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QUEENS INDEPENDENCE SEPARATION ON RECTANGULAR CHESSBOARDS

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Abstract. The famous eight queens problem with non-attacking queens placement on an 8×8 chessboard was posed in the year 1848. The *Queens separation problem* is the legal placement of the fewest number of pawns with the maximum number of independent queens placed on an $N \times N$ board which results in a separated board. Here a legal placement is defined as the separation of attacking queens by pawns. Using this concept, the current study extends the Queens separation problem onto the rectangular board $M \times N$ $(M < N)$ to result in a separated board with the maximum number of independent queens. The research work here first shows that $M + k$ queens are separated with 1 pawn and continues to prove that k pawns are required to separate $M + k$ queens. Then it focuses on finding the symmetric solutions to the $M + k$ Queens separation problem.

Key words and Phrases: Queens graph, N-queens problem, Queens separation

1. INTRODUCTION

In the *Queens graph* $Q_{M \times N}$, the squares on the board are taken as vertices, and edges are formed between two squares if they lie on the same path of the movement. A set D of squares in $Q_{M \times N}$ is independent if no two squares in D are adjacent (i.e., no two squares lie on the same diagonal, row, or column). We know that the independence number of queens on a square board $\beta(Q_N)$ is N, and Grant Cairns in [1] describes that on an $M \times N$ board where $M < N$, at most M non-attacking queens can be placed. Thus, the known queens independence number on a rectangular board is $min(M, N)$. The *N-Queens problem* is the placement of N queens on an $N \times N$ chessboard so that no queen can move to another queen's position in a single move. A placement in which any two attacking queens can be separated by a pawn is defined as a *legal placement* [4]. The *Queens separation*

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number $s_Q(M, N)$ is the minimum number of pawns that can separate some legal placement of M queens on an $N \times N$ board [4]. The Queens independence separation number $s_Q(\beta, l, M \times N)$ is the minimum number of pawns that can be placed on an $M \times N$ board to result in a separated board on which a maximum of l independent queens can be placed [3].

The studies related to the placement of multiple queens on the chessboard were started in 1848, by Max Bezzel, with the problem of placing eight queens on an 8×8 board so that none of the queens attack the other in its path. The prior study of the new problem was to find the least number of pawns required to place nine queens on an 8×8 chessboard to block the queens from attacking each other. This was proved by Zhao [6] in 1998, which explains that more than N independent queens can be placed on an $N \times N$ board by using the pawns in between the attacking queens. Using this, Chatham et al., in [4] defined the board obtained from the addition of one or more pawns to be a *separated board.* He further proved the separation of $N + k$ queens by k pawns for large enough N in [3] and then introduced some domination-related parameters to separation problems with other chess pieces.

Chatham et al., in [2, 5] defined the symmetric solutions such as *ordinary, centrosymmetric,* and *doubly centrosymmetric solutions* on a square board, and proved that all the solutions of $N + k$ queens solutions belong to one of these symmetries. Using the studies done on separation problems on the square boards, this paper extends the work onto the rectangular boards.

Our research work here presents the $M + k$ Queens separation on rectangular boards of size $M = 1, 2 \leq N$ in section 2; proves the queens separation with 1 pawn for $M \geq 3$ in section 3, and queens separation with k pawns for $M \geq 3$ in section 4, using the placement of queens. Then, we provide some results related to the symmetric solutions of $M + k$ Queens separation in section 5.

Note: Throughout the paper, we (the authors) consider $M < N$.

2. $M + k$ QUEENS SEPARATION FOR $1 \leq M \leq 2$

Lemma 2.1. *For* $1 \leq M \leq 2$*;* $s_Q(\beta, M + k, M \times N) = k$ *when* $N \geq 2(M + k) - 1$*.*

Proof. We know that the independence number of queens on a rectangular board is $min(M, N)$, where M and N denote the number of rows and columns respectively. Here, we first place the M queens one in each row. Since two queens attack each other if they are placed in the same row, column, or diagonal, each queen in a row is placed in the knight position from the previously placed queen. Hence we require $2M - 1$ columns to place M queens as $M \leq 2$. Now to increase the number of queens by 1 we need at least two extra columns one for the new queen and one for the pawn respectively. This shows that for k pawns we need $2k$ columns. Thus, we have at least $2k + 2M - 1$ columns to separate $M + k$ queens using k pawns as shown in Figure 1, 2.

Q	P	Q	P	Q	

FIGURE 1. $s_Q(\beta, 1+2, 1\times 6) = 2$

		Q		P		Q
Q		P		Q		

FIGURE 2. $s_Q(\beta, 2+2, 2\times 7) = 2$

	Q						
				Q			
Q	P						Q
					Q		
			Q				
	Q						

FIGURE 3. $s_Q(\beta, 6+1, 6\times 8) = 1$

3. EXTRA QUEEN PLACEMENT FOR $M \geq 3$

Lemma 3.1. *For $M \geq 3, N = M+2$, $s_Q(\beta, M+1, M\times N) = 1$.*

PROOF. We prove this lemma using the following three patterns. Consider an $M \times N$ board and label the rows and columns from $1, \ldots, M$ and $1, 2, \ldots, N$ respectively, from the bottom left corner.

Pattern I: let $M \geq 3$, $M \equiv 1 \mod 3$

(i) When M is even, first place the queens at $(i, 2i)$ for $1 \leq i \leq \lceil \frac{M}{2} \rceil$ and at $(i, 2i-M-1)$ for $\lceil \frac{M}{2} \rceil < i \leq M-1$. Then, place the remaining queens at $(M-1, 1)$ and $(M, 4)$ while the pawn at $(M-1, 4)$.

(ii) When M is odd, first place the queens at $(i, 2i)$ for $1 \leq i \leq \lceil \frac{M}{2} \rceil$ and at $(i, 2i-M)$ for $\lceil \frac{M}{2} \rceil < i \leq M-1$. Then, place the remaining queens at $(M-1, 1)$ and $(M, 3)$ while the pawn at $(M-1, 3)$.

Pattern II: let $M \geq 2, M \equiv 0 \mod 3$

(i) When M is even, first place M queens at $(i, 2i)$ for $1 \leq i \leq \lceil \frac{M}{2} \rceil + 1$, at $(i, 2i-M+1)$ for $\lceil \frac{M}{2} \rceil + 1 < i \leq (M-1)$. Now, place the remaining queens at $(M-2, 1), (M, 2)$, and a pawn at $(M-2, 2)$. Fig. 3 shows the placement of queens on a 6×8 board.

(ii) When M is odd, first place queens at $(i, 2i+1)$ for $1 \leq i \leq \lceil \frac{M}{2} \rceil$, at $(i, 2i-M-1)$ for $\lceil \frac{M}{2} \rceil < i \leq (M-1)$. Next, place the other queens at $(2, 1), (M, 3)$, and a pawn at $(2, 3)$.

Pattern-III: let $M \geq 3, M \equiv 2 \mod 3$

		Q				
Q		P			Q	
		Q				
				Q		
	Q					

FIGURE 4. $s_Q(\beta, 5+1, 5 \times 7) = 1$

			Q			
Q			P		Q	
			Q			
	Q					

FIGURE 5. $s_Q(\beta, 4+1, 4 \times 7) = 1$

(i) When M is even, first place M queens at $(i, 2i)$ for $1 \leq i \leq \lceil \frac{M}{2} \rceil + 1$, at $(i, 2i - M + 1)$ for $\lceil \frac{M}{2} \rceil + 1 < i \leq (M-1)$. Next, place the remaining queens at $(M-2, 1), (M, 2)$ and a pawn at $(M-2, 2)$.

(ii) When M is odd, first place the queens at $(2i-1, i+1)$ for $1 \leq i \leq \lfloor \frac{M}{2} \rfloor$. let (a, b) be the placement of the queen placed in the cell $(2i-1, i+1)$, where $i = \lfloor \frac{M}{2} \rfloor$. Now, place the queens at $(2i, b+i+1)$, where $i = 1 \leq i \leq \lfloor \frac{m}{2} \rfloor$. Then, place the remaining queens at $(\lfloor \frac{M}{2} \rfloor + 1, 1), (M, 3)$ and a pawn at $(\lfloor \frac{M}{2} \rfloor + 1, 3)$. Figure 4, shows the queens separation on a 5×7 chessboard.

Lemma 3.2. *For $M \geq 3, N \geq 5$, $s_Q(\beta, M+1, M \times N) = 1$ when $|M - N| > 2$.*

PROOF. Here we consider the following pattern:

Choose a board with $|M - N| = 2$ leaving the last $|M - N| - 2$ columns empty. Now place the queens and the pawn on the sub-board with $|M - N| = 2$ according to the patterns mentioned in Lemma 3.1. Placing the queens in this pattern would cover the entire chessboard with attacking queens being separated by a pawn. Figure 5, shows the placement of queens on a 4×7 board with sub-board 3×5 satisfying the condition $|M - N| = 2$.

Theorem 3.3. *For $M \geq 3, N \geq 5$, $s_Q(\beta, M+1, M \times N) = 1$ when $|M - N| \geq 2$.*

PROOF. From Lemma 3.1 and Lemma 3.2, it is trivial that for $M \geq 3, N \geq 5$, and $|M - N| \geq 2$, $s_Q(\beta, M+1, M \times N) = 1$.

Theorem 3.4. *For $M \geq 5$ and $N = M + 1$, $s_Q(\beta, M+1, M \times N) = 1$.*

	Q					
			Q			
					Q	
Q	P	Q				
				Q		
	Q					

FIGURE 6. $s_Q(\beta, 6+1, 6\times 7) = 1$

PROOF. If $N = M+1$, then take a square board of order $M \times M$ from the main board $M \times N$ and place $M+1$ queens on the square board using the patterns mentioned in [4], depending on M for $M \geq 6$. For $M = 5$ take a sub-board of order 5×5 and place 5 queens on it. Now, place the extra queen in the empty column and a pawn next to it inside the sub-board. Fig. 6 shows the placement of queens on 6×7 board by placing seven queens on the square board 6×6 as mentioned in [4].

4. $M+k$ QUEENS SEPARATION FOR $M \geq 3$

In this section, we wish to place $M+k$ independent queens on an $M \times N$ board. Since the independence number of queens is $min(M, N)$, only M queens can be placed on the board in such a way that no two queens attack each other. Adding a new queen to the board would attack the queens that are already placed. Hence, to place an extra queen we need at least one blocking piece like a pawn. From this it is clear that to place k extra queens at least k pawns are required i.e., $s_Q(\beta, M+k, M\times N) \geq k$. Here, we show that to place $M+k$ independent queens on an $M \times N$ board, k pawns are required.

Theorem 4.1. *For $M = 3$ and 4; $s_Q(\beta, M+k, M\times N) = k$ when $N \geq M+k+1$.*

PROOF. Suppose $M = 3$, since we know that the independence number on a rectangular board of order $M \times N$ is M we need at least $M+1$ columns to place M queens. As we cannot place 3 queens on a 3×3 board, we need at least $M+1(3+1=4)$ columns to place them. To increase the independence number by k, we place the pawns in one of the previously available columns and add a new column to place a queen in the diagonally non-attacking cell, each time whenever k value is incremented. Thus, we need at least $M+1+k$ columns.

For the case where $M = 4$, a similar argument follows. Whereas, in this case, we first place 4 queens on a 4×4 board. Here placing a queen in the $M+1^{th}$ column is not possible because every cell in the $M+1^{th}$ column would be diagonally attacked by the M queens already placed. Therefore, when $k = 1$, we add 2 columns, one for the pawn and the other for the extra queen. For the other values of k, we follow the same process as mentioned above for the case when $M = 3$. Hence, for $M = 3$ and 4, at least $M+1+k$ columns are required to separate $M+k$ queens on an

$M \times N$ board. The figure below shows the queens separation for the case $M = 3$ in (a), and $M = 4$ in (b).

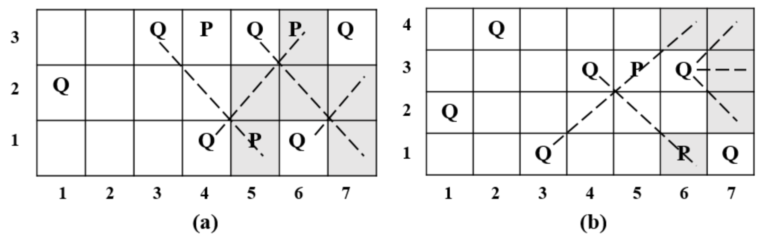

FIGURE 7. The shaded region denotes the cells that are under a direct attack by the queens that are placed diagonally along the dotted lines. $(a) s_Q(\beta, 3+3, 3\times 7) = 6$; $(b) s_Q(\beta, 4+2, 4\times 7) = 6$

Theorem 4.2. *For $M = 5$; $s_Q(\beta, M+k, M\times N) = k$ when $N \geq M+k$.*

PROOF. We begin the proof for the case $M = 5$ by placing 5 queens on a 5×5 board. When $k = 1$ i.e, to increase the independence number by one, first place a pawn on the 5×5 board. Then add a column and place the $(M+1)^{th}$ queen in one of the non-attacking cells of the new column. Thus, at least $M+1$ columns are required to place $M+1$ queens. Now repeat the same process of adding a new column each time when the k value is incremented. This concludes the proof showing that $N \geq M+k$.

Lemma 4.3. *Let $M > 3$ and $M \equiv 0 \mod 3$, then $s_Q(\beta, M+k, M\times N) = k$ for $k \leq \frac{M}{4}$ and $N \geq M + \lceil \frac{k}{2} \rceil + 1$.*

PROOF. First Number the rows and columns from $1,\dots,M$ from bottom left on an $M\times N$ board. Then place M queens on an $M\times M+1$ board. Continue to increase the independence number M to $M+K$ by using k pawns in between the attacking queens using the following placements.

(i) When M is odd, first place M queens at $(2i+1, i+1)$ for $0 \leq i \leq \lfloor \frac{M}{2} \rfloor$, and at $(2i, \lfloor \frac{M}{2} \rfloor + i + 2)$ for $0 < i \leq \lfloor \frac{M}{2} \rfloor$. Then place the pawns at $(2i-1, \lceil \frac{M}{2} \rceil + i + 2)$ for $i = 1,\dots,k$. Next place $k-1$ queens at $(2i-1, \lceil \frac{M}{2} \rceil + i + 3)$ where $i = 1,\dots,k-1$. Now place the k^{th} queen at $(2i-1, (k \mod 3) + M + 2)$.

(ii) When M is even, start placing M queens at $(2i+1, i+1)$ for $0 \leq i < \frac{M}{2}$, and at $(2i, \frac{M}{2} + i + 1)$ for $0 < i \leq \frac{M}{2}$. Then place the pawns at $(2i-1, \frac{M}{2} + i + 1)$ for $i = 1,\dots,k$. Next place $k-1$ queens at $(2i-1, \frac{M}{2} + i + 2)$ where $i = 1,\dots,k-1$. Now place the k^{th} queen at $(2i-1, ((k+1) \mod 3) + M + 1)$.

Lemma 4.4. *Let $M > 4$ and $M \equiv 1 \mod 3$, then $s_Q(\beta, M+k, M\times N) = k$ for $k < \frac{M+3}{4}$ and $N \geq M + \lceil \frac{k}{2} \rceil + 1$.*

PROOF. Number the rows and columns as mentioned in Lemma 4.3, and place the queens as follows:

(i) When M is odd, first place M queens as shown in Lemma 4.3 (i). Then place the pawns at $(2i, \lceil \frac{M}{2} \rceil + i + 1)$ for $i = 1, \ldots, k$. Next place $k-1$ queens at $(2i-1, \lceil \frac{M}{2} \rceil + i + 2)$ where $i = 1, \ldots, k-1$. Now place the k^{th} queen at $(2i-1, ((k+1) \mod 3) + M + 1)$.

(ii) When M is even, start placing M queens as mentioned in Lemma 4.3 (ii). Then place the pawns at $(2i-1, \frac{M}{2} + i + 2)$ for $i = 1, \ldots, k$. Next place $k-1$ queens at $(2i-1, \frac{M}{2} + i + 3)$ where $i = 1, \ldots, k-1$. Now place the k^{th} queen at $(2i-1, ((k+2) \mod 3) + M + 1)$.

Lemma 4.5. *Let $M > 5$ and $M \equiv 2 \mod 6$, then $s_Q(\beta, M+k, M \times N) = k$ for $k \leq \frac{M}{4}$ and $N \geq M + \lceil \frac{k}{2} \rceil + 1$.*

PROOF. Number the rows and columns using Lemma 4.3. Now start placing the M queens at $(2i+1, i+1)$ for $0 \leq i < \frac{M}{2}$, and at $(2i, \frac{M}{2}+i)$ for $0 < i \leq \frac{M}{2}$. Then place the pawns at $(2i, \frac{M}{2}+i+1)$ for $i = 1, \ldots, k$. Next place $k-1$ queens at $(2i, \frac{M}{2}+i+2)$ where $i = 1, \ldots, k-1$. Now place the k^{th} queen at $(2i, (k \mod 3) + M + 1)$.

Lemma 4.6. *Let $M > 5$ and $M \equiv 5 \mod 12$, then $s_Q(\beta, M+k, M \times N) = k$ for $k < \frac{M}{4}$ and $N \geq M + \lceil \frac{k}{2} \rceil + 1$.*

PROOF. Number the rows and columns using Lemma 4.3. First place the M queens at $(2i+1, i+1)$ for $0 \leq i \leq \lfloor \frac{M}{2} \rfloor$, and at $(2i, \frac{M}{2} + i + 1)$ for $0 < i \leq \lfloor \frac{M}{2} \rfloor$. Then place the pawns at $(2i-1, \lceil \frac{M}{2} \rceil + i + 1)$ for $i = 1, \ldots, k$. Next place $k-1$ queens at $(2i, \lceil \frac{M}{2} \rceil + i + 2)$ where $i = 1, \ldots, k-1$. Now place the k^{th} queen at $(2i, ((k+2) \mod 3) + M + 1)$.

Lemma 4.7. *Let $M > 5$ and $M \equiv 11 \mod 12$, then $s_Q(\beta, M+k, M \times N) = k$ for $k \leq \frac{M+1}{4}$ and $N \geq M + \lceil \frac{k}{2} \rceil + 1$.*

PROOF. Here we use Lemma 4.6 and place $M+k$ queens using k pawns.

To prove our next theorem we use the following result from Section.4 in [3] which explains the restriction of pawns location on an $N \times N$ board.

Theorem 4.8. [3] *If $N+k$ queens and k pawns are placed on an $N \times N$ board so that no two queens attack each other, then no pawn can be on the first or last row, first or last column, or any square adjacent to a corner.*

Theorem 4.9. *If $M+k$ non-attacking queens are separated by k pawns on an $M \times N (M < N)$ board, then no pawn can be placed on either the first or last column.*

PROOF. We know that placing a pawn divides the row or column into two independent parts, therefore it is clear that placing p pawns divides the row or column into $p+1$ independent parts.

Suppose a pawn is placed in the first column, then it must be able to separate two queens in both the row and the column at which it is placed. Since we know that the independence number of queens on an $M \times N$ board is $min(M, N)$, placing a pawn in the first or the last column allows at most two queens in that column (i.e., one above it and the other below it if the pawn is in between the column, otherwise one below or above it, if it is in the corner). Thus if we place k pawns in a column, we can place at most $k+1$ queens in that column which results in at most $2k+1$ rows occupied by some chess piece in the first column. Now the number of queens to be placed are $M + k - (k+1) = M - 1$, and the number of rows left is $M - (2k+1) = M - 2k - 1$. Now as the pawn is placed in the first column it cannot separate two queens in a row as it does not have a column towards its left, also we don't have more rows as the number of rows are less than columns. Hence, $M - 1$ queens must be placed in these $M - 2k - 1$ rows which is a contradiction, as $M - 2k - 1 < M - 1$ and which in turn results in attacking queens. The same holds when a pawn is placed in the last column. Therefore, there can be no pawn placed on either the first or last column on a rectangular board.

5. SYMMETRIC SOLUTIONS OF $M + k$ QUEENS WITH k PAWNS

R. D. Chatham in [4] describes that a square board has eight symmetries with four rotations (0, 90, 180, and 270 degrees) and four reflections (horizontal, vertical, and diagonal reflections). For any $N \geq 4$, the N-Queens solution falls into one of the three classes:

(i) *Ordinary*: symmetric only with 0^0s rotation.
(ii) *Centrosymmetric*: symmetric to 180^0 rotation but not 90^0 degree rotation.
(iii) *Doubly Centrosymmetric*: symmetric to all the rotations.

It was proved that no solution to the $N + k$ Queens problem for $N > 1$ has horizontal, vertical, or diagonal reflections of symmetry. In [5] Chatham et al. explained the existence of centrosymmetric solutions on square boards. Using this we extend the study of symmetric solutions onto the rectangular board which has only four symmetries with two rotations (0 and 180 degrees) and two reflections (horizontal and vertical reflections).

Here, we consider the horizontal and vertical reflections of symmetry in section 5.1, and the existence of centrosymmetric solutions for the $M + k$ queens separation with k pawns in section 5.2.

5.1. Reflections.

The $M + k$ queens separation with k pawns on a rectangular board has two reflections of symmetry:
1. Horizontal reflections, 2. Vertical reflections.

5.1.1. ***Horizontal reflections of symmetry.***

Theorem 5.1. *The horizontal reflections of symmetry exist only when M is odd.*

Proof. We prove this by showing that no horizontal reflections exist when the number of rows M is even. Suppose that M is even, then we have two centre rows. Now to have the horizontal reflections we have to place two queens in the same column. This results in queens attacking each other. Thus, to separate every two attacking queens we need two pawns as the board is divided equally. Hence, every queen requires one pawn. This gives k queens separation with k pawns which is a contradiction to $M+k$ queens separation with k pawns. Therefore, horizontal reflections exist only when M is odd.

Theorem 5.2. *For $M = 1$ and 3; horizontal reflections exist only when $N \geq 2k+M$.*

Proof. Note from the proof of Theorem 4.9, that k pawns can separate $k+1$ queens and therefore requires at least $2k+1$ columns to place any of these chess pieces. Therefore, the proof is trivial when $M = 1$ (i.e., $N \geq 2k+M$).

Before considering the case for $M = 3$, we claim that pawns must be placed only on the centre row to obtain horizontal reflections of symmetry. We prove this by contradiction. Suppose a pawn is placed in a row above the centre row. As we know that a pawn can separate at most 2 queens in a row, place one queen on either side of the pawn. Now to have horizontal reflections of symmetry place another pawn below the centre row in the same column in which the previous pawn is placed. Similarly, place one queen on either side of the pawn that is below the centre row, in the same columns where the previous queens are placed to have horizontal reflections. This would result in queens attacking each other in a column which is a contradiction. Therefore, horizontal reflections would exist only when pawns are placed in the center row of the board.

For the case $M = 3$, where $k = 1$ (i.e., separation with 1 pawn) we need to place $M+1$ (i.e., 3+1=4) queens. From Theorem 4.1, it is clear that 4 queens can be placed in 5 columns using 1 pawn. As we know that a pawn can separate at most 2 queens each in a row and a column, place a pawn separated by 2 queens in the alternative cells of the centre row starting from the first column.

Now placing the other 2 queens in the column where the pawn is placed would not attack the queens in the center row. This placement results in a horizontal reflection. Suppose $k > 1$, since we have already placed $M+1$ queens using 1 pawn, for the other values of k, we add $2(k-1)$ columns to the 5 columns available in which $M+1$ queens are placed. This shows that we need at least $2(k-1)+5 = 2k+3 = 2k+M$ columns for the existence of horizontal reflections.

Theorem 5.3. *For $M \geq 5$, and $k \geq \left\lfloor \frac{M}{2} \right\rfloor$; horizontal reflections exist only if $N \geq \left\lceil \frac{M}{2} \right\rceil + 2k$.*

Proof. From Theorem 5.1, we know that M is odd, and from Theorem 5.2, it is clear that pawns can be placed only on the center row in between the attacking queens, and remaining queens on either side of the center row to obtain horizontal reflections.

Note that k pawns can separate $k+1$ non-attacking queens in a row. Whereas, k pawns can separate $2\left\lfloor \frac{M}{2} \right\rfloor$ queens in a column as there are only $\left\lfloor \frac{M}{2} \right\rfloor$ rows left above and below the centre row after filling it (centre row) with some chess pieces. From this we can say that k pawns can separate $2\left\lfloor \frac{M}{2} \right\rfloor + k + 1$ queens.

We prove this by first placing $\left\lfloor \frac{M}{2} \right\rfloor + 1$ queens in the centre row. Then place the remaining queens above and below the center row in the columns in which the pawn is placed. Now to avoid queens being attacked with the queens in the adjacent rows, we place queens in the alternate columns as mentioned in Theorem 5.2. Thus, after placing $\left\lfloor \frac{M}{2} \right\rfloor$ pawns we continue to increase the independence number maintaining the horizontal reflections of symmetry inductively till k pawns by adding 2 columns each time when a pawn is placed, where one column is for the pawn and the other column is for the queen to be placed. This shows the existence of horizontal reflections for $M + k$ queens using k pawns.

5.1.2. ***Vertical reflections of symmetry:***

Theorem 5.4. *The vertical reflections of symmetry for $M + k$ Queens separation with k pawns exist only when N is odd.*

Proof. We prove this by showing that no vertical reflections exist when the number of columns N is even as proved for horizontal reflections in Theorem 5.3. Therefore, vertical reflections exist only when N is odd.

Note: From Theorem 5.1 and 5.4, we conclude that there exist both horizontal and vertical reflections of symmetry only when both M and N are odd. See Figure 8.

		Q		
Q		**P**		**Q**
		Q		

Figure 8. Horizontal reflections of symmetry on 3×5 chessboard

The following Conjectures 5.5 and 5.6, shows the existence of vertical reflections on the rectangular board.

Conjecture 5.5. *For M even and $k \geq \left\lfloor \frac{M}{2} \right\rfloor$, vertical reflections of symmetry exist only if it satisfies one of the following:*

(i) $M \equiv 2$ *mod 4 and* $N \geq 2k + \left\lceil \frac{M+3}{2} \right\rceil$
(ii) $M \equiv 0$ *mod 4 and* $N \geq 2k + \left\lceil \frac{M+2}{2} \right\rceil$

Conjecture 5.6. *For M odd, vertical reflections of symmetry exist only if it satisfies one of the following:*

(i) For $k \geq M - 2$ *and* $N \geq 2k + 3$
(ii) For $\left\lfloor \frac{M}{2} \right\rfloor \leq k < M - 2$ *and* $N \geq 3M - 4$

5.2. **Centrosymmetric solutions.** We observe that centrosymmetric solutions for $M + k$ Queens problem on an $M \times N$ board exist if it satisfies one of the following three conditions:

(i) Both M and N are odd, see Fig.9.

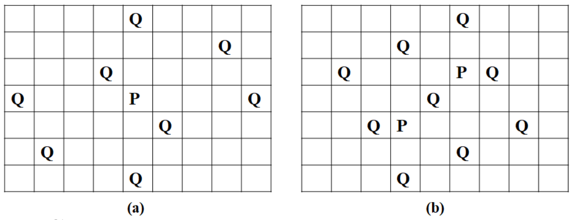

Figure 9. Centrosymmetric solutions on 7×9 chessboard with (a) 1 pawn; (b) 2 pawns

(ii) Both M and N are even and k is even, see Fig. 10(a).
(iii) M is even, N is odd and k is even, see Fig. 10(b).

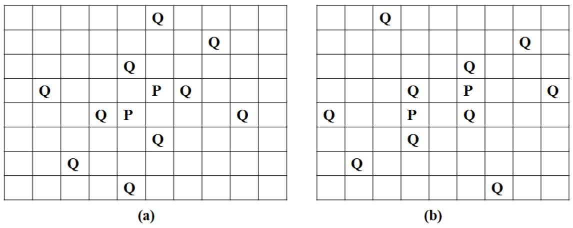

Figure 10. Centrosymmetric solutions with 2 pawns on (a) 8×10 chessboard; (b) 8×9 chessboard

Theorem 5.7. *For an $M + k$ Queens problem, no centrosymmetric solutions exist when both M and N are even, and k is odd.*

Proof. If both M and N are even, the board must have an equal number of pieces on the left half and the right half of the board as mentioned in [5]. Thus, k must be always even.

Theorem 5.8. *No centrosymmetric solutions for the $M + k$ Queens problem exist when M is even, N is odd, and k odd.*

Proof. Since we know that queens number depends on the value of M and as N is even, an equal number of queens must be placed on both the left and right half of the board. Thus, the number of pawns must also be equal as mentioned in Theorem 5.7. Hence, k must be even.

Theorem 5.9. *No centrosymmetric solutions for the $M + k$ Queens problem exist when M is odd and N is even.*

Proof. Since the number of queens depends on the value of M and as M is odd and N is even there are one central row and two central columns respectively. Thus, placement of odd number of pawns in the central column is not possible, because the board is divided into two halves. Since k is even and as M is odd, $M+k$ is odd. But as the board is divided into half an equal number of queens must be placed on either side. Therefore, no centrosymmetric solutions for the $M+k$ Queens problem exist when M is odd and N is even.

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