J. Indones. Math. Soc.
Vol. 27, No. 01 (2021), pp. 48–63.

INTERNAL AND EXTERNAL CUBIC SUBALGEBRAS OF BCK/BCI-ALGEBRAS

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Abstract. A characterization of cubic subalgebra is established. The notion of internal and external cubic subalgebra in BCK/BCI-algebra, and several properties are investigated. The R-union, R-intersection, P-union and P-intersection of internal and external cubic subalgebras in BCK/BCI-algebra are discussed.

Key words and Phrases:Cubic subalgebra, internal cubic subalgebra, external cubic subalgebra.

1. Introduction

One of the important classes of logical algebras is BCI-algebra which was introduced by Imai and Iséki for the first time in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. In the same year, these two Japanese mathematicians introduced the notion of a BCK-algebra as a proper subclass of the class of BCI-algebras. Recently BCK-algebra has been extensively investigated by several researchers. They have been applied to many branches of mathematics such as group theory, functional analysis, probability theory, fuzzy set theory, topology, and so on. For more details please refer to [3] to [16].
Moreover, there are two important classes of BCK-algebras: commutative BCK-algebras and bounded commutative BCK-algebras. Furthermore, commutative BCK-algebras have a close connection with compatible generalized difference posets.

In 2012, Jun et al. [21] combined the theory of the interval-valued fuzzy set with the fuzzy set, and introduced the notion of cubic set. Further, they defined

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2020 Mathematics Subject Classification: 06F35, 03B60, 03B52
Received: 26-10-2020, accepted: 16-02-2021.

and studied some basic operations and their properties, and this notion is applied to several algebraic structures (see [1], [2], [19], [20], [23], [24], [25], [27], [28], [29]).

In this paper, we establish a characterization of cubic subalgebra. we introduce the notion of internal and external cubic subalgebra in BCK/BCI-algebra, and investigate several properties. We discuss the R-union (resp., R-intersection, P-union, P-intersection) of internal and external cubic subalgebras in BCK/BCI-algebra.

2. Preliminaries

A *fuzzy set* in a set X is defined to be a function $\mu : X \to [0,1]$. For $I = [0,1]$, denote by I^X the collection of all fuzzy sets in a set X. Define a relation $\leq$ on I^X as follows:

$$(\forall \mu, \lambda \in I^X)\,(\mu \leq \lambda \iff (\forall x \in X)(\mu(x) \leq \lambda(x))). \tag{1}$$

For a family $\{\mu_i \mid i \in \Lambda\}$ of fuzzy sets in X, we define the join ($\vee$) and meet ($\wedge$) operations as follows:

$$\bigvee_{i\in\Lambda} \mu_i : X \to [0,1],\ x \mapsto \sup\{\mu_i(x) \mid i \in \Lambda\},$$
$$\bigwedge_{i\in\Lambda} \mu_i : X \to [0,1]\ x \mapsto \inf\{\mu_i(x) \mid i \in \Lambda\}.$$

Given two closed subintervals $D_1 = [D_1^-, D_1^+]$ and $D_2 = [D_2^-, D_2^+]$ of $[0,1]$, we define the order "$\ll$" as follows:

$$D_1 \ll D_2 \ \Leftrightarrow\ D_1^- \leq D_2^- \text{ and } D_1^+ \leq D_2^+.$$

We also define the refined minimum (briefly, rmin) and refined maximum (briefly, rmax) as follows:

$$\text{rmin}\{D_1, D_2\} = \left[\min\{D_1^-, D_2^-\}, \min\{D_1^+, D_2^+\}\right],$$
$$\text{rmax}\{D_1, D_2\} = \left[\max\{D_1^-, D_2^-\}, \max\{D_1^+, D_2^+\}\right].$$

For a family $\{D_i = [D_i^-, D_i^+] \mid i \in \Lambda\}$ of closed subintervals of $[0,1]$, we define rinf (refined infimum) and rsup (refined supermum) as follows:

$$\underset{i\in\Lambda}{\text{rinf}} D_i = \left[\inf_{i\in\Lambda} D_i^-, \inf_{i\in\Lambda} D_i^+\right] \text{ and } \underset{i\in\Lambda}{\text{rsup}} D_i = \left[\sup_{i\in\Lambda} D_i^-, \sup_{i\in\Lambda} D_i^+\right].$$

Let X be a nonempty set. A function

$$\tilde{A} : X \to D[0,1],\ x \mapsto [\tilde{A}^-(x), \tilde{A}^+(x)]$$

is called an *interval-valued fuzzy set* in X where $D[0,1]$ is the set of all closed subintervals of $[0,1]$,

$$\tilde{A}^- : X \to [0,1],\ x \mapsto \tilde{A}^-(x)$$

and

$$\tilde{A}^+ : X \to [0,1],\ x \mapsto \tilde{A}^+(x).$$

For a family $\{\tilde{A}_i \mid i \in \Lambda\}$ of interval-valued fuzzy sets in X, the union $G = \bigcup_{i\in\Lambda} \tilde{A}_i$ and the intersection $H = \bigcap_{i\in\Lambda} \tilde{A}_i$ are defined as follows:

$$G = \bigcup_{i\in\Lambda} \tilde{A}_i : X \to D[0,1],\ x \mapsto \underset{i\in\Lambda}{\mathrm{rsup}} \tilde{A}_i(x)$$

and

$$H = \bigcap_{i\in\Lambda} \tilde{A}_i : X \to D[0,1],\ x \mapsto \underset{i\in\Lambda}{\mathrm{rinf}} \tilde{A}_i(x)$$

respectively. By a *cubic set* in X we mean a structure

$$\mathcal{A} = \left\{ \langle x, \tilde{A}(x), \mu(x) \rangle \mid x \in X \right\}$$

in which $\tilde{A}$ is an interval-valued fuzzy set in X and μ is a fuzzy set in X. A cubic set $\mathcal{A} = \left\{ \langle x, \tilde{A}(x), \mu(x) \rangle \mid x \in X \right\}$ is simply denoted by $\mathcal{A} = \langle \tilde{A}, \mu \rangle$.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following axioms:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. We can define a partial ordering $\leq$ on X by $x \leq y$ if and only if $x * y = 0$. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\ (x * 0 = x), \tag{2}$$

$$(\forall x, y, z \in X)\ (x \leq\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x), \tag{3}$$

$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y), \tag{4}$$

$$(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y). \tag{5}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [17, 26] and the paper [**?**] for further information regarding BCK/BCI-algebras.

3. Cubic subalgebras

Definition 3.1 ([22]). *A cubic set $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ in X is called a cubic subalgebra of X if*

$$(\forall x, y \in X) \left(\tilde{A}(x * y) \gg \mathrm{rmin}\{\tilde{A}(x), \tilde{A}(y)\} \right), \tag{6}$$

$$(\forall x, y \in X) \left(\mu(x * y) \leq \max\{\mu(x), \mu(y)\} \right). \tag{7}$$

Theorem 3.2. *A cubic set $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ in X is a cubic subalgebra of X if and only if the nonempty sets*

$$\tilde{A}[s, t] := \{x \in X \mid \tilde{A}(x) \gg [s, t]\} \text{ and } \mu[\varepsilon] := \{x \in X \mid \mu(x) \leq \varepsilon\}$$

are subalgebras of X for all $[s, t] \in D[0, 1]$ and $\varepsilon \in [0, 1]$.

Proof. Assume that $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ is a cubic subalgebra of X. Let $x, y, a, b \in X$ be such that $x, y \in \tilde{A}[s, t]$ and $a, b \in \mu[\varepsilon]$ for all $[s, t] \in D[0, 1]$ and $\varepsilon \in [0, 1]$. Then $\tilde{A}(x) \gg [s, t]$, $\tilde{A}(y) \gg [s, t]$, $\mu(a) \leq \varepsilon$ and $\mu(b) \leq \varepsilon$. It follows from (6) and (7) that

$$\begin{aligned} &\tilde{A}(x * y) \gg \mathrm{rmin}\{\tilde{A}(x), \tilde{A}(y)\} \gg [s, t], \\ &\mu(a * b) \leq \max\{\mu(a), \mu(b)\} \leq \varepsilon \end{aligned}$$

and so that $x * y \in \tilde{A}[s, t]$ and $a * b \in \mu[\varepsilon]$. Hence $\tilde{A}[s, t]$ and $\mu[\varepsilon]$ are subalgebras of X.

Conversely, suppose that $\tilde{A}[s, t] \neq \emptyset$ and $\mu[\varepsilon] \neq \emptyset$ are subalgebras of X for all $[s, t] \in D[0, 1]$ and $\varepsilon \in [0, 1]$. For any $x, y, a, b \in X$, let $\mathrm{rmin}\{\tilde{A}(x), \tilde{A}(y)\} = [s_x, t_x]$ and $\max\{\mu(a), \mu(b)\} = \varepsilon$. Then $\tilde{A}(x) \gg [s_x, t_x]$, $\tilde{A}(y) \gg [s_x, t_x]$, $\mu(a) \leq \varepsilon$ and $\mu(b) \leq \varepsilon$, that is, $x, y \in \tilde{A}[s_x, t_x]$ and $a, b \in \mu[\varepsilon]$. Since $\tilde{A}[s_x, t_x]$ and $\mu[\varepsilon]$ are subalgebras of X, we have $x * y \in \tilde{A}[s_x, t_x]$ and $a * b \in \mu[\varepsilon]$. Hence $\tilde{A}(x * y) \gg [s_x, t_x] = \mathrm{rmin}\{\tilde{A}(x), \tilde{A}(y)\}$ and $\mu(a * b) \leq \varepsilon = \max\{\mu(a), \mu(b)\}$. Therefore $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ is a cubic subalgebra of X. □

Theorem 3.3. *If $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ are cubic subalgebras of X, then their R-intersection $\mathcal{A} \cap_R \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \vee \lambda \rangle$ is a cubic subalgebra of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned} (\tilde{A} \cap \tilde{B})(x * y) &= \mathrm{rmin}\{\tilde{A}(x * y), \tilde{B}(x * y)\} \\ &\gg \mathrm{rmin}\{\mathrm{rmin}\{\tilde{A}(x), \tilde{A}(y)\}, \mathrm{rmin}\{\tilde{B}(x), \tilde{B}(y)\}\} \\ &= \mathrm{rmin}\{\mathrm{rmin}\{\tilde{A}(x), \tilde{B}(x)\}, \mathrm{rmin}\{\tilde{A}(y), \tilde{B}(y)\}\} \\ &= \mathrm{rmin}\{(\tilde{A} \cap \tilde{B})(x), (\tilde{A} \cap \tilde{B})(y)\} \end{aligned}$$

and

$$\begin{aligned}(\mu \vee \lambda)(x * y) &= \max\{\mu(x * y), \lambda(x * y)\} \\ &\leq \max\{\max\{\mu(x), \mu(y)\}, \max\{\lambda(x), \lambda(y)\}\} \\ &= \max\{\max\{\mu(x), \lambda(x)\}, \max\{\mu(y)), \lambda(y)\}\} \\ &= \max\{(\mu \vee \lambda)(x), (\mu \vee \lambda)(y)\}.\end{aligned}$$

Therefore $\mathcal{A} \cap_R \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \vee \lambda \rangle$ is a cubic subalgebra of X. $\square$

The R-union of two cubic subalgebras of X may not be a cubic subalgebra of X as seen in the following example.

Example 3.4. *Let $X = \{0, a, b, c\}$ be a set with the following Cayley table:*

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then X is a BCI-algebra (see [17, 26]). Let $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ be cubic sets in X defined by the tabular representation in the following table:

X	$(\tilde{A}, \tilde{B})$	(μ, λ)
0	$([0.6, 0.9], [0.4, 0.7])$	$(0.3, 0.4)$
a	$([0.3, 0.4], [0.4, 0.7])$	$(0.9, 0.5)$
b	$([0.3, 0.4], [0.2, 0.3])$	$(0.9, 0.8)$
c	$([0.5, 0.8], [0.2, 0.3])$	$(0.7, 0.8)$

Then the R-union of $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ is given by the following table:

X	$\tilde{A} \cup \tilde{B}$	$\mu \wedge \lambda$
0	$[0.6, 0.9]$	0.3
a	$[0.4, 0.7]$	0.5
b	$[0.3, 0.4]$	0.8
c	$[0.5, 0.8]$	0.7

We know that $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ are cubic subalgebras of X. Note that

$$\begin{aligned}(\tilde{A} \cup \tilde{B})(a * c) &= (\tilde{A} \cup \tilde{B})(b) = [0.3, 0.4] \\ &\not\succeq [0.4, 0.7] = \text{rmin}\{(\tilde{A} \cup \tilde{B})(a), (\tilde{A} \cup \tilde{B})(c)\}\end{aligned}$$

and/or $(\mu \wedge \lambda)(a * c) = (\mu \wedge \lambda)(b) = 0.8 \not\leq 0.7 = \max\{(\mu \wedge \lambda)(a), (\mu \wedge \lambda)(c)\}$. Therefore $\mathcal{A} \cup_R \mathcal{B} = \langle \tilde{A} \cup \tilde{B}, \mu \wedge \lambda \rangle$ is not a cubic subalgebra of X.

The P-union of two cubic subalgebras of X may not be a cubic subalgebra of X as seen in the following example.

Example 3.5. *Consider the cubic sets $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ in Example 3.4. Then the P-union of $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ is given by the following table:*

X	$\tilde{A} \cup \tilde{B}$	$\mu \vee \lambda$
0	$[0.6, 0.9]$	0.4
a	$[0.4, 0.7]$	0.9
b	$[0.3, 0.4]$	0.9
c	$[0.5, 0.8]$	0.8

We can verify that $\mathcal{A} \cup_P \mathcal{B} = \langle \tilde{A} \cup \tilde{B}, \mu \vee \lambda \rangle$ is not a cubic subalgebra of X.

The P-intersection of two cubic subalgebras of X may not be a cubic subalgebra of X as seen in the following example.

Example 3.6. *Consider the cubic sets $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ in Example 3.4. Then the P-intersection $\mathcal{A} \cap_P \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \wedge \lambda \rangle$ of $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ is given by the following table:*

X	$\tilde{A} \cap \tilde{B}$	$\mu \wedge \lambda$
0	$[0.4, 0.7]$	0.3
a	$[0.3, 0.4]$	0.5
b	$[0.2, 0.3]$	0.8
c	$[0.2, 0.3]$	0.7

We can verify that $\mathcal{A} \cap_P \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \wedge \lambda \rangle$ is not a cubic subalgebra of X.

Definition 3.7. *Given two cubic sets $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ in X, the Cartesian product of $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ is denoted by $\mathcal{A} \otimes \mathcal{B} = \langle \tilde{A} \tilde{\times} \tilde{B}, \mu \times \lambda \rangle$, and is defined as follows:*

$$(\forall x, y \in X) \left((\tilde{A} \tilde{\times} \tilde{B})(x, y) = \operatorname{rmin}\{\tilde{A}(x), \tilde{B}(y)\} \right), \tag{8}$$

$$(\forall x, y \in X) \left((\mu \times \lambda)(x, y) = \max\{\mu(x), \lambda(y)\} \right). \tag{9}$$

Theorem 3.8. *If $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ are cubic subalgebras of X, then so is the Cartesian product of $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$.*

Proof. For any $(a, b), (x, y) \in X \times X$, we have

$$\begin{aligned}
(\tilde{A} \tilde{\times} \tilde{B})((a, b) * (x, y)) &= (\tilde{A} \tilde{\times} \tilde{B})(a * x, b * y) = \operatorname{rmin}\{\tilde{A}(a * x), \tilde{B}(b * y)\} \\
&\gg \operatorname{rmin}\{\operatorname{rmin}\{\tilde{A}(a), \tilde{A}(x)\}, \operatorname{rmin}\{\tilde{B}(b), \tilde{B}(y)\}\} \\
&= \operatorname{rmin}\{\operatorname{rmin}\{\tilde{A}(a), \tilde{B}(b)\}, \operatorname{rmin}\{\tilde{A}(x), \tilde{B}(y)\}\} \\
&= \operatorname{rmin}\{(\tilde{A} \tilde{\times} \tilde{B})(a, b), (\tilde{A} \tilde{\times} \tilde{B})(x, y)\}
\end{aligned}$$

and

$$\begin{aligned}(\mu\times\lambda)((a,b)*(x,y)) &= (\mu\times\lambda)(a*x,b*y)=\max\{\mu(a*x),\lambda(b*y)\}\\ &\le \max\{\max\{\mu(a),\mu(x)\},\max\{\lambda(b),\lambda(y)\}\}\\ &= \max\{\max\{\mu(a),\lambda(b)\},\max\{\mu(x),\lambda(y)\}\}\\ &= \max\{(\mu\times\lambda)(a,b),(\mu\times\lambda)(x,y)\}.\end{aligned}$$

Therefore $\mathcal{A}\otimes\mathcal{B}=\langle\tilde{A}\tilde{\times}\tilde{B},\mu\times\lambda\rangle$ is a cubic subalgebra of X. □

Definition 3.9. *By an internal cubic subalgebra of X, we mean a cubic subalgebra $\mathcal{A}=\langle\tilde{A},\mu\rangle$ of X which satisfies:*

$$(\forall x,y\in X)\left(\tilde{A}^-(x*y)\le\mu(x*y)\le\tilde{A}^+(x*y)\right). \tag{10}$$

Example 3.10. *Let X be a BCI-algebra in Example 3.4. Define a cubic set $\mathcal{A}=\langle\tilde{A},\mu\rangle$ in X by the following table:*

X	$\tilde{A}$	μ
0	$[0.5, 0.9]$	0.55
a	$[0.4, 0.7]$	0.65
b	$[0.4, 0.7]$	0.65
c	$[0.5, 0.8]$	0.65

It is routine to verify that $\mathcal{A}=\langle\tilde{A},\mu\rangle$ is an internal cubic subalgebra of X.

Definition 3.11. *By an external cubic subalgebra of X, we mean a cubic subalgebra $\mathcal{A}=\langle\tilde{A},\mu\rangle$ of X which satisfies:*

$$(\forall x,y\in X)\left(\mu(x*y)\le\tilde{A}^-(x*y)\text{ or }\mu(x*y)\ge\tilde{A}^+(x*y)\right). \tag{11}$$

Example 3.12. *Let X be a BCI-algebra in Example 3.4. Define a cubic set $\mathcal{A}=\langle\tilde{A},\mu\rangle$ in X by the following table:*

X	$\tilde{A}$	μ
0	$[0.7, 0.88]$	0.9
a	$[0.4, 0.6]$	0.99
b	$[0.4, 0.6]$	0.99
c	$[0.6, 0.8]$	0.99

It is routine to verify that $\mathcal{A}=\langle\tilde{A},\mu\rangle$ is an external cubic subalgebra of X.

Theorem 3.13. *If a cubic subalgebra $\mathcal{A}=\langle\tilde{A},\mu\rangle$ of X is not external, then there exist $x,y\in X$ such that*

$$\tilde{A}^-(x*y)<\mu(x*y)<\tilde{A}^+(x*y).$$

Proof. Straightforward. □

Theorem 3.14. *If $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ is a cubic subalgebra of X which is both internal and external, then*

$$(\forall x, y \in X) \left(\mu(x * y) \in L(\tilde{A}) \cup U(\tilde{A}) \right), \tag{12}$$

where $L(\tilde{A}) := \{\tilde{A}^-(x * y) \mid x, y \in X\}$ and $U(\tilde{A}) := \{\tilde{A}^+(x * y) \mid x, y \in X\}$.

Proof. Assume that $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ is a cubic subalgebra of X which is both internal and external. Using Definitions 3.9 and 3.11, we have

$$\tilde{A}^-(x * y) \leq \mu(x * y) \leq \tilde{A}^+(x * y)$$

and $\mu(x * y) \notin (\tilde{A}^-(x * y), \tilde{A}^+(x * y))$ for all $x, y \in X$. It follows that $\mu(x * y) = \tilde{A}^-(x * y) \in L(\tilde{A})$ or $\mu(x * y) = \tilde{A}^+(x * y) \in U(\tilde{A})$. This completes the proof. $\square$

Note that the R-intersection of two cubic subalgebras of X is a cubic subalgebra of X (see Theorem 3.3). But, the R-intersection of two internal cubic subalgebras of X may not be internal as seen in the following example.

Example 3.15. *Let X be a BCI-algebra in Example 3.4. Let $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ be cubic sets in X defined by the tabular representation in the following table:*

X	$(\tilde{A}, \tilde{B})$	(μ, λ)
0	$([0.5, 0.9], [0.4, 0.8])$	$(0.55, 0.45)$
a	$([0.4, 0.7], [0.3, 0.6])$	$(0.65, 0.50)$
b	$([0.4, 0.7], [0.4, 0.8])$	$(0.65, 0.55)$
c	$([0.5, 0.8], [0.3, 0.6])$	$(0.65, 0.55)$

Then $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ are internal cubic subalgebras of X. The R-intersection of $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ is given by the following table:

X	$\tilde{A} \cap \tilde{B}$	$\mu \vee \lambda$
0	$[0.4, 0.8]$	0.55
a	$[0.3, 0.6]$	0.65
b	$[0.4, 0.7]$	0.65
c	$[0.3, 0.6]$	0.65

We know that $\mathcal{A} \cap_R \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \vee \lambda \rangle$ is a cubic subalgebra of X. But it is not internal since $(\mu \vee \lambda)(b * c) = (\mu \vee \lambda)(a) = 0.65 \notin [0.3, 0.6] = (\tilde{A} \cap \tilde{B})(a) = (\tilde{A} \cap \tilde{B})(b * c)$.

Now, we give conditions for the R-intersection of two internal cubic subalgebras to be an internal cubic subalgebra.

Theorem 3.16. *If two internal cubic subalgebras $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ of X satisfy the condition*

$$(\forall x, y \in X) \left((\mu \vee \lambda)(x * y) \leq \min\{\tilde{A}^+(x * y), \tilde{B}^+(x * y)\} \right), \tag{13}$$

then their R-intersection $\mathcal{A} \cap_R \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \vee \lambda \rangle$ is an internal cubic subalgebra of X.

Proof. By Theorem 3.3, we know that $\mathcal{A} \cap_R \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \vee \lambda \rangle$ is a cubic subalgebra of X. Since $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ are internal, we have

$$\tilde{A}^-(x * y) \leq \mu(x * y) \leq \tilde{A}^+(x * y)$$

and

$$\tilde{B}^-(x * y) \leq \lambda(x * y) \leq \tilde{B}^+(x * y)$$

for all $x, y \in X$. It follows from (13) that

$$\begin{aligned} (\tilde{A} \cap \tilde{B})^-(x * y) &\leq (\mu \vee \lambda)(x * y) \\ &\leq \min\{\tilde{A}^+(x * y), \tilde{B}^+(x * y)\} \\ &= (\tilde{A} \cap \tilde{B})^+(x * y) \end{aligned}$$

for all $x, y \in X$. Therefore $\mathcal{A} \cap_R \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \vee \lambda \rangle$ is an internal cubic subalgebra of X. □

Note from Example 3.4 that the R-union of two cubic subalgebras may not be a cubic subalgebra. We now have the following question.

Question 3.17. *Consider two cubic subalgebras $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ of X for which their R-union is a cubic subalgebra. If $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ are internal, then is the R-union of $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ internal?*

The following example gives a negative answer to the question above.

Example 3.18. *Let X be a BCI-algebra in Example 3.4. Let $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ be cubic sets in X defined by the tabular representation in the following table:*

X	$(\tilde{A}, \tilde{B})$	(μ, λ)
0	$([0.5, 0.9], [0.3, 0.8])$	$(0.50, 0.35)$
a	$([0.3, 0.7], [0.1, 0.6])$	$(0.65, 0.55)$
b	$([0.4, 0.8], [0.2, 0.7])$	$(0.60, 0.45)$
c	$([0.3, 0.7], [0.1, 0.6])$	$(0.65, 0.55)$

Then $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ are internal cubic subalgebras of X. The R-union of $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ is given by the following table:

X	$\tilde{A} \cup \tilde{B}$	$\mu \wedge \lambda$
0	$[0.5, 0.9]$	0.35
a	$[0.3, 0.7]$	0.55
b	$[0.4, 0.8]$	0.45
c	$[0.3, 0.7]$	0.55

We know that $\mathcal{A} \cup_R \mathcal{B} = \langle \tilde{A} \cup \tilde{B}, \mu \wedge \lambda \rangle$ *is a cubic subalgebra of* X *which is not internal.*

We provide conditions for the R-union of two internal cubic subalgebras to be an internal cubic subalgebra.

Theorem 3.19. *Let* $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ *and* $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ *be cubic subalgebras of* X *such that* $\mathcal{A} \cup_R \mathcal{B} = \langle \tilde{A} \cup \tilde{B}, \mu \wedge \lambda \rangle$ *is a cubic subalgebra of* X *and*

$$(\forall x, y \in X) \left((\mu \wedge \lambda)(x * y) \geq \max\{\tilde{A}^-(x * y), \tilde{B}^-(x * y)\} \right). \tag{14}$$

If $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ *and* $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ *are internal, then their R-union* $\mathcal{A} \cup_R \mathcal{B} = \langle \tilde{A} \cup \tilde{B}, \mu \wedge \lambda \rangle$ *is an internal cubic subalgebra of* X.

Proof. If $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ are internal, then

$$\tilde{A}^-(x * y) \leq \mu(x * y) \leq \tilde{A}^+(x * y)$$

and

$$\tilde{B}^-(x * y) \leq \lambda(x * y) \leq \tilde{B}^+(x * y)$$

for all $x, y \in X$. It follows from (14) that

$$\begin{aligned} (\tilde{A} \cup \tilde{B})^-(x * y) &= \max\{\tilde{A}^-(x * y), \tilde{B}^-(x * y)\} \\ &\leq (\mu \wedge \lambda)(x * y) \leq (\tilde{A} \cup \tilde{B})^+(x * y) \end{aligned}$$

for all $x, y \in X$. Hence $\mathcal{A} \cup_R \mathcal{B} = \langle \tilde{A} \cup \tilde{B}, \mu \wedge \lambda \rangle$ is an internal cubic subalgebra of X. □

Theorem 3.20. *Let* $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ *and* $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ *be cubic subalgebras of* X *for which* $\mathcal{A} \cup_P \mathcal{B} = \langle \tilde{A} \cup \tilde{B}, \mu \vee \lambda \rangle$ *(resp.,* $\mathcal{A} \cap_P \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \wedge \lambda \rangle$*) is a cubic subalgebra of* X. *If* $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ *and* $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ *are internal, then the P-union* $\mathcal{A} \cup_P \mathcal{B} = \langle \tilde{A} \cup \tilde{B}, \mu \vee \lambda \rangle$ *(resp., P-intersection* $\mathcal{A} \cap_P \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \wedge \lambda \rangle$*) is an internal cubic subalgebra of* X.

Proof. If $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ are internal, then

$$\tilde{A}^-(x * y) \leq \mu(x * y) \leq \tilde{A}^+(x * y)$$

and

$$\tilde{B}^-(x * y) \leq \lambda(x * y) \leq \tilde{B}^+(x * y)$$

for all $x, y \in X$. It follows that

$$\begin{aligned} (\tilde{A} \cup \tilde{B})^-(x * y) &= \max\{\tilde{A}^-(x * y), \tilde{B}^-(x * y)\} \\ &\leq (\mu \vee \lambda)(x * y) \\ &\leq \max\{\tilde{A}^+(x * y), \tilde{B}^+(x * y)\} \\ &= (\tilde{A} \cup \tilde{B})^+(x * y) \end{aligned}$$

and

$$\begin{aligned}(\tilde{A}\cap\tilde{B})^{-}(x*y) &= \min\{\tilde{A}^{-}(x*y),\tilde{B}^{-}(x*y)\}\\ &\leq (\mu\wedge\lambda)(x*y)\\ &\leq \min\{\tilde{A}^{+}(x*y),\tilde{B}^{+}(x*y)\}\\ &= (\tilde{A}\cap\tilde{B})^{+}(x*y)\end{aligned}$$

for all $x, y \in X$. This completes the proof. □

Example 3.5 shows that the P-union of two cubic subalgebras of X is not a cubic subalgebra of X in general. We now take two cubic subalgebras $\mathcal{A} = \langle\tilde{A}, \mu\rangle$ and $\mathcal{B} = \langle\tilde{B}, \lambda\rangle$ of X such that their P-union is a cubic subalgebra of X. We pose a question.

Question 3.21. *If $\mathcal{A} = \langle\tilde{A}, \mu\rangle$ and $\mathcal{B} = \langle\tilde{B}, \lambda\rangle$ are external, then is the P-union of $\mathcal{A} = \langle\tilde{A}, \mu\rangle$ and $\mathcal{B} = \langle\tilde{B}, \lambda\rangle$ internal?*

The following example shows that the answer to the question above is negative.

Example 3.22. *Let $X = \{0, 1, 2\}$ be a set with the following Cayley table:*

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then X is a BCK-algebra (see [26]). Let $\mathcal{A} = \langle\tilde{A}, \mu\rangle$ and $\mathcal{B} = \langle\tilde{B}, \lambda\rangle$ be cubic sets in X defined by the tabular representation in the following table:

X	$(\tilde{A}, \tilde{B})$	(μ, λ)
0	$([0.3, 0.50], [0.5, 0.8])$	$(0.51, 0.15)$
1	$([0.2, 0.40], [0.3, 0.6])$	$(0.65, 0.25)$
2	$([0.3, 0.45], [0.4, 0.7])$	$(0.53, 0.35)$

Then $\mathcal{A} = \langle\tilde{A}, \mu\rangle$ and $\mathcal{B} = \langle\tilde{B}, \lambda\rangle$ are external cubic subalgebras of X. The P-union of $\mathcal{A} = \langle\tilde{A}, \mu\rangle$ and $\mathcal{B} = \langle\tilde{B}, \lambda\rangle$ is given by the following table:

X	$\tilde{A}\cup\tilde{B}$	$\mu\vee\lambda$
0	$[0.5, 0.8]$	0.51
1	$[0.3, 0.6]$	0.65
2	$[0.4, 0.7]$	0.53

We know that $\mathcal{A} \cup_P \mathcal{B} = \langle\tilde{A}\cup\tilde{B}, \mu\vee\lambda\rangle$ is a cubic subalgebra of X which is not internal.

We consider conditions for the P-union of external cubic subalgebras to be an internal cubic subalgebra.

Theorem 3.23. *Let $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ be external cubic subalgebras of X such that their P-union is a cubic subalgebra of X. If $\mathcal{A}^* = \langle \tilde{A}, \lambda \rangle$ and $\mathcal{B}^* = \langle \tilde{B}, \mu \rangle$ satisfy the condition* (10)*, then the P-union and P-intersection of $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ are internal cubic subalgebras of X.*

Proof. Assume that $\mathcal{A}^* = \langle \tilde{A}, \lambda \rangle$ and $\mathcal{B}^* = \langle \tilde{B}, \mu \rangle$ satisfy the condition (10). Then $\tilde{A}^-(x*y) \leq \lambda(x*y) \leq \tilde{A}^+(x*y)$ and $\tilde{B}^-(x*y) \leq \mu(x*y) \leq \tilde{B}^+(x*y)$ for all $x, y \in X$. Since $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ are external, we have $\mu(x*y) \notin \left(\tilde{A}^-(x*y), \tilde{A}^+(x*y)\right)$ and $\lambda(x*y) \notin \left(\tilde{B}^-(x*y), \tilde{B}^+(x*y)\right)$ for all $x, y \in X$. Hence we can consider four cases as follows:

$$\begin{cases} \mu(x*y) \leq \tilde{A}^-(x*y) \leq \lambda(x*y) \leq \tilde{A}^+(x*y), \\ \lambda(x*y) \leq \tilde{B}^-(x*y) \leq \mu(x*y) \leq \tilde{B}^+(x*y) \end{cases} \tag{15}$$

$$\begin{cases} \mu(x*y) \leq \tilde{A}^-(x*y) \leq \lambda(x*y) \leq \tilde{A}^+(x*y), \\ \tilde{B}^-(x*y) \leq \mu(x*y) \leq \tilde{B}^+(x*y) \leq \lambda(x*y) \end{cases} \tag{16}$$

$$\begin{cases} \tilde{A}^-(x*y) \leq \lambda(x*y) \leq \tilde{A}^+(x*y) \leq \mu(x*y), \\ \lambda(x*y) \leq \tilde{B}^-(x*y) \leq \mu(x*y) \leq \tilde{B}^+(x*y) \end{cases} \tag{17}$$

$$\begin{cases} \tilde{A}^-(x*y) \leq \lambda(x*y) \leq \tilde{A}^+(x*y) \leq \mu(x*y), \\ \tilde{B}^-(x*y) \leq \mu(x*y) \leq \tilde{B}^+(x*y) \leq \lambda(x*y) \end{cases} \tag{18}$$

for all $x, y \in X$. The first case implies that $\mu(x*y) = \tilde{A}^-(x*y) = \tilde{B}^-(x*y) = \lambda(x*y)$ for all $x, y \in X$. It follows that

$$\begin{aligned} (\tilde{A} \cup \tilde{B})^-(x*y) &= \max\{\tilde{A}^-(x*y), \tilde{B}^-(x*y)\} = (\mu \vee \lambda)(x*y) \\ &\leq \max\{\tilde{A}^+(x*y), \tilde{B}^+(x*y)\} = (\tilde{A} \cup \tilde{B})^+(x*y). \end{aligned}$$

for all $x, y \in X$. Fir the cases (16) and (17), it is clear that

$$(\tilde{A} \cup \tilde{B})^-(x*y) \leq (\mu \vee \lambda)(x*y) \leq (\tilde{A} \cup \tilde{B})^+(x*y)$$

for all $x, y \in X$. The case (18) induces $\mu(x*y) = \tilde{B}^+(x*y) = \tilde{A}^+(x*y) = \lambda(x*y)$, and so

$$\begin{aligned} (\tilde{A} \cup \tilde{B})^-(x*y) &= \max\{\tilde{A}^-(x*y), \tilde{B}^-(x*y)\} \leq (\mu \vee \lambda)(x*y) \\ &= \max\{\mu(x*y), \lambda(x*y)\} \\ &= \max\{\tilde{A}^+(x*y), \tilde{B}^+(x*y)\} = (\tilde{A} \cup \tilde{B})^+(x*y). \end{aligned}$$

for all $x, y \in X$. Therefore $\mathcal{A} \cup_P \mathcal{B} = \langle \tilde{A} \cup \tilde{B}, \mu \vee \lambda \rangle$ is an internal cubic subalgebra of X. Similarly, we can verify that $\mathcal{A} \cap_P \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \wedge \lambda \rangle$ is an internal cubic subalgebra of X. □

We consider conditions for the P-union of external cubic subalgebras to be an external cubic subalgebra.

Theorem 3.24. *Let $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ and $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ be external cubic subalgebras of X such that their P-union is a cubic subalgebra of X. If $\mathcal{A}^* = \langle \tilde{A}, \lambda \rangle$ and $\mathcal{B}^* = \langle \tilde{B}, \mu \rangle$*

satisfy the condition (11)*, then the P-union of* $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ *and* $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ *is an external cubic subalgebra of* X*.*

Proof. For any $x, y \in X$ we have

$$\begin{aligned} &\mu(x*y) \leq \tilde{A}^-(x*y) \text{ or } \mu(x*y) \geq \tilde{A}^+(x*y), \\ &\lambda(x*y) \leq \tilde{B}^-(x*y) \text{ or } \lambda(x*y) \geq \tilde{B}^+(x*y), \\ &\lambda(x*y) \leq \tilde{A}^-(x*y) \text{ or } \lambda(x*y) \geq \tilde{A}^+(x*y), \\ &\mu(x*y) \leq \tilde{B}^-(x*y) \text{ or } \mu(x*y) \geq \tilde{B}^+(x*y). \end{aligned}$$

It follows that

$$(\mu \vee \lambda)(x*y) \leq \max\{\tilde{A}^-(x*y), \tilde{B}^-(x*y)\} = (\tilde{A} \cup \tilde{B})^-(x*y)$$

or

$$(\mu \vee \lambda)(x*y) \geq \max\{\tilde{A}^+(x*y), \tilde{B}^+(x*y)\} = (\tilde{A} \cup \tilde{B})^+(x*y)$$

for all $x, y \in X$. Therefore $\mathcal{A} \cup_P \mathcal{B} = \langle \tilde{A} \cup \tilde{B}, \mu \vee \lambda \rangle$ is an external cubic subalgebra of X. □

We provide conditions for the P-intersection of external cubic subalgebras to be an external cubic subalgebra.

Theorem 3.25. *Let* $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ *and* $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ *be external cubic subalgebras of* X *such that*

(1) *The P-union of* $\mathcal{A} = \langle \tilde{A}, \mu \rangle$ *and* $\mathcal{B} = \langle \tilde{B}, \lambda \rangle$ *is a cubic subalgebra of* X*,*
(2) *The following assertion is valid.*

$$\begin{aligned} &\min\{\max\{\tilde{A}^+(x*y), \tilde{B}^-(x*y)\}, \max\{\tilde{A}^-(x*y), \tilde{B}^+(x*y)\}\} \\ &\geq (\mu \wedge \lambda)(x*y) \\ &> \max\{\min\{\tilde{A}^+(x*y), \tilde{B}^-(x*y)\}, \min\{\tilde{A}^-(x*y), \tilde{B}^+(x*y)\}\} \end{aligned} \tag{19}$$

for all $x, y \in X$*.*

Then $\mathcal{A} \cap_P \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \wedge \lambda \rangle$ *is an external cubic subalgebra of* X*.*

Proof. For any $x, y \in X$, let

$$\begin{aligned} &\varepsilon := \max\{\min\{\tilde{A}^+(x*y), \tilde{B}^-(x*y)\}, \min\{\tilde{A}^-(x*y), \tilde{B}^+(x*y)\}\}, \\ &\delta := \min\{\max\{\tilde{A}^+(x*y), \tilde{B}^-(x*y)\}, \max\{\tilde{A}^-(x*y), \tilde{B}^+(x*y)\}\}, \end{aligned}$$

in the condition (19). Then $\delta = \tilde{A}^+(x*y)$, $\delta = \tilde{B}^+(x*y)$, $\delta = \tilde{A}^-(x*y)$ or $\delta = \tilde{B}^-(x*y)$. Assume that $\delta = \tilde{A}^+(x*y)$. Then $\tilde{B}^-(x*y) \leq \tilde{A}^+(x*y) \leq \tilde{B}^+(x*y)$, and so $\varepsilon = \max\{\tilde{A}^-(x*y), \tilde{B}^-(x*y)\}$. If $\varepsilon = \tilde{A}^-(x*y)$, then

$$\tilde{B}^-(x*y) \leq \tilde{A}^-(x*y) = \varepsilon < (\mu \wedge \lambda)(x*y) \leq \delta = \tilde{A}^+(x*y) \leq \tilde{B}^+(x*y), \tag{20}$$

which implies that $(\mu \wedge \lambda)(x * y) = \tilde{A}^+(x * y)$ because if $(\mu \wedge \lambda)(x * y) < \tilde{A}^+(x * y)$ then $\mu(x * y) < \tilde{A}^+(x * y)$ or $\lambda(x * y) < \tilde{B}^+(x * y)$, a contradiction. It follows that

$$(\mu \wedge \lambda)(x * y) = \tilde{A}^+(x * y) = (\tilde{A} \cap \tilde{B})^+(x * y)$$

and so that

$$(\mu \wedge \lambda)(x * y) \notin \left((\tilde{A} \cap \tilde{B})^-(x * y), (\tilde{A} \cap \tilde{B})^+(x * y)\right).$$

If $\varepsilon = \tilde{B}^-(x * y)$, then

$$\tilde{A}^-(x * y) \leq \tilde{B}^-(x * y) = \varepsilon < (\mu \wedge \lambda)(x * y) \leq \delta = \tilde{A}^+(x * y) \leq \tilde{B}^+(x * y). \quad (21)$$

If $(\mu \wedge \lambda)(x * y) < \tilde{A}^+(x * y)$ in (21), then $\mu(x * y) < \tilde{A}^+(x * y)$ or $\lambda(x * y) < \tilde{B}^+(x * y)$, a contradiction. Hence $(\mu \wedge \lambda)(x * y) = \tilde{A}^+(x * y)$ in (21), which implies that

$$(\mu \wedge \lambda)(x * y) \notin \left((\tilde{A} \cap \tilde{B})^-(x * y), (\tilde{A} \cap \tilde{B})^+(x * y)\right).$$

Suppose that $\delta = \tilde{A}^-(x * y)$. Then

$$\tilde{B}^-(x * y) \leq \tilde{B}^+(x * y) \leq \tilde{A}^-(x * y) \leq \tilde{A}^+(x * y),$$

and so $\varepsilon = \tilde{B}^+(x * y)$. Thus

$$\tilde{B}^-(x * y) = (\tilde{A} \cap \tilde{B})^-(x * y) \leq (\tilde{A} \cap \tilde{B})^+(x * y) = \tilde{B}^+(x * y) = \varepsilon < (\mu \wedge \lambda)(x * y),$$

and hence

$$(\mu \wedge \lambda)(x * y) \notin \left((\tilde{A} \cap \tilde{B})^-(x * y), (\tilde{A} \cap \tilde{B})^+(x * y)\right).$$

By the similarly way, we can have

$$(\mu \wedge \lambda)(x * y) \notin \left((\tilde{A} \cap \tilde{B})^-(x * y), (\tilde{A} \cap \tilde{B})^+(x * y)\right)$$

for the cases $\delta = \tilde{B}^+(x * y)$ and $\delta = \tilde{B}^-(x * y)$. Therefore $\mathcal{A} \cap_P \mathcal{B} = \langle \tilde{A} \cap \tilde{B}, \mu \wedge \lambda \rangle$ is an external cubic subalgebra of X. $\square$

4. Conclusions

We have introduced the notion of internal and external cubic subalgebra in BCK/BCI-algebra. We have discussed the R-union (resp., R-intersection, P-union, P-intersection) of internal and external cubic subalgebras in BCK/BCI-algebra. We have provided conditions for

- the R-intersection of internal cubic subalgebras to be an internal cubic subalgebra.
- the R-union of internal cubic subalgebras to be an internal cubic subalgebra.
- the P-intersection of internal cubic subalgebras to be an internal cubic subalgebra.
- the P-union of internal cubic subalgebras to be an internal cubic subalgebra.
- the P-union of external cubic subalgebras to be an external cubic subalgebra.
- the P-union of external cubic subalgebras to be an internal cubic subalgebra.

- the P-intersection of external cubic subalgebras to be an external cubic subalgebra.

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