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HARMONIOUS CHROMATIC NUMBER OF CENTRAL GRAPH OF QUADRILATERAL SNAKES

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Abstract. This article shows the study about the harmonious coloring and investigation of the harmonious chromatic number of the central graph of the k-quadrilateral snake and k-alternate quadrilateral snake *i.e.* $\chi_H(C(kQ_n)) = (3k+2)n - (3k+1)$ and $\chi_H(C(k(AQ_n))) = (\frac{3k+4}{2})n - 1$.

Key words and Phrases: Harmonious coloring; harmonious chromatic number; central graph; quadrilateral and alternate quadrilateral snakes.

1. INTRODUCTION

The harmonious coloring [6, 7, 8, 17, 18] of a simple graph G is a kind of vertex coloring in which each edge of graph G has different color pair and least number of colors are to be used for this coloring is called the harmonious chromatic number, denoted by $\chi_H(G)$. For a simple graph G when we subdivide the each edge and connect all the non-adjacent vertices, such obtained graph is called the central graph [7, 17, 18] of G and it is denoted by $C(G)$. A quadrilateral snake Q_n is obtained from a path $u_1, u_2, ..., u_n$ by joining u_i and u_{i+1} to new vertices v_i and w_i respectively and adding edges v_iw_i for $(1 \leq i \leq n-1)$ in which every edge of a path is replaced by a cycle C_4. We take the following definitions from [3, 4, 9, 10, 11, 12, 13, 14, 15, 16]: quadrilateral snake, double quadrilateral snake, triple quadrilateral snake, alternate quadrilateral snake, double alternate quadrilateral snake and triple alternate quadrilateral snake and investigate the harmonious chromatic number of the central graph of these graphs. We also give the harmonious chromatic number of the central graph of k- quadrilateral snake and k- alternate quadrilateral snake, where the k-quadrilateral snake graph $k(Q_n)$ consists of k quadrilateral snakes

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with a common path and k-alternate quadrilateral snake graph $k(AQ_n)$ consists of k alternate quadrilateral snakes (alternatively) with a common path.

Throughout the paper we consider n as the number of vertices of the path P_n.

2. Harmonious Chromatic Number of $C(Q_n)$, $C(DQ_n)$, $C(TQ_n)$

Theorem 2.1. *For central graph of quadrilateral snake Q_n, the harmonious chromatic number, $\chi_H(C(Q_n)) = 5n - 4$, $n \geq 2$.*

Proof. Let us consider Q_n as the quadrilateral snake graph and P_n as the path graph contains n vertices $u_1, u_2, ..., u_n$. For obtaining central graph, we subdivide each edge u_iu_{i+1}, u_iv_i, u_iw_i and v_iw_i $(1 \leq i \leq n-1)$ of Q_n by the vertices v_i', v_i'', v_i''' and v_i'''' $(1 \leq i \leq n-1)$ of Q_n in central graph of Q_n. $V(C(Q_n)) = \{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i, v_i', v_i'', v_i''', v_i'''' : 1 \leq i \leq n-1\}$. Now coloring the vertices of $C(Q_n)$ as follows; define $c : V(C(Q_n)) \longrightarrow \{1, 2, 3, ..., 5n-4\}$ where $n \geq 2$ by $c(v_i'') = i$, $c(v_i''') = i$, $c(v_i') = n - i + 1$, $c(v_i'''') = n - 1 + i$, $c(v_i) = 2n - 2 + i$, $c(w_i) = 3n - 3 + i$ for $(1 \leq i \leq n-1)$ and $c(u_i) = 4n - 4 + i$ for $(1 \leq i \leq n)$. Now

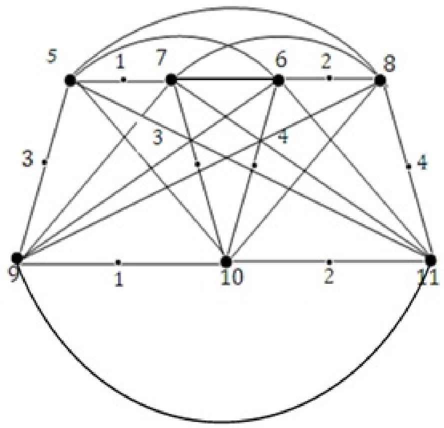

Figure 1. $C(Q_3)$ with coloring, $\chi_H(C(Q_3)) = 11$.

from above each $c(v_i)$, $c(w_i)$, $c(u_i)$ and its neighbors are assigned by different colors *i.e.* $c(v_i) \neq c(w_i) \neq c(u_i)$, although $c(v_i') = c(v_i'''')$ and $c(v_i'') = c(v_i''')$ but these vertices are at least at a distance 2, therefore it is proper. It is also obvious from above that no two edges share the same color pair and same colored vertices are at least at a distance 3, therefore it is harmonious and all the vertices are colored by $5n - 4$ colors. Now if we repeat (assign) any color on any vertex from these $5n - 4$ colors, color pairs will be repeated which leads to contradict the harmonious coloring, therefore it is minimum. Hence $\chi_H(C(Q_n)) = 5n - 4$. Figure 1 shows the central graph of Q_3 with harmonious coloring. □

Theorem 2.2. *For central graph of double quadrilateral snake DQ_n, the harmonious chromatic number, $\chi_H(C(DQ_n)) = 8n - 7$, $n \geq 2$*

Proof. Let us consider DQ_n as the double quadrilateral snake and P_n as the path graph with n vertices $u_1, u_2, ..., u_n$. Now we obtain the central graph as described in Theorem 2.1, therefore $V(C(DQ_n)) = \{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i, x_i, y_i, e'_i, e''_i, e_i, l'_i, l''_i, m'_i, m''_i : 1 \leq i \leq n-1\}$. Now coloring the vertices of $C(DQ_n)$ as follows; define $c : V(C(DQ_n)) \longrightarrow \{1, 2, 3, ..., 8n - 7\}$ for $n \geq 2$ by $c(e_i) = i$, $c(e'_i) = i$, $c(e''_i) = i$, $c(l'_i) = n - 1 + i$, $c(l''_i) = n - 1 + i$, $c(m'_i) = 2n - 2 + i$, $c(m''_i) = 2n - 2 + i$, $c(v_i) = 3n - 3 + i$, $c(w_i) = 4n - 4 + i$, $c(x_i) = 5n - 5 + i$, $c(y_i) = 6n - 6 + i$ for $(1 \leq i \leq n - 1)$ and $c(u_i) = 7n - 7 + i$ for $(1 \leq i \leq n)$. To prove c is harmonious and minimum, follow Theorem 2.1. Figure 2 shows the harmonious coloring for $C(DQ_3)$. □

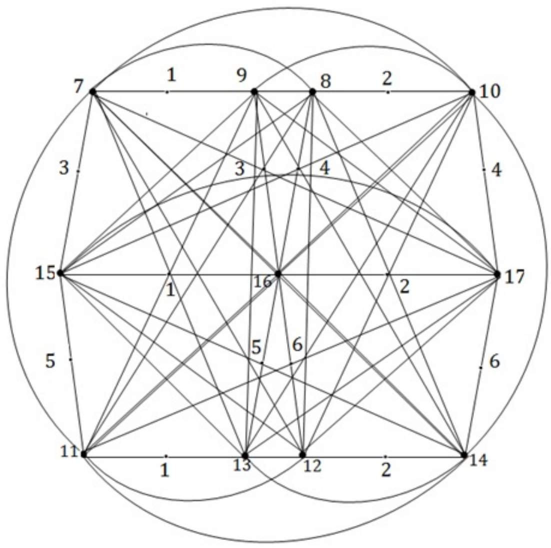

FIGURE 2. $C(DQ_3)$ with harmonious coloring, $\chi_H(C(DQ_3)) = 17$.

Theorem 2.3. *For central graph of triple quadrilateral snake TQ_n, the harmonious chromatic number, $\chi_H(C(TQ_n)) = 11n - 10$, $n \geq 2$.*

Proof. Let us consider TQ_n as the triple quadrilateral snake and P_n as the path graph with n vertices $u_1, u_2, ..., u_n$. Now we obtain the central graph as described in Theorem 2.1, therefore $V(C(TQ_n)) = \{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i, x_i, y_i, p_i, q_i, e_i, e'_i, e''_i, e'''_i, l'_i, l''_i, m'_i, m''_i, z'_i, z''_i : 1 \leq i \leq n - 1\}$. Now coloring the vertices of $C(TQ_n)$ as follows; define $c : V(C(TQ_n)) \longrightarrow \{1, 2, 3, ..., 11n - 10\}$ where $n \geq 2$ by $c(e_i) = i$,

$c(e_i') = i$, $c(e_i'') = i$, $c(e_i''') = i$, $c(l_i') = n-1+i$, $c(l_i'') = n-1+i$, $c(m_i') = 2n-2+i$, $c(m_i'') = 2n-2+i$, $c(z_i') = 3n-3+i$, $c(z_i') = 3n-3+i$, $c(v_i) = 4n-4+i$, $c(w_i) = 5n-5+i$, $c(x_i) = 6n-6+i$, $c(y_i) = 7n-7+i$, $c(p_i) = 8n-8+i$, $c(y_i) = 9n-9+i$, $c(y_i) = 10n-10+i$ for $(1 \le i \le n-1)$ and $c(u_i) = 10n-10+i$ for $(1 \le i \le n)$. For remaining proof, follow Theorem 2.1. Figure 3 shows the harmonious coloring for $C(TQ_3)$. □

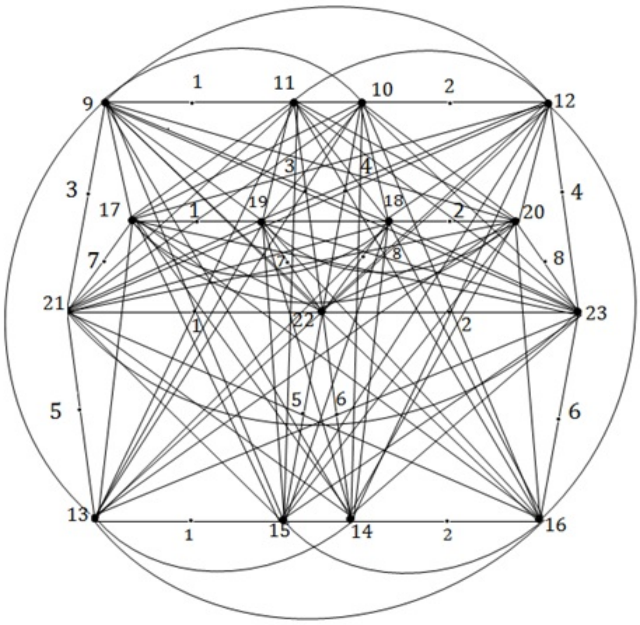

FIGURE 3. $C(TQ_3)$ with harmonious coloring, $\chi_H(C(TQ_3)) = 23$.

3. HARMONIOUS CHROMATIC NUMBER OF k-QUADRILATERAL SNAKE

Theorem 3.1. *For central graph of k-quadrilateral snake kQ_n, the harmonious chromatic number, $\chi_H(C(kQ_n)) = (3k+2)n - (3k+1)$, $n \ge 2$.*

Proof. For $k = 1$, we have $\chi_H(C(Q_n)) = 5n - 4$, this proves Theorem 2.1, for $k = 2$, we have $\chi_H(C(D(Q_n))) = 8n - 7$, this proves Theorem 2.2, for $k = 3$, we have $\chi_H(C(T(Q_n))) = 11n - 10$, this proves Theorem 2.3, which is true.

Let $\chi_H(C(kQ_n))$ be true for some positive integer r, *i.e.* $\chi_H(C(rQ_n)) = (3r+2)n - (3r+1)$. We shall now prove that $\chi_H(C((r+1)Q_n))$ is true.
Consider $\chi_H(C((r+1)Q_n)) = (3(r+1)+2)n - (3(r+1)+1) = (3r+2)n - (3r+1) + 3n - 3 = \chi_H(C(rQ_n)) + 11n - 10 - (8n-7) = \chi_H(C(rQ_n)) + \chi_H(C(TQ_n)) - \chi_H(C(DQ_n))$ which is true. Therefore it is true for $r+1$ that follows by mathematical induction, it is true for all values of k. Hence the theorem. □

4. Harmonious Chromatic Number of $C(AQ_n)$, $C(DAQ_n)$, $C(TAQ_n)$

Theorem 4.1. *For alternate quadrilateral snake AQ_n, harmonious chromatic number, $\chi_H C(AQ_n) = \frac{7n}{2} - 1$, n is even and ≥ 4.*

Proof. Let us consider AQ_n as the alternate quadrilateral snake and P_n as the path graph with n vertices $u_1, u_2, ..., u_n$. Now we obtain the central graph as described in Theorem 2.1, therefore $V(C(AQ_n)) = \{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i, v_i'', l_i', l_i'' : (1 \leq i \leq \frac{n}{2})\} \cup \{v_i' : (1 \leq i \leq n-1)\}$. Now coloring the vertices of $C(AQ_n)$ as follows; define $c : V(C(AQ_n) \longrightarrow \{1, 2, 3, ..., \frac{7n}{2} - 1\}$ where $n \geq 4$ by $c(v_i') = i$ for $(1 \leq i \leq n-1)$, $c(v_i'') = i$, $c(l_i') = n - 1 + i$, $c(l_i'') = n - 1 + i$, $c(v_i) = \frac{3n}{2} - 1 + i$, $c(w_i) = 2n - 1 + i$ for $(1 \leq i \leq \frac{n}{2})$ and $c(u_i) = \frac{5n}{2} - 1 + i$ for $(1 \leq i \leq n)$. To prove c is harmonious and minimum, follow Theorem 2.1. Figure 4 shows the central graph of AQ_n with harmonious coloring. □

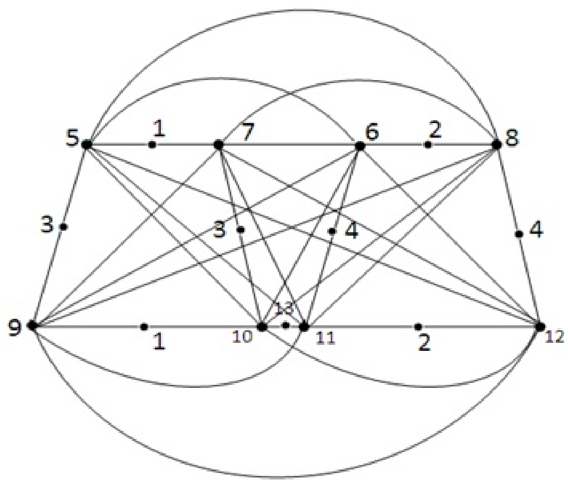

Figure 4. $C(AQ_4)$ with harmonious coloring, $\chi_H(C(AQ_4)) = 13$.

Theorem 4.2. *For central graph of double alternate quadrilateral snake $D(AQ_n)$, the harmonious chromatic number, $\chi_H(C(D(AQ_n))) = 5n - 1$, n is even and ≥ 4.*

Proof. Let us consider $D(AQ_n)$ as the double alternate quadrilateral snake and P_n as the path graph with n vertices $u_1, u_2, ..., u_n$. Now we obtain the central graph as described in Theorem 2.1, therefore $V(C(D(AQ_n))) = \{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i, x_i, y_i, e_i', e_i'', l_i', l_i'', m_i', m_i'' : (1 \leq i \leq \frac{n}{2})\} \cup \{e_i : (1 \leq i \leq n-1)\}$. Now coloring the vertices of $C(D(AQ_n))$ as follows; define $c : V(C(D(AQ_n)) \longrightarrow \{1, 2, 3, ..., 5n-1\}$ where $n \geq 4$ by $c(e_i) = i$ for $(1 \leq i \leq n-1)$, $c(e_i') = i$, $c(e_i'') = i$, $c(l_i') = n - 1 + i$, $c(l_i'') = n - 1 + i$, $c(m_i') = \frac{3n}{2} - 1 + i$, $c(m_i'') = \frac{3n}{2} - 1 + i$, $c(v_i) = 2n - 1 + i$, $c(w_i) = \frac{5n}{2} - 1 + i$, $c(x_i) = 3n - 1 + i$, $c(y_i) = \frac{7n}{2} - 1 + i$ for

$(1 \leq i \leq \frac{n}{2})$ and $c(u_i) = 4n - 1 + i$ for $(1 \leq i \leq n)$. To prove c is harmonious and minimum, follow Theorem 2.1. Figure 5 shows the central graph of $D(AQ_n)$ with harmonious coloring. □

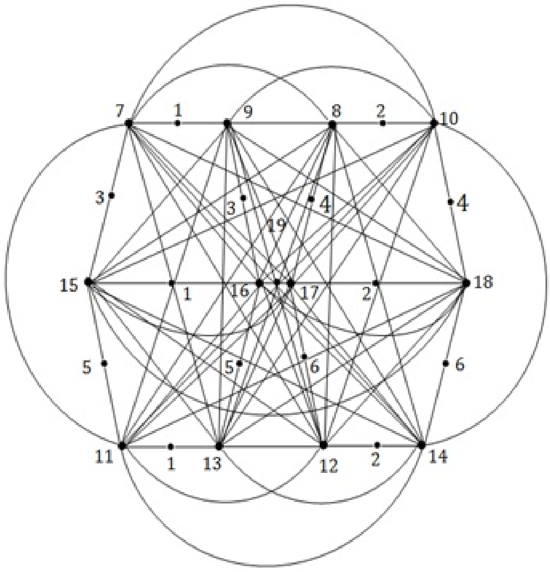

FIGURE 5. $C(D(AQ_4))$ with coloring, $\chi_H(C(D(AQ_4))) = 19$.

Theorem 4.3. *For triple alternate Quadrilateral snake $T(AQ_n)$, the harmonious chromatic number, $\chi_H C(T(AQ_n)) = \frac{13n}{2} - 1$, n is even and ≥ 4.*

Proof. Let us consider $T(AQ_n)$ as the triple alternate quadrilateral snake and P_n as the path graph with n vertices $u_1, u_2, ..., u_n$. Now we obtain the central graph as described in Theorem 2.1, therefore $V(C(T(AQ_n))) = \{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i, x_i, y_i, p_i, q_i, e'_i, e''_i, e'''_i, l_i, l'_i, l''_i, m_i, m'_i, m''_i : (1 \leq i \leq \frac{n}{2})\} \cup \{e_i : (1 \leq i \leq n - 1\}$. Now coloring the vertices of $C(T(AQ_n))$ as follows; define $c : V(C(T(AQ_n)) \longrightarrow \{1, 2, 3, ..., \frac{13n}{2} - 1\}$ for $n \geq 4$ by $c(e_i) = i$ for $(1 \leq i \leq n-1)$, $c(e'_i) = i$, $c(e''_i) = i$, $c(e'''_i) = i$, $c(l'_i) = n-1+i$, $c(l''_i) = n-1+i$, $c(m'_i) = \frac{3n}{2} - 1 + i$, $c(m''_i) = \frac{3n}{2} - 1 + i$, $c(l_i) = 2n - 1 + i$, $c(m_i) = 2n - 1 + i$, $c(v_i) = \frac{5n}{2} - 1 + i$, $c(w_i) = 3n - 1 + i$, $c(x_i) = \frac{7n}{2} - 1 + i$, $c(y_i) = 4n - 1 + i$, $c(p_i) = \frac{9n}{2} - 1 + i$, $c(q_i) = 5n - 1 + i$ for $(1 \leq i \leq \frac{n}{2})$ and $c(u_i) = \frac{11n}{2} - 1 + i$ for $(1 \leq i \leq n)$. To prove c is harmonious and minimum, follow Theorem 2.1. Figure 6 shows the central graph of $T(AQ_n)$ with harmonious coloring. □

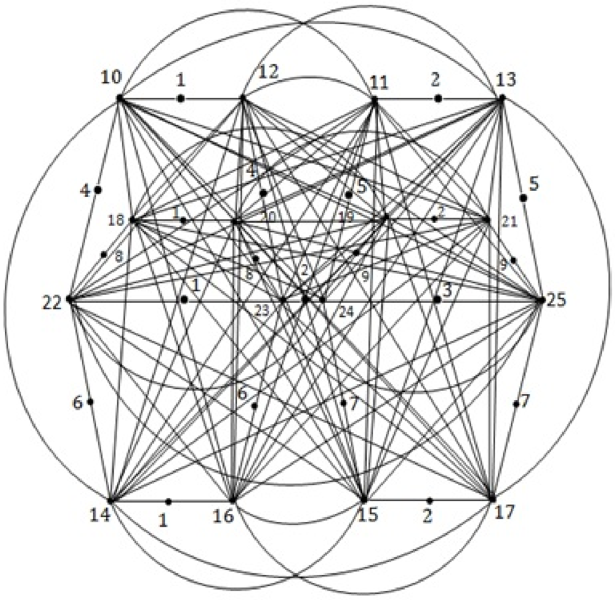

Figure 6. $C(T(AQ_4))$ with harmonious coloring, $\chi_H(C(T(AQ_4))) = 25$.

5. Harmonious Chromatic Number of k-Alternate Quadrilateral Snake

Theorem 5.1. *For central graph of k-alternate quadrilateral snake $k(AQ_n)$, the harmonious chromatic number, $\chi_H(C(k(AQ_n))) = (\frac{3k+4}{2})n - 1$, n is even and ≥ 4.*

Proof. For $k = 1$, we have $\chi_H(C(AQ_n)) = \frac{7n}{2} - 1$, this proves Theorem 4.1, for $k = 2$, we have $\chi_H(C(D(AQ_n))) = 5n - 1$, this proves Theorem 4.2, for $k = 3$, we have $\chi_H(C(T(AQ_n))) = \frac{13n}{2} - 1$, this proves Theorem 4.3 which is true. Let $\chi_H(C(k(AQ_n)))$ be true for some positive integer r, *i.e.* $\chi_H(C(r(AQ_n))) = (\frac{3r+4}{2})n - 1)$. We shall now prove that $\chi_H(C((r+1)(AQ_n)))$ is true. Consider $\chi_H(C((r+1)AQ_n)) = (\frac{3(r+1)+4}{2})n - 1) = (\frac{3r+4}{2})n - 1) + \frac{3n}{2} = \chi_H(C(rAQ_n)) + (\frac{13n}{2} - 1) - (5n - 1) = \chi_H(C(r(AQ_n))) + \chi_H(C(T(AQ_n))) - \chi_H(C(D(AQ_n)))$ which is true. Therefore it is true for $r + 1$ that follows by mathematical induction, it is true for all values of k. Hence the theorem. □

6. Conclusion

This article shows the study about the harmonious coloring and we find the harmonious chromatic number of central graph of k- quadrilateral and k-alternate quadrilateral snakes $\chi_H(C(kQ_n)) = (3k+2)n - (3k+1)$, and $\chi_H(C(k(AQ_n))) =$

$(\frac{3k+4}{2})n - 1$ respectively. For future scope we can examine the different type of colorings for these quadrilateral snakes.

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