_2 k_1 g^* x}{n^*} - \frac{\alpha_2 x k_1 g}{n^*} + \phi u \quad (252)$$

$$k_9 u = \gamma x, \quad k_{10} g = 2\mu m + \delta_1 m + \delta_2 m \quad (253)$$

$$z(0) = q^*(0) + \frac{\alpha_1 x z^*}{n^*} + \frac{\alpha_1 x^* m}{n^*} + \frac{\alpha_1 x m}{n^*} + \frac{\alpha_2 k_1 g x}{n^*} + \frac{\alpha_2 k_1 g z^*}{n^*} + \frac{\alpha_2 x k_1 g}{n^*} + \varphi m + k_7 u \quad (254)$$

where

$$k_8 = (\lambda + \mu + \gamma), \quad k_9 = (\lambda + (\mu + \phi)(1 - \phi)), \quad k_{10} = (\lambda + \theta), \quad m = \int_0^T z(\tau) d\tau. \quad (255)$$

From equation (253) we obtain

$$u = \frac{\gamma x}{k_9}, \quad g = \frac{(2\mu + \delta)m}{k_{10}} \quad (256)$$

where

$$\delta = \delta_1 + \delta_2. \quad (257)$$

Equation (257) describes the total induced death rate by EVD from $I(t)$ and $Q(t)$ respectively. Substituting equation (256) into (252) and solving for x yields

$$x = - \left[\frac{k_9 m (\alpha_1 k_{10} + \alpha_2 k_1 (2\mu + \delta))}{(k_8 k_{10} n^* - k_{10} \phi \gamma n^*) + k_9 (\alpha_1 k_{10} z^* + \alpha_2 k_1 k_{10} g^*) + \alpha_1 k_{10} m + \alpha_2 k_1 m (2\mu + \delta)} \right] x^*. \quad (258)$$

Substituting equations (216), (218), and (221) into (253) yields

$$x = - \frac{(k_9 m (\alpha_1 k_{10} + \alpha_2 k_1 (2\mu + \delta))) (k_4 n^* \theta - k_1 \alpha_2 (k_5 + k_6))}{\alpha_1 \theta (k_8 k_5 k_{10} n^* - k_{10} \phi \gamma n^*) + \alpha_1 k_9 k_5 k_{10} m + \alpha_2 k_1 k_5 k_{10} m (2\mu + \delta)} \quad (259)$$

Substituting equation (259) into (253) yields

$$x = - \left[\frac{\gamma (k_9 m (\alpha_1 k_{10} + \alpha_2 k_1 (2\mu + \delta))) (k_4 n^* \theta - k_1 \alpha_2 (k_5 + k_6))}{k_9 \alpha_1 \theta (k_8 k_{10} n^* - k_{10} \phi \gamma n^*) + \alpha_1 k_9 k_{10} m + \alpha_2 k_1 k_{10} m (2\mu + \delta)} \right] \quad (260)$$

Substituting equations (221) and (218) into (254) yields

$$z(0) = q^*(0) + \frac{\alpha_1 x^* m}{n^*} + \frac{\alpha_1 x m}{n^*} + \frac{\alpha_2 k_1 g x^*}{n^*} + \frac{\alpha_2 k_1 g x}{n^*} + \varphi m + k_7 u \quad (261)$$

Substituting equations (260), (259), and (256) into (261) yields

$$z(0) = \left\{ \begin{aligned} & q^*(0) + qm + \frac{\alpha_1 x^* m}{n^*} + \frac{\alpha_2 k_1 x^* m (2\mu + \delta)}{k_{10} n^*} \\ & - \frac{[\alpha_1 + \alpha_2 k_1 (2\mu + \delta)] [\alpha_1 k_4 k_9 k_{10} \theta n^* + \alpha_2 k_1 k_4 k_9 \theta n^* (2\mu + \delta)] m}{k_{10} n^* [k_8 k_{10} n^* - k_{10} \phi \gamma n^* + \alpha_1 k_9 k_{10} m + \alpha_2 k_1 k_9 m (2\mu + \delta)] (\alpha_2 k_1 (k_5 + k_6) + \alpha_1 \theta)} \\ & - \frac{k_7 \gamma (\alpha_1 k_4 k_{10} \theta n^* + \alpha_2 k_9 k_4 k_1 \theta n^* (2\mu + \delta)) m}{k_9 [k_8 k_{10} n^* - k_{10} \phi \gamma n^* + \alpha_1 k_9 k_5 k_{10} m + \alpha_2 k_1 k_9 m (2\mu + \delta)] (\alpha_2 k_1 (k_5 + k_6) + \alpha_1 \theta)} \end{aligned} \right\}. \quad (262)$$

Equation (262) is re-written as follows

$$z(0) = q^*(0) + \left\{ \begin{aligned} & \frac{(\varphi k_{10} n^* + \alpha_1 k_{10} x^* + \alpha_2 k_1 x^* (2\mu + \delta)) m}{k_{10} n^*} \\ & - \frac{[(\alpha_1 + \alpha_2 k_1 (2\mu + \delta)) (\alpha_1 k_4 k_9 k_{10} \theta n^* + \alpha_2 k_1 k_4 k_9 \theta n^* (2\mu + \delta))] m}{k_{10} n^* (k_8 k_{10} n^* - k_{10} \phi \gamma n^* + \alpha_1 k_9 k_{10} m + \alpha_2 k_1 k_9 m (2\mu + \delta)) (\alpha_2 k_1 (k_5 + k_6) + \alpha_1 \theta)} \\ & - \frac{k_7 \gamma (\alpha_1 k_4 k_{10} \theta n^* + \alpha_2 k_9 k_4 k_1 \theta n^* (2\mu + \delta)) m}{k_9 (k_8 k_{10} n^* - k_{10} \phi \gamma n^* + \alpha_1 k_9 k_{10} m + \alpha_2 k_1 k_9 m (2\mu + \delta)) (\alpha_2 k_1 (k_5 + k_6) + \alpha_1 \theta)} \end{aligned} \right\}. \quad (263)$$

Substituting equations (101) and (255) into (263) yields

$$z(0) = q^*(0) + \left\{ \begin{aligned} & \frac{\left(\varphi n^* ((\lambda + \theta) + \alpha_1 x^*) + \alpha_2 (1 - \theta) (2\mu + \delta) x^* \int_0^T z(\tau) d\tau \right)}{(\lambda + \theta) n^*} \\ & - \frac{\left(\frac{(\alpha_1 + \alpha_2)(1 - \theta)(2\mu + \gamma)}{(\mu + \delta_1 + \varphi)(\lambda + (\mu + \phi)(1 - \phi)) \theta n^*} \int_0^T z(\tau) d\tau \right)}{\left(\frac{(\lambda + \theta)(\lambda + \mu + \gamma) n^* - \phi n^* + \alpha_2 (\lambda + (\mu + \phi)(1 - \phi)) \int_0^T z(\tau) d\tau}{+ \alpha_2 (1 - \theta)(\lambda + (\mu + \phi)(2\mu + \delta)) \int_0^T z(\tau) d\tau} \right) \left(\frac{(\lambda + \theta) n^* (\alpha_2 (1 - \theta)) + (\mu + \delta_1)}{(\mu + \delta_2) + \alpha_1 \theta} \right)} \\ & - \frac{\left((1 - \phi)(\mu + \delta_1 + \varphi) \gamma \theta n^* \left(\frac{\alpha_1 (\lambda + (\mu + \theta)) + \alpha_2 (1 - \theta)}{(\lambda + (\mu + \phi))(2\mu + \delta)} \right) \int_0^T z(\tau) d\tau \right)}{\left(\frac{(\lambda + \theta)(\lambda + \mu + \gamma) n^* - \phi n^* + \alpha_2 (\lambda + (\mu + \phi)(1 - \phi)) \int_0^T z(\tau) d\tau}{+ \alpha_2 (1 - \theta)(\lambda + (\mu + \phi)(2\mu + \delta)) \int_0^T z(\tau) d\tau} \right) \left(\frac{(\lambda + (\mu + \gamma)(1 - \phi)(\alpha_2 (1 - \theta)))}{(\mu + \delta_1) + (\mu + \delta_2) + \alpha_1 \theta} \right)} \end{aligned} \right\}. \quad (264)$$

Equation 264 yields

$$z(0) = q^*(0) + \left\{ \begin{aligned} & \frac{\left(\varphi n^* (\lambda + \theta) + \alpha_1 x^* \right) + \alpha_2 (1 - \theta) (2\mu + \delta) x^* \int_0^T z(0) e^{-\lambda \tau} \pi(\tau) d\tau}{(\lambda + \theta) n^*} \\ & - \frac{\left(\frac{(\alpha_1 + \alpha_2 (1 - \theta)(2\mu + \gamma)(\mu + \delta_1 + \varphi)(\lambda + (\mu + \phi)(1 - \phi)) \theta n^*)}{\alpha_1 (\lambda + \theta) + \alpha_2 (1 - \theta)(2\mu + \delta)} \int_0^T z(0) e^{\lambda \tau} \pi(\tau) d\tau \right)}{\left(\frac{(\lambda + \theta)((\lambda + \mu + \gamma) n^* - \phi n^* + \alpha_1 (\lambda + ((\mu + \phi)(1 - \phi)))) \int_0^T z(0) e^{\lambda \tau} \pi(\tau) d\tau}{+ \alpha_2 (1 - \theta)(\lambda + (\mu + \phi)(2\mu + \delta)) \int_0^T z(0) e^{\lambda \tau} \pi(\tau) d\tau} \right) \left(\frac{(\lambda + \theta) n^* (\alpha_2 (1 - \theta)) + (\mu + \delta_1)}{(\mu + \delta_2) + \alpha_1 \theta} \right)} \\ & - \frac{\left((1 - \phi)(\mu + \delta_1 + \varphi) \gamma \theta n^* \left(\frac{\alpha_1 (\lambda + (\mu + \theta)) + \alpha_2 (1 - \theta)}{(\lambda + (\mu + \phi))(2\mu + \delta)} \right) \int_0^T z(0) e^{\lambda \tau} \pi(\tau) d\tau \right)}{\left(\frac{(\lambda + \theta)(\lambda + \mu + \gamma) n^* - \phi n^* + \alpha_2 (\lambda + (\mu + \phi)(1 - \phi)) \int_0^T z(0) e^{\lambda \tau} \pi(\tau) d\tau}{+ \alpha_2 (1 - \theta)(\lambda + (\mu + \phi)(2\mu + \delta)) \int_0^T z(0) e^{\lambda \tau} \pi(\tau) d\tau} \right) \left(\frac{(\lambda + (\mu + \gamma)(1 - \phi)(\alpha_2 (1 - \theta)))}{(\mu + \delta_1) + (\mu + \delta_2) + \alpha_1 \theta} \right)} \end{aligned} \right\}. \quad (265)$$

We divide both sides of equation 265 by $z(0)$ to obtain

$$1 = v_0 + \left\{ \begin{aligned} & \frac{\left(\varphi n^* (\lambda + \theta) + \alpha_1 x^* \right) + \alpha_2 (1 - \theta) (2\mu + \delta) x^* \int_0^T z(0) e^{-\lambda \tau} \pi(\tau) d\tau}{(\lambda + \theta) n^*} \\ & - \frac{\left(\frac{(\alpha_1 + \alpha_2 (1 - \theta)(2\mu + \gamma)(\mu + \delta_1 + \varphi)(\lambda + (\mu + \phi)(1 - \phi)) \theta n^*)}{\alpha_1 (\lambda + \theta) + \alpha_2 (1 - \theta)(2\mu + \delta)} \int_0^T z(0) e^{\lambda \tau} \pi(\tau) d\tau \right)}{\left(\frac{(\lambda + \theta)((\lambda + \mu + \gamma) n^* - \phi n^* + \alpha_1 (\lambda + ((\mu + \phi)(1 - \phi)))) \int_0^T z(0) e^{\lambda \tau} \pi(\tau) d\tau}{+ \alpha_2 (1 - \theta)(\lambda + (\mu + \phi)(2\mu + \delta)) \int_0^T z(0) e^{\lambda \tau} \pi(\tau) d\tau} \right) \left(\frac{(\lambda + \theta) n^* (\alpha_2 (1 - \theta)) + (\mu + \delta_1)}{(\mu + \delta_2) + \alpha_1 \theta} \right)} \\ & - \frac{\left((1 - \phi)(\mu + \delta_1 + \varphi) \gamma \theta n^* \left(\frac{\alpha_1 (\lambda + (\mu + \theta)) + \alpha_2 (1 - \theta)}{(\lambda + (\mu + \phi))(2\mu + \delta)} \right) \int_0^T z(0) e^{\lambda \tau} \pi(\tau) d\tau \right)}{\left(\frac{(\lambda + \theta)(\lambda + \mu + \gamma) n^* - \phi n^* + \alpha_2 (\lambda + (\mu + \phi)(1 - \phi)) \int_0^T z(0) e^{\lambda \tau} \pi(\tau) d\tau}{+ \alpha_2 (1 - \theta)(\lambda + (\mu + \phi)(2\mu + \delta)) \int_0^T z(0) e^{\lambda \tau} \pi(\tau) d\tau} \right) \left(\frac{(\lambda + (\mu + \gamma)(1 - \phi)(\alpha_2 (1 - \theta)))}{(\mu + \delta_1) + (\mu + \delta_2) + \alpha_1 \theta} \right)} \end{aligned} \right\} \quad (266)$$

where

$$v_0 = \frac{q^*(0)}{z(0)} \quad (267)$$

Suppose that λ is a real value function, then from the characteristics equation (267) with equation (184), and the fact that $R_{eff} > 1$, we have shown that $\lambda < 0$, Since for $\lambda \geq 0$, in the last part of equation (230), the RHS is negative while the LHS is positive, which suggests that the $EE - E^*$ is Locally Asymptotically Stable (LAS). Thus, the $EE - E^*$ is LAS since the real root has the dominant real part. This completes the proof of Lemma 3.7 $\square$

3.8. Global Stability of the EE State.

Theorem 3.11. *If $\mathcal{R}_{eff} > 1$, then the endemic equilibrium E^* is (GAS).*

Proof. Assume that $\mathcal{R}_{eff} > 1$, then there exists a unique endemic equilibrium $E^* = (x^*, u^*, y^*, g^*, r^*, q^*(\tau))$ where $x^*, u^*, y^*, g^*, r^*, q^*(\tau)$ satisfies the equilibrium equations (197) to (203).

Let $G_2(x, u) = x - u - u \ln \frac{x}{u}$ for $x, u > 0$, then we construct the following Lyapunov functional.

$$V(t) = \sum_{i=1}^n \left[x_i(t) - x_i^* - x_i^* \ln \left(\frac{x_i(t)}{x_i^*} \right) \right]$$

$$V = V_1 + V_2 + V_3 + V_4 + V_5 + V_6 \quad (268)$$

$$V_1 = \left(x - x^* - x^* \ln \frac{x}{x^*} \right) \quad (269)$$

$$V_2 = \left(u - u^* - u^* \ln \frac{u}{u^*} \right) \quad (270)$$

$$V_3 = \left(y - y^* - y^* \ln \frac{y}{y^*} \right) \quad (271)$$

$$V_4 = \left(g - g^* - g^* \ln \frac{g}{g^*} \right) \quad (272)$$

$$V_5 = \left(r - r^* - r^* \ln \frac{r}{r^*} \right) \quad (273)$$

$$V_6 = \int_0^T q^*(t) G_3(q(t, \tau), q^*(t)) d\tau. \quad (274)$$

$\square$

The following Lemmas evaluate the derivatives of V_1, V_2, V_3, V_4, V_5 and V_6 along the solutions of equations (1) to (6), respectively.

Lemma 3.12. *Let V_1 be defined as in equation (269), then*

$$V_1' = \frac{-(\mu + \gamma)}{x} (x - x^*)^2 - \frac{\alpha_1}{n^* n x} (x z n^* - x^* z^* n) + \left\{ \varphi(u - u^*) - \frac{\alpha_2(1 - \theta)}{n^* n} (x g n^* - x^* g^* n) \right\} \left(\frac{x - x^*}{x} \right). \quad (275)$$

Proof. Direct differentiation of equation (269) yields

$$V_1' = \left(\frac{x - x^*}{x} \right) \frac{dx}{dt}. \quad (276)$$

Using equation (1) in equation (276) yields

$$V_1' = \left(\frac{x - x^*}{x} \right) \left\{ \Lambda - \frac{\alpha_1 x}{n} - \frac{\alpha_2(1 - \theta)}{n} gx + \phi u - (\mu + \gamma)x \right\}. \quad (277)$$

From (277) we obtain

$$\Lambda = \frac{\alpha_1 x^* z^*}{n^*} + \frac{\alpha_2(1 - \theta)x^* g^*}{n^*} - \phi u^* + (\mu + \gamma)x^*. \quad (278)$$

Substituting equation (278) into equation (277) yields

$$\begin{aligned} V_1' = & \left(\frac{x - x^*}{x} \right) \left\{ \frac{\alpha_1 x^* z^*}{n^*} + \frac{\alpha_2(1 - \theta)x^* g^*}{n^*} - \phi u^* \right\} \\ & + (\mu + \gamma)x^* - \frac{\alpha_1 zx}{n} - \frac{\alpha_2(1 - \theta)x^* g^*}{n} + \phi u^* - (\mu + \gamma)x^*. \end{aligned} \quad (279)$$

Expanding equation (279) yields

$$\begin{aligned} V_1' = & -\frac{\alpha_1}{n^* n x} (x z n^* - x^* z^* n) (x - x^*) - \frac{\alpha_2(1 - \theta)}{n n^* x} (x g n^* - x^* g^* n) (x - x^*) \\ & - \frac{(\mu + \gamma)}{x} (x - x^*)(x - x^*) + \frac{\phi}{x} (u - u^*)(x - x^*). \end{aligned} \quad (280)$$

Further simplification of equation (280) yields

$$\begin{aligned} V_1' = & \left[-\frac{(\mu + \gamma)}{x} (x - x^*)^2 - \frac{\alpha_1}{n^* n x} (x z n^* - x^* z^* n) (x - x^*) \right] \\ & - \left[\frac{\alpha_2(1 - \theta)}{n^* n x} (x g n^* - g^* x^* n) - \frac{\phi}{x} (u - u^*)(x - x^*) \right]. \end{aligned} \quad (281)$$

Equation (281) gives the same result as in equation (275). This completes the proof of Lemma 3.12 $\square$

Lemma 3.13. *Let V_2 be defined as in equation (270), then*

$$V_2' = -\frac{x^*}{u} (u - u^*) + \frac{\gamma x}{u} (u - u^*). \quad (282)$$

Proof. Direct differentiation of equation (270) gives

$$V_2' = \left(\frac{u - u^*}{u} \right) \frac{du}{dt}. \quad (283)$$

Substituting equation (2) into equation (283) yields

$$V_2' = \left(\frac{u - u^*}{u} \right) \{ \gamma x - ((\mu + \phi)(1 - \phi)) u \}. \quad (284)$$

Using equation (198) in equation (284) gives

$$V_2' = -\frac{\gamma x^*}{u}(u - u^*) + \frac{\gamma x}{u}(u - u^*). \quad (285)$$

This completes the proof of Lemma 3.13. $\square$

Lemma 3.14. *Let V_3 be defined as in equation (271), then*

$$V_3' = -\frac{\alpha_2(1-\theta)}{nz^*}g^*x^*(z - z^*) + \frac{\alpha_2(1-\theta)}{nz}gx(z - z^*). \quad (286)$$

Proof. Direct differentiation of equation (271) yields

$$V_3' = \left(\frac{z - z^*}{z} \right) \frac{dz}{dt}, \quad z = y, \quad z^* = y^*. \quad (287)$$

Substituting equation (3) into equation (287) yields

$$V_3' = \left(\frac{z - z^*}{z} \right) \left\{ \frac{\alpha_1 zx}{n} + \frac{\alpha_2(1-\theta)gx^*}{n} - (\mu + \delta_1 + \varphi)z \right\}. \quad (288)$$

Using equation (199) in equation (288) yields

$$V_3' = -\frac{\alpha_2(1-\theta)}{n^*z^*}g^*x^*(z - z^*) + \frac{\alpha_2(1-\theta)}{nz}gx(z - z^*). \quad (289)$$

This completes the proof of Lemma 3.14. $\square$

Lemma 3.15. *Let V_4 be defined as in equation (272), then*

$$V_4' = -\frac{(\mu + \delta_1) + (\mu + \delta_2)}{g^*}z^*(g - g^*) + \frac{(\mu + \delta_1) + (\mu + \delta_1)}{g}(g - g^*). \quad (290)$$

Proof. Direct differentiation of equation (272) yields

$$V_4' = \left(\frac{g - g^*}{g} \right) \frac{dg}{dt}. \quad (291)$$

Substituting equation (4) in equation (291) yields

$$V_4' = \left(\frac{g - g^*}{g} \right) \{(\mu + \delta_1) + (\mu + \delta_2)\} - \theta g. \quad (292)$$

Using equation (200) in equation (292) yields

$$V_4' = -\frac{(\mu + \delta_1) + (\mu + \delta_2)}{g^*}z^*(g - g^*) + \frac{(\mu + \gamma) + (\varphi + \delta_1)}{g}z(g - g^*). \quad (293)$$

This completes the proof of Lemma 3.15. $\square$

Lemma 3.16. *Let V_5 be defined as in equation (273), then*

$$V_5' = -\frac{\omega z^*}{r^*}(r - r^*) + \frac{\omega z}{r}(r - r^*). \quad (294)$$

Proof. Direct differentiation of equation (273) yields

$$V'_5 = \left(\frac{r - r^*}{r} \right) \frac{dr}{dt}. \quad (295)$$

Substituting equation (5) in equation (295) yields

$$V'_5 = \left(\frac{r - r^*}{r} \right) (\omega z - \mu r). \quad (296)$$

Using equation (201) in equation (296) gives

$$V'_5 = -\frac{\omega z^*}{r^*} (r - r^*) + \frac{\alpha z}{r} (r - r^*). \quad (297)$$

This completes the proof of Lemma 3.16. $\square$

We use the lemma by [41] as stated to aid in establishing the proof of Lemma 3.18.

Lemma 3.17. *Let $V_* = \int_0^\infty \Phi(a) G_3(i(t, a), i^*(a)) da$ then*

$$V := \Phi(0) G_3(i(t, 0), i^*(0)) + \int_0^\infty [\Phi'(a) - \theta(a) \Phi(a)] G_3(i(t, a), i^*(a)) da. \quad (298)$$

Lemma 3.18. *Let V_6 be defined as in (274), then*

$$V'_6 = q'(0) \left(1 - \frac{q(t, 0)}{q'(0)} + \ln \frac{q(t, 0)}{q'(0)} \right) + 2 \int_0^\tau \frac{dq'}{dt} \left(1 - \frac{q^*(t, 0)}{q(t, 0)} + \ln \frac{q^*(t, 0)}{q(t, 0)} \right) dt. \quad (299)$$

Proof. Applying the result in Lemma 3.17, we obtain the direct differentiation of equation (274) as

$$V'_6 = q^*(0) G_3(q(t, 0), q^*(0)) + \int_0^\tau \left(\frac{dq^*(\tau)}{dt} - \sigma(\tau) q^*(\tau) \right) G_3(q(t, \tau), q^*(t, \tau)) d\tau. \quad (300)$$

Using $G_2(x, u) = x - u - u \ln \frac{x}{u}$ for $x, u > 0$ in (300) gives

$$= q^*(0) \left(1 - \frac{q(t, 0)}{q^*(0)} + \ln \frac{q(t, 0)}{q^*(0)} \right) + 2 \int_0^\tau \frac{dq^*(\tau)}{dt} \left(1 - \frac{q(t, 0)}{q^*(\tau)} + \ln \frac{q(t, 0)}{q^*(\tau)} \right) dt. \quad (301)$$

Using equation (202) in equation (301) gives

$$V'_6 = q^*(0) \left(1 - \frac{q(t, 0)}{q^*(0)} + \ln \frac{q(t, 0)}{q^*(0)} \right) + 2 \int_0^\tau \frac{dq^*(\tau)}{dt} \left(1 - \frac{q(t, 0)}{q^*(\tau)} + \ln \frac{q(t, 0)}{q^*(\tau)} \right) dt. \quad (302)$$

The addition of equations (281), (285), (289), (293), (297), and (302) takes the form

$$\frac{dV(t)}{dt} = X - U + Z \quad (303)$$

where

$$X = \left[\begin{array}{l} \frac{\phi}{x}(u - u^*)(x - x^*) + \frac{x\gamma}{u}(u - u^*) \\ + \frac{\alpha_2(1 - \theta)gx}{nz}(z - z^*) + \frac{(\mu + \delta_1) + (\mu + \delta_2)}{g}z(g - g^*) + \frac{\omega z}{r}(r - r^*) \end{array} \right]. \quad (304)$$

$$U = \left[\begin{array}{l} \frac{\alpha_1}{nn^*x}(xzn^* - x^*z^*n)(x - x^*) - \frac{\alpha_2(1 - \theta)}{nn^*x}(gxn^* - g^*x^*n)(x - x^*) \\ + \frac{\alpha_2(1 - \theta)}{n^*z^*}g^*x^*(z - z^*) + \frac{\omega z^*}{r^*}(r - r^*) + \frac{(\mu + \delta_1) + (\mu + \delta_2)}{g^*}z^*(g - g^*) + \frac{\mu + \gamma}{x}(x - x^*)^2 \end{array} \right]. \quad (305)$$

$$Z = q^*(0) \left(1 - \frac{q(t, 0)}{q^*(0)} + \ln \frac{q(t, 0)}{q^*(0)} \right) + 2 \int_0^\tau \frac{dq^*(\tau)}{dt} \left(1 - \frac{q(t, \tau)}{q^*(\tau)} + \ln \frac{q(t, \tau)}{q^*(\tau)} \right) d\tau. \quad (306)$$

From equation (306), we see that $Z \leq 0$ since $1 - a + Ina \leq 0$ for all $a > 0$ with equality holding if and only if $a = 1$. From equation (303), if $X < U$ then $\frac{dV(t)}{dt}$ is negative and definite, meaning $\frac{dV(t)}{dt} < 0$. Also it follow that, $\frac{dV(t)}{dt} = 0$ if and only if $x = x^*$, $u = u^*$, $y = y^*$, $g = g^*$, $r = r^*$, $n = n^*$, and $\frac{q(t; 0)}{q^*(0)} = \frac{q(t; \tau)}{q^*(\tau)}$, for all $\tau \geq 0$. Therefore, the largest compact invariant set in $\left\{ (x^*, u^*, y^*, g^*, r^*, q^*(t)) \in \Omega : \frac{dV(t)}{dt} = 0 \right\}$ is the singleton E^* , if $R_{eff} > 1$, then by LaSalles's invariant principle E^* is globally asymptotically stable in Ω if $X < U$. This completes the proof of Lemma 3.18 $\square$

4. RESULTS AND DISCUSSION

In this section, we present results and discuss the effects of sensitive parameters on the effective reproduction number. The sensitivity analysis is extremely important because it tells the researcher the parameter that is most sensitive to disease transmission and which of the parameters should be targeted by intervention strategy. Thus, we normalize forward sensitivity indices with respect to a parameter value of Y as defined in equations (307) and (308)

$$S_Y^{R_{eff}} = \frac{\partial R_{eff}}{\partial Y} \times \frac{Y}{R_{eff}}, \quad (307)$$

$$Y = \{\Lambda, \theta, \varphi, \phi, \alpha_1, \alpha_2, \delta_1, \delta_2, \gamma, \mu\}. \quad (308)$$

The sensitivity indices for parameter values of the effective reproduction number were calculated.

TABLE 2. Parameter Values for Sensitivity Analysis

Variable	Value	Source
S	6,346,564	Calculated
P	12,067	[5]
I	37,692	[5]
G	10,669	[5]
Q	24,764	[5]
R	21,244	[1]
N	6,453,000	[1]
α_1	0.00000197	Calculated
α_2	0.00000148	Calculated
Λ	10,970	Calculated
μ	0.0017	Calculated
δ_1	0.0212	Calculated
δ_1	0.0036	Calculated
δ	0.0248	Calculated
γ	1.429	Calculated
$\theta, \varphi, \phi, k, \tau, T, \omega$	(0–1)	Calculated

The sensitivity analysis was conducted for the cases when the infection age is one month, two months, and three months, respectively. The control parameters are considered at $k = \theta = \varphi = 0.75$. This is because safe burial practices of dead bodies and quarantine with treatment can reduce disease transmission drastically for at least 0.75 and above.

TABLE 3. Sensitivity Indices for one month, two months and three months

Parameter	SA values (1 Month infection)	SA value for (2 months infection)	SA values for (3 Month infection)
θ	$-1.937356130 \times 10^{-14}$	$-1.919568652 \times 10^{-14}$	$-1.919568652 \times 10^{-14}$
ϕ	$-1.499900237 \times 10^{-1}$	-1.499900252×10^0	-1.499900252×10^0
μ	-1.000857222×10^0	-1.000857231×10^0	-1.000857231×10^0
Λ	$9.999989374 \times 10^{-1}$	$9.999989475 \times 10^{-1}$	$9.999989475 \times 10^{-1}$
γ	$2.785375945 \times 10^{-1}$	$2.785375973 \times 10^{-1}$	$2.785375973 \times 10^{-1}$
φ	$1.062415857 \times 10^{-6}$	$1.052661482 \times 10^{-6}$	$1.052661482 \times 10^{-6}$
α_1	$7.772912690 \times 10^{-13}$	$7.701547123 \times 10^{-13}$	$7.701547123 \times 10^{-13}$
δ_1	$2.184678192 \times 10^{-15}$	$2.164619969 \times 10^{-15}$	$2.164619969 \times 10^{-15}$
α_2	$6.987391506 \times 10^{-15}$	$2.879352976 \times 10^{-15}$	$2.879352976 \times 10^{-15}$
δ_2	$3.709830889 \times 10^{-16}$	$3.675769758 \times 10^{-16}$	$3.675769758 \times 10^{-16}$

The values in Table 4 were obtained through simulation experiment on R_{eff} with the aid of Maple codes. Table 3 shows that all the parameters have either positive or negative effects on R_{eff} . Table 4 shows that θ , ϕ and μ have the highest sensitivity index, followed by Λ , γ , φ , α_1 , δ_2 , and α_2 . This is a practical indication

that R_{eff} is an increasing function of Λ , γ , φ , α_1 , δ_2 , and α_2 . Thus, decreasing Λ , γ , φ , α_1 , δ_1 , and α_2 will reduce the spread of EVD and vice versa. Figures 3 to 5 are the graphical simulations of R_{eff} versus sensitive parameters of Λ , γ , φ , and μ .

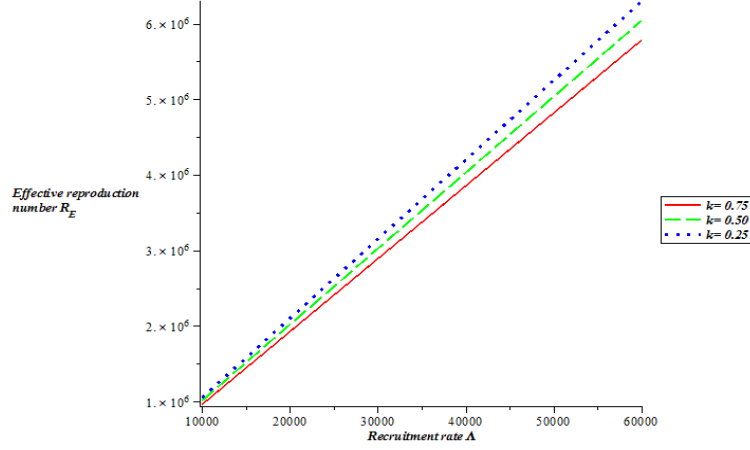


FIGURE 3. Effect of effective reproduction number on recruitment number

Figure 3 shows a linear relationship between the effective reproduction number and the recruitment number when the control measures were 0.75, 0.50 and 0.25 respectively. This clearly shows that R_{eff} is an increasing function of the recruitment number. This agrees with the reality that as people are recruited into the population, they are either susceptible or infected, and if there is no control measure in the endemic regions, the more their chances of being infected or spreading the infection to others.

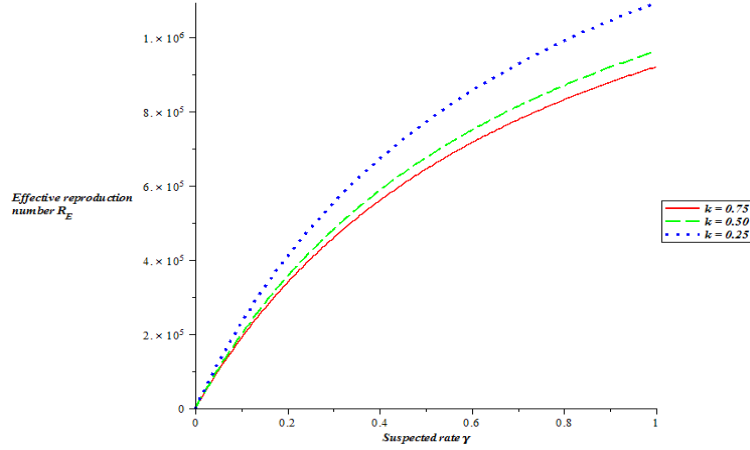


FIGURE 4. Effect of suspected rate on the effective reproduction number

Figure 4, shows a relationship between the effective reproduction number versus the suspect rate of susceptible population when the control measures are 0.75, 0.50 and 0.25 respectively. The effective reproduction number shows a logistic increase as control measures are being placed on 0.25 and continues to decrease from 0.50 to 0.75. This agrees with the reality of EVD transmission dynamics whereby, as control measures increase, there is a lower infection rate, which reduces the number of suspected EVD cases.

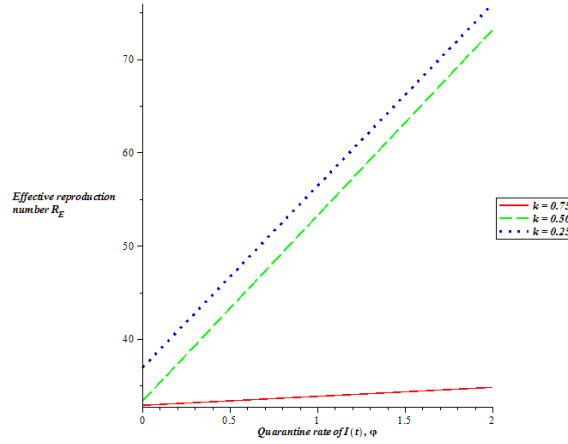


FIGURE 5. Effective reproduction number versus quarantine rate

Figure 5 shows the relationship between the effective reproduction number and quarantine rate of the infected population $I(t)$. The effective reproduction number is a function of the suspected rate of infection. This is a common fact that; the effective reproduction number remains an increasing or a decreasing function of the quarantine rate when the control parameters are peg at $k = 0.25, 0.50, 0.75$

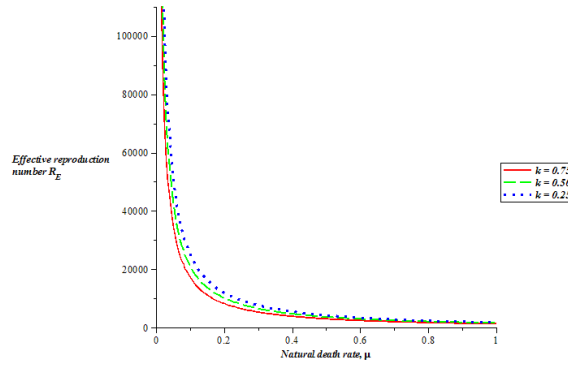


FIGURE 6. Effective reproduction number versus population natural death rate

Figure 6 shows the relationship between the effective reproduction number and the population natural death for different control measures. The simulation shows that the higher the population's natural death, the lower the effective reproduction number. This agrees with the reality of Ebola dynamics.

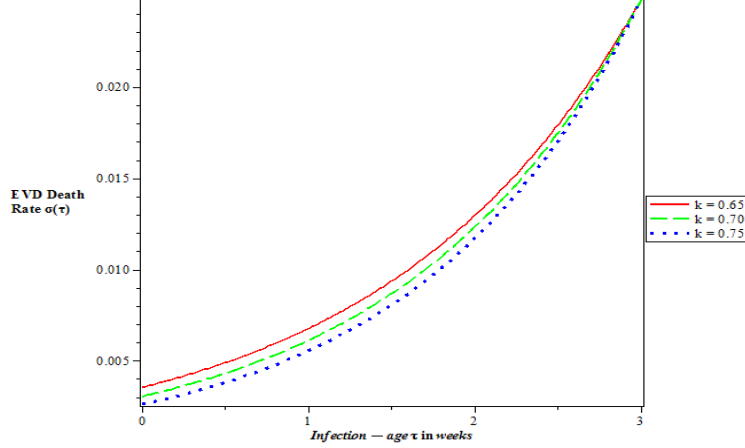


FIGURE 7. EVD induce death rate versus the Infection-age

Figure 7 shows that the death rate due to EVD increases with the infection age and converges to an endemic point. This implies that if the initial infection age has reached its maximum age where $\tau = T$ an infected individual would hardly survive even if the victim is quarantined with effective medical treatment.

5. CONCLUSION

In this research study, an infection-age structure mathematical model for Ebola disease dynamics was developed using ordinary differential equations, partial differential equations and integro-differential equations. We incorporate the effects of safe burial practices and two quarantine compartments which are quarantine under observation $P(t)$ and quarantine with treatment $Q(t)$ as effective control measures. We observe that the estimated equation (70) by classical Gronwall's inequality provides continuity of the solution q with respect to the initial age of infection ψ . Hence, equations (6), (7) and (13) exist and are unique and well posed biologically. We illustrate the existence and uniqueness of model equations (6), (7) and (13) in the norm space of $L^1(0, \tau)$, which shows the biological meaning of an infection-age density $q(t, \tau)$ for the infected population. We equally prove the existence and uniqueness of solutions to the model equations. We show that solutions of the model equations depend continuously on the initial age distributions of EVD. We determine the existence and uniqueness of equations (1) to (5) as given by equations (89) to (93) which are all bounded for each of the absolute

derivation. Figure 7 gives a practical effect of death rate due to EVD and infection-age distributions. We determine the Disease-Free Equilibrium (DFE) and Endemic Equilibrium (EE) state of the model. The effective reproduction number (R_{eff}) of the model was obtained and used to determine the local and global stability of DFE, giving the condition for DFE to be locally asymptotically stable if $(R_{eff}) < 1$. and unstable if $(R_{eff}) > 1$. A Suitable Lyapunov function was constructed and used to analyze the global stability of DFE, it was shown that the DFE is globally asymptotically stable if $(R_{eff}) \leq 1$ while unstable if $(R_{eff}) > 1$.

The model is oriented to inform public health epidemiologists by simulating outbreak dynamics, evaluating the effectiveness of interventions like safe burial practices and multiple quarantine stages, and identifying critical parameters for the control of EVD. The model is crucial for understanding the complex transmission cycles of EVD, including factors like transmission from deceased individuals and contaminated environments. The incorporation of safe burial practices of EVD victims in the model is a vital public health intervention that must be carried out properly by trained personnel in a way that is also sensitive to local cultural and religious beliefs to effectively control the spread of Ebola. The result of mathematical model analysis has shown that multiple quarantine stages combined with effective medical treatment for EVD are important for disease containment, improved patient survival rates, better resource management, and building community trust. In essence, moving suspected individuals of Ebola virus infection from quarantine under observation $P(t)$ to quarantine with treatment $Q(t)$ upon confirmation of infection, the policy prevents an infected person from transmitting the virus to others in the community. Optimally, infection age with quarantine and effective medical treatment allow more targeted, effective, and resource-efficient interventions. The model approach enhances public health control measures in several key ways, including precise and case-evidence-based strategy, which improves individual patient care and strengthens overall public health trust.

Acknowledgement. We would like to thank all the authors, whose works are cited in this research studies; and we also wish to acknowledge the support we received from Microbiology Laboratory, Postgraduate School, Federal University of Technology, Minna, Niger State, Nigeria.

REFERENCES

- [1] E. Azuaba and N. I. Akinwande, "Analytical solution of the mathematical model of ebola disease dynamics incorporating infection-age structure in the quarantined compartment with treatment," *Journal of the Nigerian Association of Mathematical Physics*, vol. 45, no. 3, pp. 369–378, 2018. https://www.researchgate.net/publication/348192303_Analytical_Solution_of_the_Mathematical_Model_of_Ebola_Disease_Dynamics_Incorporating_Infection-Age_Structure_in_the_Quarantined_Compartment_with_Treatment.
- [2] Centre for Disease Control and Prevention (CDC), "Ebola monthly epidemiological documentary," 2018. <http://www.cdc/africa/disease/ebola/reports2018>.

- [3] C. Adebamowo, O. Bah-Sow, and F. Binka, "Randomized controlled trials for ebola: practical and ethical issues," *Lancet*, vol. 384, pp. 1423–1424, 2014. [https://doi.org/10.1016/S0140-6736\(14\)61734-7](https://doi.org/10.1016/S0140-6736(14)61734-7).
- [4] C. I. Kamalu, N. S. Dozie, P. Oghome, M. S. Nwakaudu, F. N. Uzondo, and J. C. Obijiagu, "S-i-s compartmental modeling and predictive analysis of ebola outbreak in central and west africa," *World Journal of Engineering Research and Technology*, vol. 2, no. 3, pp. 13–36, 2016. https://www.wjert.org/home/article_abstract/57.
- [5] World Health Organization (WHO), "Ebola response team, ebola virus disease in west africa - the first 9 months of the epidemic and forward projections," *New Engl. J. Med.*, vol. 371, pp. 1481–1495, 2014. https://www.msf.org/drc-ebola-outbreaks?gad_campaignid=23265353485&gad_source=1&gbraid=0AAAAADs2cdbk1_93si2V0-Sb2A_CyldYv.
- [6] World Health Organization, "Ebola situation report - 4 september 2025/ebola." <http://apps.who.int/ebola/en/current-situation/ebola-situation-report-4-september-2025>, 2025. Geneva: World Health Organization.
- [7] D. G. Bausch, J. S. Towner, S. F. Dowell, F. Kaducu, M. Lukwiya, and A. Sanchez, "Assessment of the risk of ebola virus transmission from bodily fluids and fomites," *Journal of Infectious Diseases*, vol. 196, no. 2, pp. 142–172, 2007. 10.2143/journals.2006.18.01794-0942.
- [8] S. H. Lee-Kwan, R. C. Locsin, and M. Nanyonga, "Support services for survivors of ebola virus disease - sierra leone," *Morbidity and Mortality Weekly Report (MMWR)*, vol. 6, no. 3, pp. 6–1205, 2014. <https://pmc.ncbi.nlm.nih.gov/articles/PMC5779533/>.
- [9] R. T. Abah, N. I. Akinwande, I. A. Enagi, F. Kuta, S. Abdulrahman, and S. A. Somma, "Stability analysis of the disease-free equilibrium state of a mathematical model of ebola fever diseases epidemic," *International Journal of Innovation in Science and Mathematics*, vol. 3, no. 2, pp. 118–123, 2015. <http://irepo.futminna.edu.ng:8080/jspui/handle/123456789/11557>.
- [10] R. I. Gweryina, A. R. Kimbir, and T. Aboiyar, "Numerical study and sensitivity analysis of ebola model with multi-intervention strategies in a heterogenous population," *Asian Research Journal of Mathematics*, vol. 10, no. 3, pp. 1–19, 2018. <https://doi.org/10.9734/ARJOM/2018/43921>.
- [11] C. E. Madubueze, A. R. Kimbir, and T. Aboiyar, "Global stability of ebola virus disease model with contact tracing and quarantine," *Applied and Applied Mathematics*, vol. 13, no. 1, pp. 382–403, 2018. https://www.researchgate.net/publication/325925372_Global_Stability_of_Ebola_Virus_Disease_Model_with_Contact_Tracing_and_Quarantine.
- [12] R. M. Anderson and R. M. May, *Infectious diseases of humans, dynamics and control*. Oxford: Oxford University, 1991. <https://doi.org/10.1017/S0950268800059896>.
- [13] C. Castillo-Chavez and F. Brauer, *Mathematical Models in Population Biology and Epidemiology*. New York: Springer, 2001. <https://doi.org/10.1007/978-1-4614-1686-9>.
- [14] D. Greenhalgh, "Analytical results on the stability of age-structured recurrent epidemic models," *J. Math. Appl. Med. Biol.*, vol. 4, pp. 109–144, 1987. [https://doi.org/10.1016/0040-5809\(88\)90016-0](https://doi.org/10.1016/0040-5809(88)90016-0).
- [15] D. Greenhalgh, "Vaccination campaigns for common childhood diseases," *Math. Biosci.*, vol. 100, pp. 201–240, 1990. [https://doi.org/10.1016/0025-5564\(90\)90040-6](https://doi.org/10.1016/0025-5564(90)90040-6).
- [16] D. Greenhalgh, "Existence, threshold and stability results for an age-structured epidemic model with vaccination and a non-separable transmission coefficient," *International Journal of Information Systems in Science*, vol. 244, pp. 641–668, 1993. <https://doi.org/10.1093/imammb/5.2.81>.
- [17] V. Andreasen, "Disease regulation of age-structured host population theorem," *Population*, 1989. [https://doi.org/10.1016/0040-5809\(89\)90031-2](https://doi.org/10.1016/0040-5809(89)90031-2).
- [18] V. Andreasen, "The effect of age-dependent host mortality on the dynamics of an endemic disease," *Mathematical Biosciences*, vol. 114, pp. 29–58, 1993. <https://thiele.ruc.dk/imfufatekster/pdf/233.pdf>.
- [19] K. Dietz, "Transmission and control of arbovirus disease," in *Epidemiology* (D. Ludwig and K. L. Cooke, eds.), vol. 28, pp. 104–121, Philadelphia: Society for Industrial and

- Applied Mathematics (SIAM), 2 ed., 1975. https://www.researchgate.net/publication/265366109_Transmission_and_control_of_arbovirus_diseases.
- [20] K. Dietz and D. Schenzle, "Proportionate mixing models for age-dependent infection transmission," *Journal of Mathematical Biology*, vol. 22, pp. 117–120, 1985. <https://doi.org/10.1007/bf00276550>.
 - [21] D. W. Tudor, "An age-dependent epidemic model with application to measles," *Math. Biosci.*, vol. 73, pp. 131–147, 1985. [https://doi.org/10.1016/0025-5564\(85\)90081-1](https://doi.org/10.1016/0025-5564(85)90081-1).
 - [22] H. W. Hethcote, "The mathematics of infectious disease," *SIAM Review*, vol. 42, no. 4, pp. 599–653, 2000. <https://doi.org/10.1137/S0036144500371907>.
 - [23] H. Inaba, "Threshold and stability results for an age-structured epidemic model," *J. Math. Biol.*, vol. 28, pp. 411–434, 1990. <https://doi.org/10.1007/BF00178326>.
 - [24] N. I. Akinwande, "A mathematical model of the chaotic dynamics of the aids disease pandemic," *Journal of Nigerian Mathematical Society*, vol. 24, pp. 8–16, 2005. <http://irepo.futminna.edu.ng:8080/jspui/bitstream/123456789/20186/1/OCRM.TECH%20MATHS2280.pdf>.
 - [25] N. I. Akinwande, "A mathematical model of the dynamics of the hiv/aids disease pandemic," 2006. <http://irepo.futminna.edu.ng:8080/jspui/bitstream/123456789/20186/1/OCRM.TECH%20MATHS2280.pdf>.
 - [26] T. T. Ashezua, N. I. Akinwande, S. Abdulrahman, R. O. Olayiwola, and F. A. Kuta, "Local stability analysis of an infection-age mathematical model for tuberculosis disease dynamics," *Journal of Applied Sciences and Environmental Management*, vol. 19, no. 4, pp. 665–669, 2015. <https://doi.org/10.4314/jasem.v19i4.14>.
 - [27] W. Jinliang, Z. Ran, and K. Toshikazu, "Mathematical analysis for an age-structured hiv infection model with saturation infection rate," *Electronic Journal of Differential Equations*, vol. 33, pp. 1–19, 2015. https://www.researchgate.net/publication/271907813_Mathematical_analysis_for_an_age-structured_HIV_infection_model_with_saturation_infection_rate.
 - [28] P. W. Nelson, M. A. Gilchrist, D. Coombs, J. M. Hyman, and A. S. Perelson, "An age-structured model of hiv infection that allows for variations in the production rate of viral particles and the death rate of productively infected cells," *Math. Biosci. Engr.*, vol. 1, no. 2, pp. 267–288, 2004. <https://doi.org/10.3934/mbe.2004.1.267>.
 - [29] G. Huang, X. Liu, and Y. Takeuchi, "Lyapunov functions and global stability for age structured hiv infection model," *SIAM J. Appl. Math.*, vol. 72, no. 1, pp. 25–38, 2012. <https://doi.org/10.1137/110826588>.
 - [30] R. Grimshaw, *Nonlinear Ordinary Differential Equations*. Oxford, London, Edinburgh, Boston, Melbourne: Blackwell Scientific Publications, 1990. <https://doi.org/10.1201/9780203745489>.
 - [31] I. Mimmo and M. Fabio, *The Basic Approach to Age-Structured Population Dynamics*. Berlin: Springer, 2 ed., 2010. <https://doi.org/10.1007/978-94-024-1146-1>.
 - [32] I. Mimmo and M. Fabio, "Lecture notes on mathematical modelling in the life sciences: The basic approach to age-structured population dynamics," 2017. Retrieved from www.springer-ebook/eletronic/doi10.1007/978-94-024-1146-1.
 - [33] S. Anita, *Analysis and Control of Age-Dependent Population Dynamics*. Dordrecht, Boston, London: Kluwer Academic Publishers, 2000. https://doi.org/10.1007/978-94-015-9436-3_2.
 - [34] G. F. Webb, *Theory of Nonlinear Age-Dependent Population Dynamics*. 270 Madison Avenue, New York: Marcel Dekker, Inc., 1985. <https://www.scirp.org/reference/referencespapers?referenceid=1571270>.
 - [35] M. El-Doma, "Analysis of an sirs age-structured epidemic model with vaccination and vertical transmission of disease," *Applications and Applied Mathematics: An International Journal (AAM)*, vol. 1, no. 1, pp. 36–61, 2006. <https://digitalcommons.pvamu.edu/aam/vol1/iss1/4/>.

- [36] Z. Shuai and P. Van Den Driessche, “Global stability of infectious disease models using lyapunov functions,” *J. Appl. Math.*, vol. 73, pp. 1513–1532, 2013. <https://doi.org/10.1137/120876642>.
- [37] A. L. Olutimo, O. J. Oni, and O. C. Ayeni, “Ultimate boundedness result for a certain system of third order nonlinear differential equations,” *Advances in Mathematics: Scientific Journal*, no. 2, pp. 959–970, 2021. <https://doi.org/10.37418/Amsj.10.2.25>.
- [38] J. Medlock, “Mathematical analysis of epidemiological models,” *Journal of Emerging Infectious Diseases*, vol. 11, no. 9, pp. 1355–1362, 2010. http://dimacs.rutgers.edu/Workshops/ASIEconEpi/Slides/Intro_to_Math_Epi_II.pdf.
- [39] J. Wang and T. Zhang, “Mathematical analysis for an age-structured hiv infection model with saturation infection rate,” *Electronic Journal of Differential Equations*, vol. 33, pp. 1–19, 2016. Retrieved from <http://www.ejde.math.unt.edu/tpejde.math.txstate.edu>.
- [40] I. D. Omoko, O. Oni, T. G. Alabi, and M. B. Taoreed, “Dynamical models in intrinsic lyapunov methods,” *Journal of Applied Mathematics and Physics*, vol. 13, no. 4, pp. 1352–136, 2025. <https://doi.org/10.4236/jamp.2025.134073>.
- [41] F. Brauer, Z. Shuai, and P. Van Den Driessche, “Dynamics of an age of infection cholera model,” *Mathematical Biosciences and Engineering*, vol. 10, no. 6, pp. 1335–1349, 2013. <https://doi.org/10.3934/mbe.2013.10.1335>.