ON THE SUPER EDGE-MAGIC DEFICIENCY AND $\alpha$-VALUATIONS OF GRAPHS

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Abstract. A graph $G$ is called super edge-magic if there exists a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(u) + f(v) + f(uv)$ is a constant for each $uv \in E(G)$ and $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. The super edge-magic deficiency, $\mu_s(G)$, of a graph $G$ is defined as the smallest nonnegative integer $n$ with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer $n$. In this paper, we prove that if $G$ is a graph without isolated vertices that has an $\alpha$-valuation, then $\mu_s(G) \leq |E(G)| - |V(G)| + 1$. This leads to $\mu_s(G) = |E(G)| - |V(G)| + 1$ if $G$ has the additional property that $G$ is not sequential. Also, we provide necessary and sufficient conditions for the disjoint union of isomorphic complete bipartite graphs to have an $\alpha$-valuation. Moreover, we present several results on the super edge-magic deficiency of the same class of graphs. Based on these, we propose some open problems and a new conjecture.

Key words: Super edge-magic labeling, super edge-magic deficiency, sequential labeling, sequential number, $\alpha$-valuation.
Abstrak. Suatu graf $G$ disebut sisi-ajaib super jika terdapat sebuah fungsi bijektif $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ sedemikian sehingga $f(u) + f(v) + f(uv)$ adalah sebuah konstanta untuk tiap $uv \in E(G)$ dan $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. Defisiensi sisi-ajaib super, $\mu_s(G)$, dari sebuah graf $G$ didefinisikan sebagai bilangan bulat non negatif terkecil $n$ dengan sifat yaitu graf $G \cup nK_1$ adalah sisi-ajaib super atau $+\infty$ jika tidak terdapat bilangan bulat $n$ yang demikian.

Pada paper ini, kami membuktikan bahwa jika $G$ adalah sebuah graf tanpa titik terisolasi yang mempunyai sebuah nilai-$\alpha$, maka $\mu_s(G) \leq |E(G)| - |V(G)| + 1$. Hal ini menghasilkan $\mu_s(G) = |E(G)| - |V(G)| + 1$ jika $G$ mempunyai sifat tambahan yaitu $G$ adalah tidak berurutan. Kami juga memberikan syarat perlu dan cukup untuk gabungan disjoin dari graf bipartit lengkap isomorfik untuk mempunyai sebuah nilai-$\alpha$. Lebih jauh, kami menunjukkan beberapa hasil pada defisiensi sisi-ajaib dari kelas graf yang sama. Berdasarkan hal-hal tersebut, kami mengusulkan beberapa masalah terbuka dan sebuah konjektur baru.

Kata kunci: Pelabelan sisi-ajaib super, defisiensi sisi-ajaib super, pelabelan secara berurutan, bilangan secara berurutan, nilai-$\alpha$.

1. Introduction

The notation and terminology of this paper will generally follow closely that of [4]. All graphs considered here are finite, simple and undirected. The vertex set of a graph $G$ is denoted by $V(G)$, while the edge set is denoted by $E(G)$. A complete bipartite graph with partite sets $X$ and $Y$, where $|X| = s$ and $|Y| = t$, is denoted by $K_{s,t}$. For any graph $G$, the graph $mG$ denotes the disjoint union of $m$ copies of $G$. For two integers $a$ and $b$ with $b \geq a$, the set $\{x \in \mathbb{Z} | a \leq x \leq b\}$ will be denoted by simply writing $[a,b]$, where $\mathbb{Z}$ denotes the set of all integers.

The first paper in edge-magic labelings was published in 1970 by Kotzig and Rosa [20], who called these labelings: magic valuations; these were later rediscovered by Ringel and Lladó [22], who coined one of the now popular terms for them: edge-magic labelings. More recently, they have also been referred to as edge-magic total labelings by Wallis [24]. For a graph $G$ of order $p$ and size $q$, a bijective function $f : V(G) \cup E(G) \rightarrow [1, p + q]$ is called an edge-magic labeling of $G$ if $f(u) + f(v) + f(uv)$ is a constant $k$ (called the valence of $f$) for each $uv \in E(G)$. If such a labeling exists, then $G$ is called an edge-magic graph. In 1998, Enomoto et al. [5] defined an edge-magic labeling $f$ of a graph $G$ to be a super edge-magic labeling if $f$ has the additional property that $f(V(G)) = [1,p]$. Thus, a graph possessing a super edge-magic labeling is a super edge-magic graph. Lately, super edge-magic labelings and super edge-magic graphs are called by Wallis [24] strong edge-magic total labelings and strongly edge-magic graphs, respectively.

The following lemma taken from [6] provides necessary and sufficient conditions for a graph to be super edge-magic.
Lemma 1.1. A graph $G$ of order $p$ and size $q$ is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow [1, p]$ such that the set

$$S = \{ f(u) + f(v) \mid uv \in E(G) \}$$

consists of $q$ consecutive integers. In such a case, $f$ extends to a super edge-magic labeling of $G$ with valence $k = p + q + s$, where $s = \min(S)$ and

$$S = [k - (p + q), k - (p + 1)].$$

For every graph $G$, Kotzig and Rosa [20] proved that there exists an edge-magic graph $H$ such that $H \cong G \cup nK_1$ for some nonnegative integer $n$. This motivated them to define the edge-magic deficiency of a graph. The edge-magic deficiency, $\mu(G)$, of a graph $G$ is the smallest nonnegative integer $n$ for which the graph $G \cup nK_1$ is edge-magic. Motivated by the concept of edge-magic deficiency, Figueroa-Centeno et al. [10] analogously defined the super edge-magic deficiency of a graph. The super edge-magic deficiency, $\mu_s(G)$, of a graph $G$ is either the smallest nonnegative integer $n$ with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer $n$. Thus, the super edge-magic deficiency of a graph $G$ is a measure of how close $G$ is to being super edge-magic.

An alternative term exists for the super edge-magic deficiency, namely, the vertex dependent characteristic. This term was coined by Hedge and Shetty [16]. In [16], they gave a construction of polygons having same angles and distinct sides using the result on the super edge-magic deficiency of cycles provided in [10].

In 1967, Rosa [23] initiated the study of $\beta$-valuations. They were later studied by Golomb [14], who called them graceful labelings, which is the term used in the current literature of graph labelings. A graph $G$ of size $q$ is called graceful if there exists an injective function $f : V(G) \rightarrow [0, q]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. Such a function is called a graceful labeling. In [23], Rosa also introduced the notion of $\alpha$-valuations stemming from his interest in graph decompositions. A graceful labeling $f$ is called an $\alpha$-valuation if there exists an integer $\lambda$ (called the critical value of $f$) so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$. Moreover, he pointed out that a graph that admits an $\alpha$-valuation $f$ is necessarily bipartite and has the partite sets $\{v \in V(G) \mid f(v) \leq \lambda\}$ and $\{v \in V(G) \mid f(v) > \lambda\}$.

The notion of sequential graphs was introduced by Grace [15]. He defined a graph $G$ of size $q$ to be sequential if there exists an injective function $f : V(G) \rightarrow [0, q - 1]$ (with the label $q$ allowed if $G$ is a tree) such that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting set of edge labels is $[m, m + q - 1]$ for some positive integer $m$. Such a function is called a sequential labeling.

We now consider a concept that is somehow related to the super edge-magic deficiency of graphs without isolated vertices as well as $\alpha$-valuations and sequential labelings. The notion of the sequential number was recently introduced by Figueroa-Centeno and Ichishima [11]. The sequential number, $\sigma(G)$, of a graph $G$ of size $q$ without isolated vertices is defined to be either the smallest positive integer $n$ for which it is possible to label the vertices of $G$ with distinct elements from the set $[0, n]$ in such a way that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and
the resulting edge labels are $q$ consecutive integers or $+\infty$ if there exists no such integer $n$. Thus, the sequential number of a graph $G$ is a measure of how close $G$ is to being sequential.

Figueroa-Centeno and Ichishima [11] found the following formula for the sequential number of a graph without isolated vertices in terms of its super edge-magic deficiency and order. As a consequence of this theorem, they also determined the exact value of the super edge-magic deficiency of the complete bipartite graph, which is stated in the succeeding corollary. These will later serve as the bases for some remarks and a new conjecture.

**Theorem 1.2.** If $G$ is a graph of order $p$ without isolated vertices, then
\[ \sigma(G) = \mu_s(G) + p - 1. \]

Due to Theorem 1.2, the sequential number plays an important role in the study of super edge-magic deficiency of a graph without isolated vertices.

**Corollary 1.3.** For all integers $s$ and $t$ with $s \geq 2$ and $t \geq 2$,
\[ \mu_s(K_{s,t}) = (s - 1)(t - 1). \]

In this paper, we prove that if $G$ is a graph of order $p$ and size $q$ without isolated vertices that has an $\alpha$-valuation, then $\mu_s(G) \leq q - p + 1$. Additionally, if $G$ is not sequential, then $\mu_s(G) = q - p + 1$. Also, we provide necessary and sufficient conditions for the disjoint union of isomorphic complete bipartite graphs to have an $\alpha$-valuation. Moreover, we present several results on the super edge-magic deficiency of the same class of graphs. These lead to some open problems and a new conjecture.

The survey by Gallian [12] on graph labeling problems is an excellent source of additional information. More information on super edge-magic graphs and related subjects can be found in the books by Bača and Miller [2], and Wallis [24].

2. Main Results

Our goal of this section is to establish a general formula for the super edge-magic deficiency of graphs without isolated vertices that have $\alpha$-valuations, but not sequential. To achieve this, we start with the following result.

**Theorem 2.1.** If $G$ is a graph of order $p$ and size $q$ without isolated vertices that has an $\alpha$-valuation, then
\[ \mu_s(G) \leq q - p + 1. \]

**Proof.** First, assume that $G$ is a graph of size $q$ without isolated vertices that has an $\alpha$-valuation $f$ with critical value $\lambda$. Then $G$ is bipartite and has the partite sets
\[ X = \{ x \in V(G) \mid f(x) \leq \lambda \} \quad \text{and} \quad Y = \{ y \in V(G) \mid f(y) > \lambda \}. \]

Next, define the vertex labeling $g : V(G) \to [0,q]$ such that
\[ g(v) = \begin{cases} f(v), & \text{if } v \in X; \\
\lambda + q + 1 - f(v), & \text{if } v \in Y. \end{cases} \]
Now, notice that $g(X) \subseteq [0, \lambda]$ and $g(Y) \subseteq [\lambda + 1, q]$.

This implies that $g$ is an injective function and

$$g(x) + g(y) = \lambda + q + 1 - (f(y) - f(x))$$

for each $xy \in E(G)$, where $x \in X$ and $y \in Y$. Thus,

$$\lambda + 1 \leq g(x) + g(y) \leq \lambda + q$$

since $1 \leq f(y) - f(x) \leq q$. Finally, notice that since $f$ is an $\alpha$-valuation of $G$, it follows that

$$\{ f(y) - f(x) | x \in X \text{ and } y \in Y \} = [1, q],$$

implying that $\{ g(x) + g(y) | xy \in E(G) \}$ is a set of $q$ consecutive integers. This implies that $\sigma(G) \leq q$; hence, it follows from Theorem 1.2 that $\mu_s(G) \leq q - p + 1$. □

If $G$ is a graph of order $p$ and size $q$ without isolated vertices that is not sequential, then it is clearly true that $\sigma(G) \geq q$. Thus, it follows from Theorem 1.2 that $\mu_s(G) \geq q - p + 1$. Combining this with Theorem 2.1, we have the following result.

**Corollary 2.2.** If $G$ is a graph of order $p$ and size $q$ without isolated vertices that has an $\alpha$-valuation and is not sequential, then

$$\mu_s(G) = q - p + 1.$$  

3. ON THE DISJOINT UNION OF COMPLETE BIPARTITE GRAPHS

In this section, we study the super edge-magic deficiency of the disjoint union of isomorphic complete bipartite graphs. To do this, we first present necessary and sufficient conditions for such graphs to have an $\alpha$-valuation.

Rosa [23] observed that all complete bipartite graphs have $\alpha$-valuations. This result is now extended in the following theorem.

**Theorem 3.1.** Let $m$, $s$ and $t$ be integers with $m \geq 1$, $s \geq 2$ and $t \geq 2$. Then the graph $mK_{s,t}$ has an $\alpha$-valuation if and only if $(m, s, t) \neq (3, 2, 2)$.

**Proof.** For every two positive integers $s$ and $t$, the complete bipartite graph $K_{s,t}$ has shown to admit an $\alpha$-valuation by Rosa [23]. Also, Abrham and Kotzig [1] have proved that $m = 3$ is the only integer such that the 2-regular graph $mC_4 \cong mK_{2,2}$ does not have an $\alpha$-valuation. Thus, it suffices to show that for all integers $m$, $s$ and $t$ such that $m \geq 2$ and $t > s \geq 2$ except $(m, s, t) = (3, 2, 2)$, there exists an $\alpha$-valuation of $mK_{s,t}$. Let $mK_{s,t}$ have partite sets $X = \bigcup_{i=1}^m X_i$ and $Y = \bigcup_{i=1}^m Y_i$, where $X_i = \{ x_{i,j} | i \in [1, m] \text{ and } j \in [1, s] \}$ and $Y_i = \{ y_{i,j} | i \in [1, m] \text{ and } j \in [1, t] \}$ are the partite sets of the $i$-th component of $mK_{s,t}$. Then define the vertex labeling $f : V(mK_{s,t}) \to [0, mst]$ such that

$$f(x_{i,j}) = (s + 1)(i - 1) - 2 + j,$$
if \( i \in [1, m] \) and \( j \in [1, s] \); and

\[
f(y_{i,j}) = mst - 1 - (st - s - 1)i + s(j - 1),
\]

if \( i \in [1, m] \) and \( j \in [1, t] \).

To show that \( f \) is indeed an \( \alpha \)-valuation of \( mK_{s,t} \), notice first that for each \( i \in [1, m] \),

\[
f(X_i) = \{a_i, a_i + 1, \ldots, a_i + s - 1\}
\]
is a sequence of \( s \) consecutive integers, and

\[
f(Y_j) = \{b_j, b_j + s, \ldots, b_j + s(t - 1)\}
\]
is an arithmetic progression with \( t \) terms and common difference \( s \), where \( a_i = (s + 1)(i - 1) \) and \( b_i = mst - 1 - (st - s - 1)i \). Now, it follows that not only \( f(X_i) \neq f(X_j) \) for \( i \neq j \) and \( f(Y_k) \neq f(Y_l) \) for \( k \neq l \), but also \( f(X_i) \neq f(Y_j) \) for \( i \neq j \). Moreover, it follows that

\[
f(X) \subseteq [a_1, a_m + s - 1] \text{ and } f(Y) \subseteq [b_1, b_1 + s(t - 1)]
\]
or, equivalently,

\[
f(X) \subseteq [0, m(s + 1) - 2] \text{ and } f(Y) \subseteq [m(s + 1) - 1, mst].
\]

This implies that \( f \) is an injective function. Finally, notice that for each \( i \in [1, m] \), the induced edge labels in the \( i \)-th component of \( mK_{s,t} \) are \( st \) consecutive integers of the set

\[
[b_i - a_i, b_i - a_i + st - 1] = [(m - i)st + 1, (m - i + 1)st].
\]

Thus, the induced edge labels are precisely \([1, mst]\). Therefore, \( f \) is an \( \alpha \)-valuation of \( mK_{s,t} \) with critical value \( m(s + 1) - 2 \).

An illustration of Theorem 3.1 is given in Figure 1 for \( m = 2, s = 3 \) and \( t = 4 \).

The remaining part of this section contains results on the super edge-magic deficiency of the graph \( mK_{s,t} \).

We first consider the super edge-magic deficiency of the forest \( mK_{1,n} \). For all positive integers \( m \) and \( n \) such that \( m \) is odd, Figueroa-Centeno et al. [8] have shown that \( \mu_s(mK_{1,n}) = 0 \). When \( m \) is even, we only know that \( \mu_s(mK_{1,1}) = 1 \) for \( m \geq 2 \) (see [10]), and \( \mu_s(mK_{1,2}) = 0 \) for \( m \geq 4 \) (see [3]). Thus, the only instance that needs to be settled is when \( m \) is even and \( n \geq 2 \). For this, we have found the following result.

**Theorem 3.2.** For all positive integers \( m \) and \( n \) such that \( m \) is even,

\[
\mu_s(mK_{1,n}) \leq 1.
\]

**Proof.** Let \( F \cong mK_{1,n} \cup K_1 \) be the forest with

\[
V(F) = \{x_i | i \in [1, m]\} \cup \{y_{i,j} | i \in [1, m] \text{ and } j \in [1, n]\} \cup \{z\}
\]

and

\[
E(F) = \{x_i y_{i,j} | i \in [1, m] \text{ and } j \in [1, n]\},
\]

and consider two cases.
Case 1: For \( m = 2 \), define the vertex labeling \( f : V(F) \rightarrow [1, 2n + 3] \) such that
\[
f(x_i) = 2n + 5 - 2i, \quad \text{if } i \in [1, 2];
f(y_{i,j}) = i + 2j - 2, \quad \text{if } i \in [1, 2] \text{ and } j \in [1, n];
\]
and
\[
f(z) = 2n + 2. \quad \text{Notice then that}
\]
\[
\{ f(y_{i,j}) \mid i \in [1, 2] \text{ and } j \in [1, n] \} = [1, 2n]
\]
and
\[
\{ f(x_1), f(x_2), f(z) \} = [2n + 1, 2n + 3],
\]
which implies that \( f \) is a bijective function. Notice also that
\[
\{ f(x_1) + f(y_{i,j}) \mid j \in [1, n] \} = \{ 2n + 2 + 2j \mid j \in [1, n] \}
\]
and
\[
\{ f(x_2) + f(y_{2,j}) \mid j \in [1, n] \} = \{ 2n + 1 + 2j \mid j \in [1, n] \},
\]
implying that
\[
\{ f(u) + f(v) \mid uv \in E(F) \} = [2n + 3, 4n + 2]
\]
is a set of \( 2n \) consecutive integers. Thus, by Lemma 1.1, \( f \) extends to a super edge-magic labeling of \( F \) with valence \( 6n + 6 \).

Case 2: For \( m = 2k \), where \( k \) is an integer with \( k \geq 2 \), define the vertex labeling \( f : V(F) \rightarrow [1, 2kn + 2k + 1] \) such that
\[
f(x_i) = \begin{cases} 2k(n + 1) + 3 - 2i, & \text{if } i \in [1, k]; \\ 2k(n + 2) + 2 - 2i, & \text{if } i \in [k + 1, 2k]; \end{cases}
\]
\[
f(y_{i,j}) = \begin{cases} i + k(j - 1), & \text{if } i \in [1, k] \text{ and } j \in [1, n]; \\ i + k(n + j - 2) + 1, & \text{if } i \in [k + 1, 2k] \text{ and } j \in [1, n]; \end{cases}
\]
and
\[
f(z) = kn + 1. \quad \text{Notice then that}
\]
\[
\{ f(y_{i,j}) \mid i \in [1, k] \text{ and } j \in [1, n] \} \cup \{ f(z) \} = [1, kn + 1],
\]
\[
\{ f(y_{i,j}) \mid i \in [k + 1, 2k] \text{ and } j \in [1, n] \} = [kn + 2, 2kn + 1],
\]
and
\[
\{ f(x_i) \mid i \in [1, 2k] \} = [2kn + 2, 2kn + 2k + 1],
\]
which implies that \( f \) is a bijective function. Notice also that
\[
\{ f(x_i) + f(y_{i,j}) \mid i \in [1, k] \text{ and } j \in [1, n] \} = [2kn + k + 3, 3kn + k + 2]
\]
and
\[
\{ f(x_i) + f(y_{i,j}) \mid i \in [k + 1, 2k] \text{ and } j \in [1, n] \} = [3kn + k + 3, 4kn + k + 2],
\]
implying that
\[
\{ f(u) + f(v) \mid uv \in E(F) \} = [2kn + k + 3, 4kn + k + 2]
\]
is a set of \( 2kn \) consecutive integers. Thus, by Lemma 1.1, \( f \) extends to a super edge-magic labeling of \( F \) with valence \( 6kn + 3k + 4 \).

Therefore, we conclude that \( \mu_s(mK_{1,n}) \leq 1 \) for all positive integers \( m \) and \( n \) such that \( m \) is even. \( \square \)
Ivančo and Lučkanícová [18] proved that the forest $K_{1,m} \cup K_{1,n}$ is super edge-magic if and only if either $m$ is a multiple of $n+1$ or $n$ is a multiple of $m+1$. Thus, $\mu_s (2K_{1,n}) \geq 1$ for every positive integer $n$. Combining this with Theorem 3.2, we obtain the following result.

**Corollary 3.3.** For every positive integer $n$, 

$$\mu_s (2K_{1,n}) = 1.$$ 

The previous result supports the validity of the conjecture of Figueroa-Centeno et al. [9] that if $F$ is a forest with two components, then $\mu_s (F) \leq 1$.

Ringel and Lladó [22] proved that a graph of order $p$ and size $q$ is not edge-magic if $q$ is even, $p+q \equiv 2 \pmod{4}$ and each vertex has odd degree. This together with Theorem 3.2 leads us to conclude the following result.

**Corollary 3.4.** For all positive integers $m$ and $n$ such that $m \equiv 2 \pmod{4}$ and $n$ is odd, 

$$\mu_s (mK_{1,n}) = 1.$$ 

Our final result on the super edge-magic deficiency of forests concerns $mK_{1,3}$.

**Corollary 3.5.** For every positive integer $m$, 

$$\mu_s (mK_{1,3}) = \begin{cases} 0, & \text{if } m \equiv 4 \pmod{8} \text{ or } m \text{ is odd;} \\ 1, & \text{if } m \equiv 2 \pmod{4}. \end{cases}$$

**Proof.** Define the forest $4K_{1,3}$ with 

$$V (4K_{1,3}) = \{x_i | i \in [1,4]\} \cup \{y_{i,j} | i \in [1,4] \text{ and } j \in [1,3]\}$$

and 

$$E (4K_{1,3}) = \{x_i y_{i,j} | i \in [1,4] \text{ and } j \in [1,3]\}.$$ 

Then the vertex labeling $f : V (4K_{1,3}) \to [1,16]$ such that 

$$(f (x_i))_{i=1}^4 = (13, 12, 10, 8);$$

$$(f (y_{1,j}))_{j=1}^3 = (1, 2, 7); \quad (f (y_{2,j}))_{j=1}^3 = (4, 5, 6);$$

$$(f (y_{3,j}))_{j=1}^3 = (3, 9, 11); \quad (f (y_{4,j}))_{j=1}^3 = (14, 15, 16)$$

induces a super edge-magic labeling of $4K_{1,3}$ with valence 41. Now, recall the result presented in [8] that if $G$ is a (super) edge-magic bipartite or tripartite graph and $m$ is odd, then $mG$ is (super) edge-magic. Since the forests $K_{1,3}$ and $4K_{1,3}$ are super edge-magic bipartite graphs, it follows from the mentioned result that $\mu_s (mK_{1,3}) = 0$ when $m \equiv 4 \pmod{8}$ or $m$ is odd. The remaining case is an immediate consequence of Corollary 3.4. $\square$

The preceding results in this section motivate us to propose the following problem.

**Problem 1.** For even $m \geq 4$ and $n \geq 3$, determine the exact value of $\mu_s (mK_{1,n})$. 
We now direct our attention briefly to the super edge-magic deficiency of the 2-regular graph $mK_{2,2}$. For every positive integer $m$, Ngurah et al. [21] proved that if $m$ is odd, then $\mu_s(mK_{2,2}) \leq m$ while if $m$ is even, then $\mu_s(mK_{2,2}) \leq m-1$. They also posed the problem of finding a better upper bound for $\mu_s(mK_{2,2})$. However, with the aid of Corollary 2.2, we are able to provide the exact value of $\mu_s(mK_{2,2})$ which we determine to be 1.

**Corollary 3.6.** For every positive integer $m$,

$$\mu_s(mK_{2,2}) = 1.$$  

*Proof.* As we mentioned in the proof of Theorem 3.1, the 2-regular graph $3C_4 \cong 3K_{2,2}$ does not admit an $\alpha$-valuation. Also, Gnanajothi [13] has shown that the 2-regular graph $mC_n$ is sequential if and only if $m$ and $n$ are odd. By adding these facts to Corollary 2.2, we obtain that $\mu_s(mK_{2,2}) = 1$ except for $m = 3$, and $\mu_s(3K_{2,2}) \geq 1$. However, the graph $3K_{2,2} \cup K_1$ is super edge-magic by labeling the vertices in its cycles with $1-8-3-9-1, 2-6-7-12-2, 4-11-5-13-4$, and its isolated vertex with 10 to obtain a valence of 33, which implies that $\mu_s(3K_{2,2}) \leq 1$. Consequently, $\mu_s(mK_{2,2}) = 1$ for every positive integer $m$. \hfill $\square$

The previous result adds credence to the conjecture of Figueroa-Centeno et al. [9] that for all integers $m \geq 1$ and $n \geq 3$, $\mu_s(mC_n) = 1$, if $mn \equiv 0 \pmod{4}$.

The final result of this section concerns an upper bound for $\mu_s(mK_{s,t})$. For all integers $m$, $s$ and $t$ with $m \geq 1$, $s \geq 4$ and $t \geq 4$, Ngurah et al. [21] discovered an upper bound for $\mu_s(mK_{s,t})$, namely, $\mu_s(mK_{s,t}) \leq m(st - s - t) + 1$. Actually, the conditions on $s$ and $t$ in their result can be relaxed as we will see next.

**Corollary 3.7.** For all integers $m$, $s$ and $t$ with $m \geq 1$, $s \geq 2$ and $t \geq 2$,

$$\mu_s(mK_{s,t}) \leq m(st - s - t) + 1.$$  

*Proof.* It has already been verified in the proof of Corollary 3.6 that $\mu_s(3K_{2,2}) \leq 1$. Thus, the desired result readily follows from this, and Theorems 2.1 and 3.1. \hfill $\square$

By Corollaries 1.3, 3.6 and 3.7, we suspect the following conjecture to be true.

**Conjecture 1.** For all integers $m$, $s$ and $t$ with $m \geq 1$, $s \geq 2$ and $t \geq 2$,

$$\mu_s(mK_{s,t}) = m(st - s - t) + 1.$$  

Of course, if it is true that the graph $mK_{s,t}$ is not sequential for all integers $m$, $s$ and $t$ with $m \geq 1$ and $s \geq 2$ and $t \geq 2$, so is Conjecture 1 by Corollary 2.2 and Theorem 3.1. However, we do not know whether or not the mentioned statement is true. Thus, we propose the following problem.

**Problem 2.** For all integers $m$, $s$ and $t$ with $m \geq 1$, $s \geq 2$ and $t \geq 2$, determine whether or not the graph $mK_{s,t}$ is sequential.
4. Concluding Remarks

We conclude this paper with some remarks on bounds for the super edge-magic deficiency of bipartite graphs and open problems.

Figueras-Centeno et al. [9] have shown that if \( G \) is a bipartite or tripartite graph and \( m \) is odd, then \( \mu_s(mG) \leq m\mu_s(G) \). Unfortunately, this bound is not sharp. For instance, we can easily see that \( \mu_s(K_{2,2}) = 1 \), which implies that \( \mu_s(3K_{2,2}) \leq 3 \); however, we know by Corollary 3.6 that \( \mu_s(3K_{2,2}) = 1 \). Also, the same bound does not hold for even \( m \), since we know that \( \mu_s(K_{1,n}) = 0 \) (see [5]) and \( \mu_s(2K_{1,n}) = 1 \) (see Corollary 3.3). On the other hand, by Corollaries 1.3 and 3.7, we obtain that \( \mu_s(mG) \leq m\mu_s(G) - m + 1 \) when \( G \cong K_{s,t} \). This leads us to ask in the next problem whether a similar upper bound is obtained for any bipartite graph.

**Problem 3.** Given a bipartite graph \( G \) and an integer \( m \geq 2 \), find a good upper bound for \( \mu_s(mG) \) in terms of \( m \) and \( \mu_s(G) \).

To proceed further, another definition is required here. For two graphs \( G_1 \) and \( G_2 \) with disjoint vertex sets, the cartesian product \( G \cong G_1 \times G_2 \) has \( V(G) = V(G_1) \times V(G_2) \), and two vertices \((u_1, u_2) \) and \((v_1, v_2) \) of \( G \) are adjacent if and only if either \( u_1 = v_1 \) and \( u_2v_2 \in E(G_2) \) or \( v_2 = v_2 \) and \( u_1v_1 \in E(G_1) \). An important class of graphs is defined in terms of cartesian product. The \( n \)-dimensional cube \( Q_n \) is the graph \( K_2 \) if \( n = 1 \), while for \( n \geq 2 \), \( Q_n \) is defined recursively as \( Q_n \times K_2 \). It is easily observed that \( Q_n \) is an \( n \)-regular bipartite graph of order \( 2^n \) and size \( n2^{n-1} \).

We now discuss briefly lower and upper bounds for \( \mu_s(Q_n) \). Figueras-Centeno et al. [6] pointed out that \( Q_n \) is super edge-magic if and only if \( n = 1 \). Kotzig [19] has shown that \( Q_n \) has an \( \alpha \)-valuation for all \( n \), whereas the authors proved that \( Q_n \) is sequential for \( n \geq 4 \) (see [17]). Combining these with Corollary 2.2 and Theorem 2.1, we obtain exact values \( \mu_s(Q_1) = 0 \), \( \mu_s(Q_2) = 1 \) and \( \mu_s(Q_3) = 5 \), and the upper bound \( \mu_s(Q_n) \leq (n-2)2^{n-1} + 1 \) for \( n \geq 4 \). It is now important to mention that the largest vertex labeling of the sequential labeling found in [17] is \( n2^{n-1} - 5 \), which implies that \( \sigma(Q_n) \leq n2^{n-1} - 5 \). This together with Theorem 2.1 gives us the upper bound \( \mu_s(Q_n) \leq (n-2)2^{n-1} - 4 \) for \( n \geq 4 \). This bound is certainly better than the above bound obtained by applying an \( \alpha \)-valuation of \( Q_n \) provided in [19] to Theorem 2.1. Figueras-Centeno et al. [8] found an upper bound for the size of a super edge-magic triangle-free graph of order \( p \geq 4 \) and size \( q \), namely, \( q \leq 2p - 5 \). By utilizing this, we obtain the lower bound \( \mu_s(Q_n) \geq (n - 4)2^{n-2} + 3 \) for \( n \geq 2 \). In the light of the mentioned bounds and exact values for \( \mu_s(Q_n) \), we finally propose the following two problems.

**Problem 4.** For every integer \( n \geq 4 \), find better lower and upper bounds for \( \mu_s(Q_n) \).

**Problem 5.** For every integer \( n \geq 4 \), determine the exact value of \( \mu_s(Q_n) \).

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References


