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STRONG CONVERGENCE OF A HYBRID METHOD FOR INFINITE FAMILY OF NONEXPANSIVE MAPPING AND VARIATIONAL INEQUALITY

Savita Rathee[1], Monika Swami[2]

Maharshi Dayanand University, Rohtak, 124001, India,
[1]dr.savitarathee@gmail.com, [2]monikaswami06@gmail.com

Abstract. The motivation behind this paper is to use hybrid method for searching a typical component of the set of fixed points of an infinite family of nonexpansive mappings and the set of monotone, Lipschtiz continuous variational inequality problem. The contemplated method is combination of two method one is extragradient method and the other one is DQ method. Also, we demonstrate the strong convergence of the designed iterative technique, under some warm conditions.

Key words and Phrases: Non expansive mapping, Fixed point problem, Projection method, Variational inequality problem, extragradient method.

1. INTRODUCTION

Throughout this paper, let H be a real Hilbert Space with norm $||\cdot||$ and inner product $<\cdot,\cdot>$. Let D be a non empty closed convex subset of H. Let $A : D \to H$ be a non linear mapping then the problem of the variational inequality is to find a point $x \in D$ such that

$$< Ax, y - x > \geq \ 0, \quad \forall \ y \in D. \tag{1}$$

The solution set of the variational inequality is represented by Ω.
A point $x \in D$ is said to be a fixed point if $Tx = x$. We adopt $F(T)$ to represent the set of fixed points of T. A self mapping T on D is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y||, \forall x, y \in D$.

In 1964, Stampacchia [15] introduced and studied Variational inequality problem. It is notable that a wide classes of issue emerging in pure and applied sciences can be illuminated with the assistance of variational inequality problem or in other words we say that many problems are proved equivalent to variational inequality problem such as optimization problem, maximisation problem. Several researcher

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works on common solution of the variational inequality and the optimization problem. In fact, there are different approach to study variational inequality problems. Based on distinct approaches, many algorithms for solving variational inequality problem is considered and proposed ([3], [5], [6], [7], [8], [13], [17], [16]) Let us start with one of the method which is used in our paper, i.e., Korpelevich's extragradient method which was popularized by Korpelevich [9] in 1976 and which initiate a sequence $\{a_n\}$ defined as:

$$\begin{aligned} b_n &= P_D(a_n - \lambda A a_n) \\ a_{n+1} &= P_D(a_n - \lambda A b_n), n \geq 0 \end{aligned}$$

where P_D is the metric projection from R^n onto $D, A : D \to H$ is a monotone operator and λ is a constant. Korpelelevich [9] proved that the sequence $\{a_n\}$ converges strongly to a solution of VI(D,A).

Korpelevich's extragradient technique has widely been read for the solution of finding common point which belong to the solution set of fixed points of a nonexpansive mapping and variational inequality. In 2006, Nadezhkina and Takahashi [10] introduced the following method:

$$\begin{aligned} x_0 &= x \in D, \\ y_n &= P_D(x_n - \lambda_n A x_n), \\ z_n &= \alpha_n x_n + (1 - \alpha_n) T P_D(x_n - \lambda_n A y_n), \\ D_n &= \{z \in D : ||z_n - z|| \leq ||x_n - z||\}, \\ Q_n &= \{z \in D :< x_n - z, x - x_n >\geq 0\}, \\ x_{n+1} &= P_{D_n \cap Q_n} x, n \geq 0, \end{aligned}$$

where P_D is the metric projection from H onto D and $T : D \to D$ be a nonexpansive mapping with $A : D \to H$ monotone k-Lipschitz continuous mapping having two sequences $\{\alpha_n\}$ and $\{\lambda_n\}$. They established the strong convergence of the sequences, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ to the same element of $F(T) \cap \Omega$.

Influenced from the work of Ceng. et al. [1], in which they proved the weak convergence of the iterative method to $\cap_{i=n}^{N} F(T_i) \cap \Omega$ while considering finite family of nonexpansive mapping, Ceng. et al. [2], discovered hybrid extragradient like approximation method for proving strong convergence of this method to $P_{F(T)\cap\Omega}$, Yao. et al. [19], suggested a hybrid method for variational inequality and fixed point of infinite family of nonexpansive mapping and prove its strong convergence to $\cap_{n=i}^{\infty} F(T_n) \cap \Omega$, in this paper, we use hybrid method for finding a typical component of the arrangement of fixed points of an infinite family of nonexpansive mapping and the set of monotone, Lipschtiz continuous variational inequalities problem. The planned method is combination of two methods, one is the extragradient method and the other one is DQ method. Also, we demonstrated the strong convergence of the designed iterative technique, under some warm conditions.

2. PRELIMINARIES

In this section, we recall some basics definitions and lemmas which are further used in our proof. Let D be a nonempty closed convex subset of a real Hilbert space H. A mapping $A : D \to H$ is called monotone if $< Au - Av, u - v > \geq 0, \forall u, v \in D$ and a mapping $T : D \to D$ is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y||, \forall x, y \in D$. $F(T)$ denote the set of fixed points of T, that is, $F(T) = \{x \in D : Tx = x\}$. For every point $x \in H$, $P_D(x)$ represent the unique nearest point in D and P_D is called the metric projection of H onto convex subset D and also, a nonexpansive mapping from H onto D. It has the following properties:

(i) $P_D(x) \in D$

$$||P_D(x) - P_D(y)|| \ \leq \ ||x - y||, \ \forall \ x, y \in H$$

(ii) $< x - P_D(x), y - P_D(x) > \leq \ 0, \ \forall \ x \in H, y \in D$

(iii) The property (ii) is equivalent to

$$||x - P_D(x)||^2 + ||y - P_D(x)||^2 \ \leq \ ||x - y||, \ \forall \ x \in H, y \in D$$

(iv) In the variational inequality problem, projection implies that

$$u \in \Omega, \iff u = P_D(u - \lambda Au), \ \forall \ \lambda > 0.$$

As Opial's condition [11], implies for, any sequence $\{x_n\}$ with x_n converges weakly to x and the inequality

$$\lim_{n\to\infty} \inf ||x_n - x|| < \lim_{n\to\infty} \inf ||x_n - y||$$

holds $\forall \ y \in H$ with $y \neq x$.

As D be a nonempty closed and convex subset of H. Let $\{T_i\}_{i=1}^{\infty}$ be infinite family of nonexpansive mappings of D into itself and $\{\mu_i\}_{i=1}^{\infty}$ be a real number sequence such that $0 \leq \mu_i \leq 1$ for every $i \in N$.
Here, we use the mapping W_n [12] defined as

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \mu_n T_n U_{n,n+1} + (1 - \mu_n) I, \\
U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1 - \mu_{n-1}) I, \\
U_{n,k} &= \mu_k T_k U_{n,k+1} + (1 - \mu_k) I, \\
U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1 - \mu_{k-1}) I, \\
U_{n,2} &= \mu_2 T_2 U_{n,3} + (1 - \mu_2) I, \\
W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1 - \mu_1) I,
\end{aligned}$$

where $\mu_1, \mu_2, \cdots$ are real numbers such that $0 \leq \mu_n \leq 1$ for each $n \geq 0$ and $T_1, T_2, \cdots$ are nonexpansive mappings from D into itself. Non expansivity of T_i gives us the non expansivity of W_n. We have the following pivotal lemmas related to W_n [12] which are stated as:

Lemma 2.1. [11] *Let D be a non empty closed convex subset of real Hilbert space H. Let $T_1, T_2, T_3, \cdots$ are nonexpansive mappings of D into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\mu_1, \mu_2, \mu_3, \cdots$ are real numbers such that $0 \leq \mu_i \leq 1$ for every $i \in N$. Then for every $x \in D$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Lemma 2.2. [11] *Let D be a nonempty closed convex subset of H. Let $T_1, T_2, T_3, \cdots$ are nonexpansive mappings of D into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\mu_1, \mu_2, \mu_3, \cdots$ are real numbers such that $0 \leq \mu_i \leq 1$ for every $i \in N$. Then, $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.3. [14] *Using the above two lemmas, W is defined from D to itself as: $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \forall\ x \in D$. If $\{x_n\}$ is bounded sequence in D, then*

$$\lim_{n\to\infty} ||Wx_n - W_n x_n|| = 0.$$

Lemma 2.4. [18] *Let D be a nonempty closed convex subset of H. Let $T : D \to D$ be a nonexpansive mapping with $F(T) \neq \phi$. Then T is demiclosed on D, i.e., if $y_n \to z \in D$ weakly and $y_n - Ty_n \to y$ strongly then $(I - T)z = y$.*

Lemma 2.5. [4] *Let D be nonempty closed and convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $s = P_D(u)$. If $\{x_n\}$ is such that $w_w(x_n) \subset D$ and satisfy the equation*

$$||x_n - u|| \leq ||u - s||,\ \forall\ n.$$

Then $\{x_n\} \to s$.

Here $w_w(x_n)$ respresents the weak convergence of sequence x_n.

3. Main Result

Theorem 3.1. *Let H be real Hilbert Space and D be its nonempty closed and convex subset. Let $A : D \to H$ be a k- Lipschitz continuous, monotone, nonexpansive mapping and $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of D into itself such that $\cap_{n=1}^{\infty} F(T_n) \cap \Omega \neq \phi$. Let $a_1 = a_0 \in D$. For $D_1 = D$, let $\{t_n\}$, $\{c_n\}$, $\{b_n\}$, and $\{a_n\}$ be sequences generated by:*

$$\begin{aligned} t_n &= P_D(a_n - \lambda_n A a_n) \\ c_n &= \beta_n a_n + (1 - \beta_n) W_n P_D(a_n - \lambda_n A t_n) \\ b_n &= \alpha_n a_n + (1 - \alpha_n) W_n P_D(a_n - \lambda_n A c_n), \\ D_n &= \{s \in D : ||b_n - s|| \leq ||a_n - s||, ||c_n - s|| \leq ||a_n - s||\}, \\ Q_n &= \{s \in D :< a_n - s, a_0 - a_n >\geq 0\}, \\ a_{n+1} &= P_{D_n \cap Q_n} a_0, n \geq 0, \end{aligned} \tag{2}$$

where W_n is mapping defined above. Assume the following conditions hold:

(i) $\{\lambda_n\} \subset [l, m]$ for some $l, m \in (0, 1/k)$

(ii) $\{\alpha_n\}, \{\beta_n\} \subset [0, c]$ *for some* $c \in [0, 1)$.

Then the sequences $\{t_n\}$, $\{c_n\}$, $\{b_n\}$, *and* $\{a_n\}$ *converges strongly to the same point* $P_{\cap_{n=1}^{\infty} F(T_n) \cap \Omega}(a_0)$.

PROOF. We divide the proof into several parts while proving the main result. We assume that all the assumptions of theorem are satisfied.
Step 1: Firstly, we prove that $D_n \cap Q_n$ is closed and convex for every $n \in N \cup \{0\}$. As it is obvious that D_n is closed and Q_n is closed and convex . We only need to prove that D_n is convex.
Since, $||b_n - s|| \leq ||a_n - s||$ is equivalent to $||b_n - a_n||^2 + 2 < b_n - a_n, a_n - s > \leq 0$ it gives us that D_n is convex . Hence, $D_n \cap Q_n$ is a closed convex subset of H for any $n \in N$
Step 2: We show that $\cap_{n=1}^{\infty} F(T_n) \cap \Omega \subseteq D_n \cap Q_n$.
For this purpose, set $u_n = P_D(a_n - \lambda_n A t_n)$ and $v_n = P_D(a_n - \lambda_n A c_n)$ for all $n \geq 1$. Choose $a \in \cap_{n=1}^{\infty} F(T_n) \cap \Omega$. From property (iii) of P_D, we have

$$\begin{aligned}
||u_n - a||^2 &\leq ||a_n - \lambda_n t_n - a||^2 - ||a_n - \lambda_n A t_n - u_n||^2 \\
&= ||a_n - a||^2 - ||a_n - u_n||^2 + 2\lambda_n < A t_n, a - u_n > \\
&= ||a_n - a||^2 - ||a_n - u_n||^2 + 2\lambda_n < A t_n, a - t_n > + 2\lambda_n < A t_n, t_n - u_n >
\end{aligned} \tag{3}$$

As $a \in \Omega$ and $t_n \in D_n \subset D$, we get

$$< Au, t_n - a > \geq 0.$$

From the monotonicity of A, we have

$$< A t_n, t_n - a > \geq 0. \tag{4}$$

Combining (3) and (4), we get

$$\begin{aligned}
||u_n - a||^2 &\leq ||a_n - a||^2 - ||a_n - u_n||^2 + 2\lambda_n < A t_n, t_n - u_n > \\
&= ||a_n - a||^2 - ||a_n - t_n||^2 - 2 < a_n - t_n, t_n - u_n > - ||t_n - u_n||^2 \\
&+ 2\lambda_n < A t_n, t_n - u_n > \qquad (5) \\
&= ||a_n - a||^2 - ||a_n - t_n||^2 - ||t_n - u_n||^2 + 2 < a_n - t_n - \lambda_n A t_n, u_n - t_n >
\end{aligned}$$

as $t_n = P_D(a_n - \lambda_n A a_n)$ and $u_n \in D_n$. Then by the property (iii) of P_D, we have

$$< a_n - \lambda_n A a_n - t_n, u_n - t_n > \leq 0.$$

Hence,

$$\begin{aligned}
&< a_n - \lambda_n A t_n - t_n, u_n - t_n > \\
&= < a_n - \lambda_n A a_n - t_n, u_n - t_n > + < \lambda_n A a_n - \lambda_n A t_n, u_n - t_n > \\
&\leq < \lambda_n A a_n - \lambda_n A t_n, u_n - t_n > < \lambda_n k ||a_n - t_n||.||u_n - t_n||.
\end{aligned} \tag{6}$$

From (5) and (6), we have

$$\begin{aligned}
||u_n - a||^2 &\leq ||a_n - a||^2 - ||a_n - t_n||^2 - ||t_n - u_n||^2 + 2 < a_n - t_n - \lambda_n A t_n, u_n - t_n > \\
&\leq ||a_n - a||^2 - ||a_n - t_n||^2 - ||t_n - u_n||^2 + 2\lambda_n k ||a_n - t_n||.||u_n - t_n|| \\
&\leq ||a_n - a||^2 - ||a_n - t_n||^2 - ||t_n - u_n||^2 + \lambda_n^2 k^2 ||a_n - t_n||^2 + ||u_n - t_n||^2 \\
&\leq ||a_n - a||^2 + (\lambda_n^2 k^2 - 1)||a_n - t_n||^2 \\
&\leq ||a_n - a||^2.
\end{aligned} \tag{7}$$

In a similar manner, we can show that

$$\begin{aligned} ||v_n - a||^2 &\le ||a_n - a||^2 + ((\lambda_n^2 k^2 - 1)||a_n - c_n||^2 \\ &\le ||a_n - a||^2. \end{aligned} \tag{8}$$

From (7) together with $c_n = \beta_n a_n + (1-\beta_n)W_n u_n$ and $a = W_n a$, we get

$$\begin{aligned} ||c_n - a||^2 &= ||\beta_n(a_n - a) + (1-\beta_n)(W_n u_n - a)||^2 \\ &\le \beta_n ||a_n - a||^2 + (1-\beta_n)||W_n u_n - a||^2 \\ &\le \beta_n ||a_n - a||^2 + (1-\beta_n)||u_n - a||^2 \\ &\le \beta_n ||a_n - a||^2 + (1-\beta_n)(||a_n - a||^2 + (\lambda_n^2 k^2 - 1)||a_n - u_n||^2) \\ &\le ||a_n - a||^2 + (1-\beta_n)(\lambda_n^2 k^2 - 1)||a_n - u_n||^2 \\ &\le ||a_n - a||^2. \end{aligned} \tag{9}$$

Again, from (8) together with $b_n = \alpha_n a_n + (1-\alpha_n)W_n v_n$ and $a = W_n a$, we obtain

$$\begin{aligned} ||b_n - a||^2 &= ||\alpha_n(a_n - a) + (1-\alpha_n)(W_n v_n - a)||^2 \\ &\le ||a_n - a||^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)||a_n - c_n||^2 \\ &\le ||a_n - a||^2. \end{aligned} \tag{10}$$

Thus, $a \in D_n$ for $\{b_n\}$.

From, (9) and (10), we get

$$a \in D_n$$

and hence

$$\cap_{n=1}^{\infty} F(T_n) \cap \Omega \subseteq D_n \ \forall \ n \in N. \tag{11}$$

Next, we show that

$$\cap_{n=1}^{\infty} F(T_n) \cap \Omega \subset D_n \cap Q_n \ \forall \ n \in N.$$

We prove this by induction. For $n = 1$, we have

$$\cap_{n=1}^{\infty} F(T_n) \cap \Omega \subset D_1 \text{ and } Q_1 = H,$$

and we get

$$\cap_{n=1}^{\infty} F(T_n) \cap \Omega \subset D_1 \cap Q_1.$$

Assume a_k is defined and $\cap_{n=1}^{\infty} F(T_n) \cap \Omega \subset D_k \cap Q_k$ for some $k \ge 0$. Then D_k and Q_k is closed and convex due to well defined nature of c_k and b_k as elements of D. Thus $D_k \cap Q_k$ is closed and convex subset, which is nonempty since by assumption of $\cap_{n=1}^{\infty} F(T_n) \cap \Omega$. Consequently, $a_{n+1} \in D_n \cap Q_n$ such that

$$a_{n+1} = P_{D_n \cap Q_n} a_0,$$

then

$$< a_{n+1} - z, a_0 - a_{n+1} > \geq 0 \text{ for each } z \in D_n \cap Q_n.$$

As especial

$$< a_{n+1} - z, a_0 - a_{n+1} > \geq 0 \text{ for each } z \in \cap_{n=1}^{\infty} F(S_n) \cap \Omega$$

and hence $z \in Q_{n+1}$. It follows that $\cap_{n=1}^{\infty} F(T_n) \cap \Omega \subset Q_{n+1}$. This together with (11) gives

$$\cap_{n=1}^{\infty} F(T_n) \cap \Omega \subset D_n \cap Q_n, \ \forall \ n.$$

Thus $\{a_n\}$ is well-defined.

Step 3: Now, we show that $\{a_n\}$ is bounded.

Since $\cap_{n=1}^{\infty} F(T_n) \cap \Omega$ is a nonempty closed convex subset of H, then there exist a unique $z_0 \in \cap_{n=1}^{\infty} F(T_n) \cap \Omega$ such that $z_0 = P_{\cap_{n=1}^{\infty} F(T_n) \cap \Omega} a_0$. From $a_{n+1} = P_{D_n \cap Q_n} a_0$, we have

$$||a_{n+1} - a_0|| \leq ||z - a_0|| \text{ for every } z \in D_n \cap Q_n \text{ and for every } n \in N.$$

Since $z_0 \in \cap_{n=1}^{\infty} F(T_n) \cap \Omega \subset D_n \cap Q_n$, we have

$$||a_{n+1} - a_0|| \leq ||z_0 - a_0||, \ \forall \ n \in N. \tag{12}$$

Thus, we obtain $\{a_n\}$ is bounded.

As $a_{n+1} \in D_n \cap Q_n$ and $a_n = P_{Q_n} a_0$, we have

$$< a_0 - a_n, a_n - a_{n+1} > \geq 0 \tag{13}$$

$$0 \leq -||a_0 - a_n||^2 + ||a_0 - a_n||.||a_0 - a_{n+1}||,$$

and therefore

$$||a_{n+1} - a_0|| \geq ||a_n - a_0||.$$

This together with the boundedness of the sequence $\{a_n\}$ imply that $\lim\limits_{n\to\infty} ||a_n - a_0||$ exists.

Step 4: Now, we obtain the following equalities

$$\lim_{n\to\infty} ||a_{n+1} - a_n|| = \lim_{n\to\infty} ||a_n - c_n|| = \lim_{n\to\infty} ||a_n - t_n|| = 0$$

and

$$\lim_{n\to\infty} ||a_n - W_n a_n|| = \lim_{n\to\infty} ||b_n - W_n b_n|| = 0.$$

Consider

$$\begin{aligned} ||a_{n+1} - a_n||^2 &= ||(a_{n+1} - a_0) - (a_n - a_0)||^2 \\ &= ||a_{n+1} - a_0||^2 - ||a_n - a_0||^2 - 2 < a_{n+1} - a_0, a_n - a_0 > \\ &\leq ||a_{n+1} - a_0||^2 - ||a_n - a_0||^2. \end{aligned}$$

As $\lim_{n\to\infty} ||a_n - a_0||$ exists, we get

$$||a_{n+1} - a_0||^2 - ||a_n - a_0||^2 \to 0.$$

Therefore, $\lim_{n\to\infty} ||a_{n+1} - a_n|| = 0$.

Since $a_{n+1} \in D_n$, we have

$$||b_n - a_{n+1}|| \le ||a_n - a_{n+1}||$$

and

$$\begin{aligned}||a_n - b_n|| &\le ||a_n - a_{n+1}|| + ||a_{n+1} - b_n||\\ &\le 2||a_{n+1} - a_n|| \to 0.\end{aligned}$$

Similarly, we get $||a_n - c_n|| \to 0$.
Now, for each $u \in \cap_{n=1}^{\infty} F(T_n) \cap \Omega$ from (8), we have

$$\begin{aligned}||a_n - t_n||^2 &\le \frac{1}{(1-\beta_n)(\lambda_n^2 k^2 - 1)}(||a_n - a||^2 - ||c_n - a||^2)\\ &\le \frac{1}{(1-\beta_n)(\lambda_n^2 k^2 - 1)}(||a_n - a|| + ||c_n - a||)(||a_n - c_n||).\end{aligned}$$

Since $||a_n - c_n|| \to 0$, sequence $\{a_n\}$ and $\{c_n\}$ are bounded, we obtain

$$||a_n - t_n|| \to 0.$$

By the same idea as in (7), we obtain that

$$||u_n - a||^2 \le ||a_n - a||^2 + (\lambda_n^2 k^2 - 1)||t_n - u_n||^2.$$

Hence,

$$\begin{aligned}||c_n - a||^2 &\le \beta_n||a_n - a||^2 + (1-\beta_n)||u_n - a||^2\\ &\le \beta_n||a_n - a||^2 + (1-\beta_n)(||a_n - a||^2 + (\lambda_n^2 k^2 - 1)||t_n - u_n||^2)\\ &= ||a_n - a||^2 + (1-\beta_n)(\lambda_n^2 k^2 - 1)||t_n - u_n||^2.\end{aligned}$$

It follows that

$$\begin{aligned}||t_n - u_n||^2 &\le \frac{1}{(1-\beta_n)(\lambda_n^2 k^2 - 1)}(||a_n - a|| - ||c_n - a||)||a_n - c_n||\\ &\to 0.\end{aligned}$$

From the k-Lipschitz continuity of A, we have $||Ab_n - At_n|| \to 0$, from

$$||a_n - u_n|| \le ||a_n - t_n|| + ||t_n - u_n||,$$

we get

$$||a_n - u_n|| \to 0.$$

In a similar manner and from the idea as in (7), we get

$$||v_n - a||^2 \le ||a_n - a||^2 + (\lambda_n^2 k^2 - 1)||t_n - v_n||.$$

Hence,

$$\begin{aligned}||b_n - a||^2 &\leq \alpha_n||a_n - a||^2 + (1-\alpha_n)(||a_n - a||^2 + (\lambda_n^2 k^2 - 1)||t_n - v_n||^2) \\ &\leq ||a_n - a||^2 + (1-\alpha_n)(\lambda_n^2 k_2 - 1)||t_n - v_n||^2.\end{aligned}$$

It follows that

$$\begin{aligned}||t_n - v_n||^2 &\leq \frac{1}{(1-\alpha_n)(\lambda_n^2 k^2 - 1)}(||a_n - a||^2 - ||b_n - a||^2) \\ &\leq \frac{1}{(1-\alpha_n)(\lambda_n^2 k^2 - 1)}(||a_n - a|| + ||b_n - a||)||a_n - b_n|| \\ &\to 0,\end{aligned}$$

and from

$$||a_n - v_n|| \leq ||a_n - b_n|| + ||b_n - v_n||,$$

we have

$$||a_n - v_n|| \to 0.$$

Since $c_n = \beta_n a_n + (1-\beta_n)W_n u_n$,
we find

$$(1-\beta_n)(W_n u_n - u_n) = \beta_n(u_n - a_n) + (c_n - u_n)$$

and

$$\begin{aligned}(1-c)||W_n u_n - u_n|| &\leq (1-\beta_n)||W_n u_n - u_n|| \\ &\leq \beta_n||u_n - a_n|| + ||c_n - u_n|| \\ &\leq (1+\beta_n)||u_n - a_n|| + ||c_n - a_n||.\end{aligned}$$

Hence,

$$||u_n - W_n u_n|| \to 0. \tag{14}$$

In a similar way, as

$$b_n = \alpha_n a_n + (1-\alpha_n)v_n,$$

we have

$$(1-\alpha_n)(W_n v_n) = \alpha_n(v_n - a_n) + (b_n - v_n)$$

and

$$\begin{aligned}(1-c)||W_n v_n - v_n|| &\leq (1-\alpha_n)||W_n v_n - v_n|| \\ &\leq \alpha_n||v_n - a_n|| + ||b_n - a_n|| \\ &\leq (1+\alpha_n)||v_n - a_n|| + ||b_n - a_n||.\end{aligned}$$

Hence,

$$||v_n - W_n v_n|| \to 0. \tag{15}$$

To conclude

$$\begin{aligned} ||a_n - W_n a_n|| &\le ||a_n - u_n|| + ||u_n - W_n u_n|| + ||W_n u_n - W_n a_n|| \\ &\le ||a_n - u_n|| + ||u_n - W_n u_n|| + ||u_n - a_n|| \\ &\le 2||a_n - u_n|| + ||u_n - W_n u_n|| \end{aligned}$$

so

$$||a_n - W_n a_n|| \to 0.$$

Similarly, from (15), we get

$$||b_n - W_n b_n|| \to 0.$$

As, $||a_n - W_n a_n||$ and $||b_n - W_n b_n|| \to 0$

On the other hand, since $\{a_n\}$ and $\{b_n\}$ are bounded and from lemma (2.3), we have

$$\lim_{n\to\infty} ||W_n a_n - W a_n|| = 0$$

and

$$\lim_{n\to\infty} ||W_n b_n - W b_n|| = 0$$

Therefore, we have

$$\lim_{n\to\infty} ||a_n - W a_n|| = 0$$

and

$$\lim_{n\to\infty} ||b_n - W b_n|| = 0.$$

Step 5: Strong convergence of $\{t_n\}, \{b_n\}, \{c_n\}$ and $\{a_n\}$ to $P_{\cap_{n=1}^{\infty} F(T_n)\cap\Omega} a_0$. Furthermore, since $\{a_n\}$ and $\{b_n\}$ is bounded and has a subsequence $\{a_{n_i}\}$ and $\{b_{n_i}\}$ which converges weakly to some $a \in D$, hence we have

$$\lim_{j\to\infty} ||a_{n_j} - W a_{n_j}|| = 0 \text{ and } \lim_{j\to\infty} ||b_{n_j} - W b_{n_j}|| = 0.$$

From lemma (2.4), which gives that $I - W$ is demiclosed at zero. Thus, $a \in F(W)$. Since $u_n = P_{D_n}(a_n - \lambda_n A t_n)$ and $v_n = P_{D_n}(a_n - \lambda_n A c_n)$, for every $x \in D_n$, we have

$$< a_n - \lambda_n A t_n - u_n, u_n - x > \ge 0 \text{ and } < a_n - \lambda_n A c_n - v_n, v_n - x > \ge 0$$

hence,

$$< x - u_n, A t_n > \ge < x - u_n, \frac{a_n - u_n}{\lambda_n} > \text{ and } < x - v_n, A c_n > \ge < x - v_n, \frac{a_n - v_n}{\lambda_n} >$$

Combinig with monotonicity of A and consider $s_n = u_n + v_n$, we have

$$\begin{aligned} < x - s_n, Ax > &=< x - (u_n + v_n), Ax > \\ &=< x - u_n, Ax > + < x - v_n, Ax > \\ &\geq< x - u_n, Au_n > + < x - v_n, Av_n > \\ &\geq< x - u_n, Au_n - Ac_n > + < x - u_n, Ac_n > \\ &\quad + < x - v_n, Av_n - Ab_n > + < x - v_n, Ab_n > \\ &\geq< x - u_n, Au_n - Ac_n > + < x - v_n, Av_n - Ab_n > \\ &\quad + < x - u_n, \frac{a_n - u_n}{\lambda_n} > + < x - v_n, \frac{a_n - v_n}{\lambda_n} > \end{aligned}$$

Since $\lim_{n\to\infty}(a_n - u_n) = \lim_{n\to\infty}(c_n - u_n) = 0$ and $\lim_{n\to\infty}(a_n - v_n) = \lim_{n\to\infty}(b_n - v_n) = 0$, A is Lipschitz continuous and $\lambda_n \geq l > 0$, we deduce that

$$< x - a, Ax >= \lim_{n_i\to\infty} < x - s_{n_i}, Ax >\geq 0$$

This implies that $a \in \Omega$. Consequently, $a \in \cap_{n=1}^{\infty} F(T_n) \cap \Omega$ that is $w_w(a_n) \subset \cap_{n=1}^{\infty} F(T_n) \cap \Omega$.
In (12), if we assume $u = P_{\cap_{n=1}^{\infty} F(T_n)\cap\Omega} a_0$, we get

$$||a_0 - a_{n+1}|| \leq ||a_0 - P_{\cap_{n=1}^{\infty} F(T_n)\cap\Omega} a_0|| \tag{16}$$

Notice that $w_w(a_n) \subset \cap_{n=1}^{\infty} F(T_n) \cap \Omega$. Then, (16) and lemma (2.5) ensure the strong convergence of $\{a_{n+1}\}$ to $P_{\cap_{n=1}^{\infty} F(T_n)\cap\Omega} a_0$.
Consequently, $\{t_n\}$,$\{b_n\}$ and $\{c_n\}$ also converges strongly to $P_{\cap_{n=1}^{\infty} F(T_n)\cap\Omega} a_0$.
Hence the result.
Remark 1: We obtain the result of [9], if infinite family of mappings reduces to single mapping with $\beta_n = 0$ and $W_n = I$.
Remark 2: If $Q_n = 0$ and equation (2) reduces to two step iteration, then we obtain Theorem 3.1 of [19].

4. Numerical Example

In this part, we give an example which supports our result.
Example: Let $H = R$ and $D = [0, 2]$. Let $\alpha_n = n/(n + 2)$, $\beta_n = n/(n + 3)$, $\lambda_n = 1 + 1/n$, $Ax = 1/3(x - 1)$ and $W_n(x) = 2x/n$.
For $\{a_n\}$ defined in (2), we divide this procedure into 3 steps:
Setp 1: Find D_n. Since $D_n = \{s \in D : ||b_n - s|| \leq ||a_n - s||, ||c_n - s|| \leq ||a_n - s||\}$, we obtain $(2s - (a_n + b_n))(a_n - b_n) \leq 0$ and $(2s - (a_n + c_n))(a_n - c_n) \leq 0$. We have the different cases:
Case1: If $a_n - b_n = 0$, then $D_n = D \ \forall\ n \geq 1$.
Case 2: If $a_n - b_n > 0$ then $s \leq \frac{a-n+b_n}{2}$. Thus $D_n = [0, \frac{a_n+b_n}{2}]$, $\forall n \geq 1$.
Case 3: If $a_n - b_n < 0$ then $s \geq \frac{a-n+b_n}{2}$. Thus $D_n = [\frac{a_n+b_n}{2}, 2]$, $\forall n \geq 1$.
Similarly, we have $D_n = [0, \frac{a_n+c_n}{2}]$ for $a_n - c_n > 0$ and $D_n = [\frac{a_n+c_n}{2}, 2]$ for $a_n - c_n < 0$ $\forall n \geq 1$.

Thus D_n is intersection of possible cases generated due to $\{b_n\}$ and $\{c_n\}$.
Step 2: Find $Q_n = \{s \in Q :< a_n - s, x - a_n >\geq 0\} = \{s \in Q : (a_n - s)(x - a_n) \geq 0\}$.
We obtain the following cases:
Case 1: If $x - a_n = 0 \Rightarrow Q_n = D$.
Case 2: $x - a_n > 0 \rightarrow a_n - s \geq 0 \Rightarrow s \leq a_n \Rightarrow Q_n = D \cap [0, a_n]$.
Case 3: $x - a_n < 0 \rightarrow a_n - s \leq 0 \Rightarrow s \geq a_n \Rightarrow Q_n = D \cap [a_n, 2]$.
Step 3: Calculate the numerical result of $a_{n+1} = P_{D_n \cap Q_n} a_1$. Take $a_1 = 0.19$, we obtain the Table 1 and we observe that 0 is the solution of our iteration.

TABLE 1. Numerical result of iteration when $a_1 = 0.19$

n	λ_n	α_n	β_n	t_n	c_n	b_n	$D_n \cap Q_n$	a_n
1	2	0.333	0.25	0.73	0.6025	0.67	[0.24,2]	0.19
2	1.5	0.5	0.4	0.62	0.354	0.4015	[0.08075,2]	0.24
3	1.33	0.6	0.5	0.489	0.143	0.172	[0.0454,0.10925]	0.08075
4	1.25	0.67	0.5714	0.443	0.085	0.101	[0.028,0.145]	0.0454
5	1.2	0.714	0.625	0.417	0.0567	0.066	[0.0192,0.162]	0.028
6	1.167	0.75	0.67	0.4006	0.0408	0.047	[0.0139,0.171]	0.019
12	1.083	0.852	0.8	0.364	0.012	0.013	[0.0039,0.1853]	0.005
16	1.0625	0.89	0.84	0.35	0.006	0.007	[0.0023,0.187]	0.002
19	1.053	0.904	0.86	0.352	0.0048	0.0052	[0.0017,0.188]	0.0018
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
28	1.035	0.93	0.903	0.346	0.00234	0.00244	[0.00079,0.00081]	0.00086
29	1.034	0.935	0.906	0.345	0.00218	0.0022	[0.00058,0.00066]	0.00079

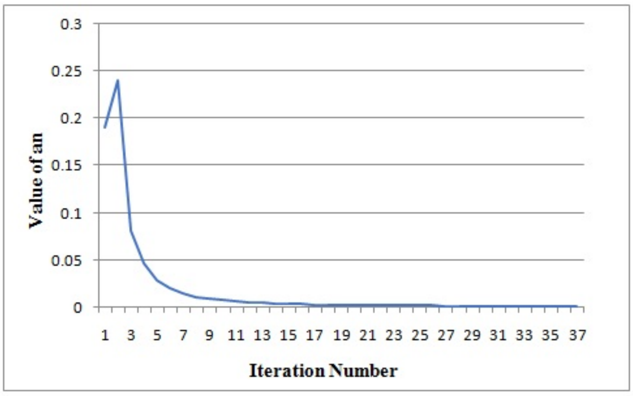

FIGURE 1. Convergence of $\{a_n\}$

5. CONCLUSION

In this paper, we proposed an algorithm to obtain its strong convergence and calculate the common solution of the infinite family of nonexpansive mappings with the variational inequality problem under some imposed conditions over algorithm. The efficiency of the proposed algorithm has also been illustrated by numerical example.

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