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PROJECTIVE CURVATURE TENSOR WITH RESPECT TO ZAMKOVOY CONNECTION IN LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract. The purpose of the present paper is to study some properties of Projective curvature tensor with respect to Zamkovoy connection in Lorentzian Para Sasakian manifold(briefly, LP-Sasakian manifold). We obtain some results on Lorentzian Para-Sasakian manifold with the help of Zamkovoy connection and Projective curvature tensor. Moreover, we study the LP-Sasakian manifold satisfying $P^* (\xi, U) \circ W_0^* = 0$ and $P^* (\xi, U) \circ W_2^* = 0$, where P^*, W_0^* and W_2^* are Projective curvature tensor, W_0- curvature tensor and W_2-curvature tensor with respect to Zamkovoy connection respectively.

Key words and Phrases: LP-Sasakian manifolds, Zamkovoy Connection, Projective Curvature tensor

1. Introduction

In 1989, K. Matsumoto [7] first introduced the notion of Lorentzian Para-Sasakian manifolds. Also, in 1992, I. Mihai and R. Rosca [8] introduced independently the notion of Lorentzian Para Sasakian manifolds(briefly, LP-Sasakian Manifolds) in classical analysis. In an $n-$ dimensional metric manifold the signature of the metric tensor is the number of positive and negative eigenvalues of the metric. If the metric has s positive eigenvalues and t negative eigenvalues then the signature of the metric is (s, t). For a non-degerate metric tensor $s + t = n$. A Lorentzian manifold is a special case of a semi Riemannian manifold, in which

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the signature of the metric is $(1, n-1)$ or $(n-1, 1)$. And the metric g is called here a Lorentzian metric, which is named after the physicist Hendrik Lorentz. The LP-Sasakian manifold was further studied by several authors. We cite ([3], [9]) and their references.

The notion of Projective curvature tensor was first introduced by K. Yano and S. Bochner [13] in 1953. This curvature tensor was further studied by U. C. De and J. Sengupta [4], S. Ghosh [5]. If there exists a one -one mapping between each co-ordinate neighbourhood of a manifold M to a domain of R^n such that any geodesic of M corresponds to a straight line in R^n, then the manifold M is said to be locally projectively flat. Due to [4], the Projective curvature tensor P of rank four for an n-dimensional Riemannian Manifold M is given by

$$\begin{aligned} & P(X,Y,Z,V) \\ = \; & R(X,Y,Z,V) - \frac{1}{n-1}\left[S(Y,Z)\,g(X,V) - S(X,Z)\,g(Y,V)\right] \end{aligned} \tag{1}$$

for all X, Y, V & $Z \in \chi(M)$, set of all vector fields of the manifold M, where P denotes the Projective curvature tensor of type $(0,4)$ and R denotes the Riemannian curvature tensor of type $(0,4)$ defined by

$$P(X,Y,Z,V) = g(P(X,Y)Z, V) \tag{2}$$

$$R(X,Y,Z,V) = g(R(X,Y)Z, V) \tag{3}$$

where R is the Riemannian curvature tensor of type $(0,3)$, P is the Projective curvature tensor of type $(0,3)$ and S denotes the Ricci tensor of type $(0,2)$.

In 2008, the notion of Zamkovoy connection on para contact manifold was introduced by S. Zamkovoy [14]. Zamkovoy connection was defined as a canonical paracontact connection whose torsion is the obstruction of paracontact manifold to be a para sasakian manifold. This connection was further studied by many researcher. For instance, we see ([2], [1], [6]). For an n-dimensional almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, the Zamkovoy connection (∇^*) in terms of Levi-Civita connection (∇) is given by

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\,\xi - \eta(Y)\,\nabla_X \xi + \eta(X)\,\phi Y \tag{4}$$

for all $X, Y \in \chi(M)$.

In a LP-Sasakian manifold M of dimension $(n > 2)$, the Projective curvature tensor P, W_0 Curvature tensor [10], W_2-Curvature tensor [12] with respect to the Levi-Civita connection are given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\left[S(Y,Z)X - S(X,Z)Y\right] \tag{5}$$

$$W_0(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\left[S(Y,Z)X - g(X,Z)QY\right] \tag{6}$$

$$W_2(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\left[g(Y,Z)QX - g(X,Z)QY\right] \tag{7}$$

The Projective curvature tensor, W_0-Curvature tensor and W_2-Curvature tensor with respect to the Zamkovoy connection are given by,

$$P^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{n-1}[S^*(Y,Z)X - S^*(X,Z)Y] \quad (8)$$

$$W_0^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{n-1}[S^*(Y,Z)X - g(X,Z)Q^*Y] \quad (9)$$

$$W_2^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{n-1}[g(Y,Z)Q^*X - g(X,Z)Q^*Y] \quad (10)$$

where R^*, S^* and Q^* are Riemannian curvature tensor, Ricci tensor and Ricci operator with respect to Zamkovoy connection ∇^* respectively.

Definition 1.1. *An n-dimensional LP -Sasakian manifold M is said to be generalized $\eta-$Einstein manifold if the Ricci tensor of type (0,2) is of the form*

$$S(Y,Z) = k_1 g(Y,Z) + k_2\eta(Y)\eta(Z) + k_3\omega(Y,Z) \quad (11)$$

for all $Y,Z \in \chi(M)$, set of all vector fields of the manifold M and k_1 , k_2 and k_3 are scalars and ω is a 2-form.

Definition 1.2. *An n-dimensional LP-Sasakian manifold M is said to be Projectively flat if $P(X,Y)Z = 0$ for all $X,Y,Z \in \chi(M)$.*

Definition 1.3. *An n-dimensional LP-Sasakian manifold M is said to be $\xi-$ Projectively flat if $P(X,Y)\xi = 0$ for all $X,Y,Z \in \chi(M)$.*

This paper is structured as follows: after introduction, a short description of LP-Sasakian manifold is given in section (2). In section (3), we have discussed LP-Sasakian manifold admitting Zamkovoy connection ∇^* and obtain curvature tensor R^*, Ricci tensor S^*, Scalar curvature tensor r^*, in LP-Sasakian manifold. Section (4) contains Projectively flat LP-Sasakian manifold with respect to the connection ∇^*. In section (5) we have discussed Locally Projectively $\phi-$symmetric LP-Sasakian manifold M with respect to ∇^*. In section (6) we have discussed a LP-Sasakian manifold satisfying $P^*(\xi,U)\circ W_0^* = 0$. In section (7) we have discussed a LP-Sasakian manifold satisfying $P^*(\xi,U)\circ W_2^* = 0$.

2. Preliminaries

An n-dimensional differentiable manifold is called a LP-Sasakian manifold if it admits a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Lorentzian metric g which satisfies

$$\phi^2 Y = Y + \eta(Y)\xi, \eta(\xi) = -1, \eta(\phi X) = 0, \ \phi\xi = 0 \quad (12)$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \quad (13)$$

$$g(X, \phi Y) = g(\phi X, Y), \eta(Y) = g(Y,\xi) \quad (14)$$

$$\nabla_X \xi = \phi X, \quad g(X,\xi) = \eta(X) \quad (15)$$

$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \quad (16)$$

$$\forall X,Y \in \chi(M)$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let us introduced a symmetric $(0,2)$ tensor field ω such that $\omega(X,Y) = g(X,\phi Y)$. Also, since the vector field η is closed in LP- Sasakian manifold, we have

$$(\nabla_X \eta) Y = \omega(X,Y), \omega(X,\xi) = 0, \forall X, Y \in \chi(M) \tag{17}$$

In LP- Sasakian manifold, the following relations also hold:

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \tag{18}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{19}$$

$$R(\xi,Y)Z = g(Y,Z)\xi - \eta(Z)Y \tag{20}$$

$$R(\xi,Y)\xi = \eta(Y)\xi + Y \tag{21}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{22}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{23}$$

$$Q\xi = (n-1)\xi, Q\phi = \phi Q, S(X,Y) = g(QX,Y), S^2(X,Y) = S(QX,Y) \tag{24}$$

3. Some Properties of LP-Sasakian manifolds with respect to Zamkovoy connection

Using (15) and (17) in (4), we get

$$\nabla^*_X Y = \nabla_X Y + g(X,\phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y \tag{25}$$

with torsion tensor

$$T^*(X,Y) = 2[\eta(X)\phi Y - \eta(Y)\phi X] \tag{26}$$

In view of (4) and (17), we have

$$(\nabla^*_X g)(Y,Z) = -2g(Y,\phi Z)\eta(X) \tag{27}$$

Putting $Y = \xi$ in (25)

$$\nabla^*_X \xi = 2\phi X \tag{28}$$

Using (14), (15) and (16) in (25), we obtain

$$\begin{aligned}\nabla^*_X(\phi Y) &= \phi(\nabla_X Y) + 2g(X,Y)\xi + \eta(Y)X \\ &\quad + \eta(X)Y + 4\eta(X)\eta(Y)\xi\end{aligned} \tag{29}$$

$$\nabla^*_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{30}$$

$$\begin{aligned}\nabla^*_X g(Y,\phi Z) &= g(\nabla_X Y,\phi Z) + g(Y,\phi\nabla_X Z) + g(X,Z)\eta(Y) \\ &\quad + g(X,Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z)\end{aligned} \tag{31}$$

In view of (25), (29), (30) and (31), we have

$$\begin{aligned}\nabla^*_X\nabla^*_Y Z &= \nabla_X\nabla_Y Z + g(X,\phi\nabla_Y Z)\xi - \eta(\nabla_Y Z)\phi X + \eta(X)\phi\nabla_Y Z \\ &+g(\nabla_X Y,\phi Z)\xi + g(Y,\phi\nabla_X Z)\xi + g(X,Z)\eta(Y)\xi \\ &+g(X,Y)\eta(Z)\xi + 2\eta(X)\eta(Y)\eta(Z)\xi + 2g(Y,\phi Z)\phi X \\ &-g(X,\phi Z)\phi Y - \eta(\nabla_X Z)\phi Y - \phi(\nabla_X Y)\eta(Z) - 2g(X,Y)\eta(Z)\xi \\ &-\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - 4\eta(X)\eta(Y)\eta(Z)\xi \\ &+g(X,\phi Y)\phi Z + \eta(\nabla_X Y)\phi Z + \phi(\nabla_X Z)\eta(Y) + 2g(X,Z)\eta(Y)\xi \\ &+\eta(Y)\eta(Z)X + \eta(X)\eta(Y)Z + 4\eta(X)\eta(Y)\eta(Z)\xi \qquad (32)\end{aligned}$$

Interchanging X and Y

$$\begin{aligned}\nabla^*_Y\nabla^*_X Z &= \nabla_Y\nabla_X Z + g(Y,\phi\nabla_X Z)\xi - \eta(\nabla_X Z)\phi Y + \eta(Y)\phi\nabla_X Z \\ &+g(\nabla_Y X,\phi Z)\xi + g(X,\phi\nabla_Y Z)\xi + g(Y,Z)\eta(X)\xi \\ &+g(Y,X)\eta(Z)\xi + 2\eta(Y)\eta(X)\eta(Z)\xi + 2g(X,\phi Z)\phi Y \\ &-g(Y,\phi Z)\phi X - \eta(\nabla_Y Z)\phi X - \phi(\nabla_Y X)\eta(Z) - 2g(Y,X)\eta(Z)\xi \\ &-\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X - 4\eta(Y)\eta(X)\eta(Z)\xi \\ &+g(Y,\phi X)\phi Z + \eta(\nabla_Y X)\phi Z + \phi(\nabla_Y Z)\eta(X) + 2g(Y,Z)\eta(X)\xi \\ &+\eta(X)\eta(Z)Y + \eta(Y)\eta(X)Z + 4\eta(Y)\eta(X)\eta(Z)\xi \qquad (33)\end{aligned}$$

Also we have

$$\begin{aligned}\nabla^*_{[X,Y]}Z &= \nabla_{[X,Y]}Z + g(\nabla_X Y,\phi Z)\xi - g(\nabla_Y X,\phi Z)\xi - \eta(Z)\phi\nabla_X Y \\ &+\eta(Z)\phi\nabla_Y X + \eta(\nabla_X Y)\phi Z - \eta(\nabla_Y X)\phi Z \qquad (34)\end{aligned}$$

Let R^* be the Riemannian curvature tensor with respect to Zamkovoy connection and it is defined as

$$R^*(X,Y)Z = \nabla^*_X\nabla^*_Y Z - \nabla^*_Y\nabla^*_X Z - \nabla^*_{[X,Y]}Z \qquad (35)$$

Using (25), (32), (33) and (34) in (35), we get

$$\begin{aligned}&R^*(X,Y)Z \\ =\ &R(X,Y)Z + 3g(X,Z)\eta(Y)\xi - 3g(Y,Z)\eta(X)\xi + 3g(Y,\phi Z)\phi X \\ &-3g(X,\phi Z)\phi Y - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X \qquad (36)\end{aligned}$$

Consequently one can easily bring out the followings:

$$\begin{aligned}S^*(Y,Z) &= S(Y,Z) + (n-1)\eta(Y)\eta(Z) + 3\psi g(Y,\phi Z) \qquad (37)\\ S^*(\xi,Z) &= S^*(Z,\xi) = 0 \qquad (38)\\ Q^*Y &= QY + (n-1)\eta(Y)\xi + 3\psi\phi Y \qquad (39)\\ Q^*\xi &= 0 \qquad (40)\\ r^* &= r - n + 1 + 3\psi^2 \qquad (41)\end{aligned}$$

$$R^*(X,Y)\xi = 0 \tag{42}$$
$$R^*(\xi,Y)Z = 4g(\phi Y,\phi Z)\xi \tag{43}$$
$$R^*(X,\xi)Z = -4g(\phi X,\phi Z)\xi \tag{44}$$

for all $X,Y,Z \in \chi(M)$, where $\psi = trace(\phi)$

Thus we can state the followings:

Proposition 3.1. *Let M be an n-dimensional LP-Sasakian manifold admitting Zamkovoy connection ∇^*, then*

(i) *The curvature tensor R^* of ∇^* is given by (36)*
(ii) *The Ricci tensor S^* of ∇^* is given by (37)*
(iii) *The scalar curvature r^* of ∇^* is given by (41)*
(iv) *The Ricci tensor S^* of ∇^* is symmetric.*
(v) *R^* satisfies: $R^*(X,Y)Z + R^*(Y,Z)X + R^*(Z,X)Y = 0$.*

4. Projectively flat LP-Sasakian manifold with respect to the Zamkovoy connection

Theorem 4.1. *If an $n-$dimensional LP-Sasakian manifold M is Projectively flat with respect to Zamkovoy connection, then it is a generalized $\eta-$Einstein manifold.*

Proof. In view of (8), (36) and (37), the Projective curvature tensor P^* with respect to the Zamkovoy connection ∇^* on a LP-Sasakian manifold M of dimension $(n > 2)$ takes the form

$$\begin{aligned} P^*(X,Y)Z &= R(X,Y)Z + 3g(X,Z)\eta(Y)\xi - 3g(Y,Z)\eta(X)\xi + 3g(Y,\phi Z)\phi X \\ &\quad -3g(X,\phi Z)\phi Y - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X \\ &\quad -\frac{1}{n-1}[S(Y,Z)X + (n-1)\eta(Y)\eta(Z)X + 3\psi g(Y,\phi Z)X] \\ &\quad +\frac{1}{n-1}[S(X,Z)Y + (n-1)\eta(X)\eta(Z)Y + 3\psi g(X,\phi Z)Y] \end{aligned} \tag{45}$$

Let M be projectively flat with respect to Zamkovoy connection, then from (45), we get

$$\begin{aligned} &R(X,Y)Z \\ =\ &-3g(X,Z)\eta(Y)\xi + 3g(Y,Z)\eta(X)\xi - 3g(Y,\phi Z)\phi X \\ &+3g(X,\phi Z)\phi Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+\frac{1}{n-1}[S(Y,Z)X + (n-1)\eta(Y)\eta(Z)X + 3\psi g(Y,\phi Z)X] \\ &-\frac{1}{n-1}[S(X,Z)Y + (n-1)\eta(X)\eta(Z)Y + 3\psi g(X,\phi Z)Y] \end{aligned} \tag{46}$$

Taking inner product of (46) with a vector field V, we have

$$\begin{aligned}
& R(X,Y,Z,V) \\
= \; & -3g(X,Z)\eta(Y)\eta(V)+3g(Y,Z)\eta(X)\eta(V)-3g(Y,\phi Z)g(\phi X,V) \\
& +3g(X,\phi Z)g(\phi Y,V)+\eta(X)\eta(Z)g(Y,V)-\eta(Y)\eta(Z)g(X,V) \\
& +\frac{1}{n-1}\left[S(Y,Z)+(n-1)\eta(Y)\eta(Z)+3\psi g(Y,\phi Z)\right]g(X,V) \\
& -\frac{1}{n-1}\left[S(X,Z)+(n-1)\eta(X)\eta(Z)+3\psi g(X,\phi Z)\right]g(Y,V) \qquad (47)
\end{aligned}$$

Setting $X=V=\xi$ and using (12), (22) in (47), we get

$$S(Y,Z)=4(n-1)g(Y,Z)+3(n-1)\eta(Y)\eta(Z)-3\psi\omega(Y,Z)$$

where $\omega(Y,Z)=g(\phi Y,Z)$.

which shows that M is an $\eta-$Einstein manifold. Hence the theorem is proved. □

Corollary 4.2. *An $n-$ dimensional LP-Sasakian manifold M is $\xi-$ Projectively flat with respect to Zamkovoy connection iff it is so with respect to Levi-Civita connection.*

Proof. Using (5) in (45), we get

$$\begin{aligned}
P^*(X,Y)Z \;=\; & P(X,Y)Z+3g(X,Z)\eta(Y)\xi-3g(Y,Z)\eta(X)\xi \\
& +3g(Y,\phi Z)\phi X-3g(X,\phi Z)\phi Y \\
& -\frac{3\psi}{n-1}\left[g(Y,\phi Z)X+g(X,\phi Z)Y\right] \qquad (48)
\end{aligned}$$

Setting $Z=\xi$ in (48), we get

$$P^*(X,Y)\xi=P(X,Y)\xi$$

Therefore, *M is $\xi-$Projectively flat with respect to Zamkovoy connection iff it is so with respect to Levi-Civita connection.* □

5. Locally Projectively $\phi-$symmetric LP-Sasakian manifolds with respect to Zamkovoy connection

In 1977, Takahashi [11] first studied the concept of locally ϕ-symmetry on Sasakian manifold. In this section we consider a locally projectively ϕ-symmetric LP-Sasakian manifolds with respect to the connection ∇^*.

Definition 5.1. *An $n-$dimensional LP-Sasakian manifold M is said to be locally projectively ϕ-symmetric with respect to Zamkovoy connection ∇^* if the projective curvature tensor P^* with respect to the connection ∇^* satisfies*

$$\phi^2\left(\nabla^*_W P^*\right)(X,Y)Z=0 \qquad (49)$$

where X,Y,Z and W are horizontal vector fields on M, i.e X,Y,Z and W are orthonormal to ξ on the manifold M.

Theorem 5.2. *An $n-$dimensional LP-Sasakian manifold M $(n>3)$ is locally projectively ϕ-symmetric with respect to Zamkovoy connection if and only if it is so with respect to the Levi-Civita connection, provided trace $(\phi)=0$.*

Proof. In view of (25), we have

$$\begin{aligned}(\nabla^*_W P^*)(X,Y)Z &= (\nabla_W P^*)(X,Y)Z + g(W,\phi P^*(X,Y)Z)\xi \\ &\quad -\eta(P^*(X,Y)Z)\phi W + \eta(W)\phi P^*(X,Y)Z \qquad (50)\end{aligned}$$

Taking covariant differentiation of (48) in the direction of W and considering $trace\,(\phi)=0$, we obtain

$$\begin{aligned}(\nabla_W P^*)(X,Y)Z &= (\nabla_W P)(X,Y)Z \\ &\quad +3[g(X,Z)g(W,\phi Y)-g(Y,Z)g(W,\phi X)]\xi \\ &\quad +3[g(W,Z)\eta(Y)+g(Y,W)\eta(Z)+2\eta(W)\eta(Y)\eta(Z)]\phi X \\ &\quad +3g(Y,\phi Z)[g(W,X)\xi+\eta(X)W+2\eta(W)\eta(X)\xi] \\ &\quad -3[g(W,Z)\eta(X)+g(X,W)\eta(Z)+2\eta(W)\eta(X)\eta(Z)]\phi Y \\ &\quad -3g(X,\phi Z)[g(W,Y)\xi+\eta(Y)W+2\eta(W)\eta(Y)\xi] \qquad (51)\end{aligned}$$

In view of (12), (18) and (45), we obtain

$$\begin{aligned}&\eta(P^*(X,Y)Z) \\ &= g(Y,Z)\eta(X)-g(X,Z)\eta(Y)-3g(X,Z)\eta(Y)+3g(Y,Z)\eta(X) \\ &\quad -\frac{1}{n-1}[S(Y,Z)+(n-1)\eta(Y)\eta(Z)+3\psi g(Y,\phi Z)]\eta(X) \\ &\quad +\frac{1}{n-1}[S(X,Z)+(n-1)\eta(X)\eta(Z)+3\psi g(X,\phi Z)]\eta(Y) \qquad (52)\end{aligned}$$

Using (51) and (52) in (50), we get

$$\begin{aligned}(\nabla^*_W P^*)(X,Y)Z &= (\nabla_W P)(X,Y)Z+3[g(X,Z)g(W,\phi Y)-g(Y,Z)g(W,\phi X)]\xi \\ &\quad +3[g(W,Z)\eta(Y)+g(Y,W)\eta(Z)+2\eta(W)\eta(Y)\eta(Z)]\phi X \\ &\quad +3g(Y,\phi Z)[g(W,X)\xi+\eta(X)W+2\eta(W)\eta(X)\xi] \\ &\quad -3[g(W,Z)\eta(X)+g(X,W)\eta(Z)+2\eta(W)\eta(X)\eta(Z)]\phi Y \\ &\quad -3g(X,\phi Z)[g(W,Y)\xi+\eta(Y)W+2\eta(W)\eta(Y)\xi] \\ &\quad +g(W,\phi P^*(X,Y)Z)\xi-g(Y,Z)\eta(X)\phi W+g(X,Z)\eta(Y)\phi W \\ &\quad +3g(X,Z)\eta(Y)\phi W-3g(Y,Z)\eta(X)\phi W \\ &\quad +\frac{1}{n-1}[S(Y,Z)+(n-1)\eta(Y)\eta(Z)+3\psi g(Y,\phi Z)]\eta(X)\phi W \\ &\quad -\frac{1}{n-1}[S(X,Z)+(n-1)\eta(X)\eta(Z)+3\psi g(X,\phi Z)]\eta(Y)\phi W \\ &\quad +\phi P(X,Y)Z\eta(W)+3g(Y,\phi Z)X\eta(W)-3g(X,\phi Z)Y\eta(W) \\ &\quad +3g(Y,\phi Z)\eta(X)\eta(W)\xi-3g(X,\phi Z)\eta(Y)\eta(W)\xi \qquad (53)\end{aligned}$$

Applying ϕ^2 on both sides of (53) and using (12), we obtain

$$\phi^2(\nabla^*_W P^*)(X,Y)Z=\phi^2(\nabla_W P)(X,Y)Z \qquad (54)$$

where X, Y, Z, W are horizontal vector fields and $trace\,(\phi) = 0$. Hence the theorem is proved. □

6. LP-Sasakian manifold satisfying $P^*(\xi, U) \circ W_0^* = 0$

Theorem 6.1. *In an n-dimensional $(n > 3)$ LP-Sasakian manifold M admitting Zamkovoy connection ∇^*, if the condition $P^*(\xi, U) \circ W_0^* = 0$ holds, then the equation*

$$\begin{aligned} S^2(X, Y) &= 4(n-1)S(X, Y) - 6\psi S(\phi X, Y) + 12(n-1)\psi g(X, \phi Y) \\ &\quad - 9\psi^2 g(X, Y) + \left[3(n-1)^2 - 9\psi^2\right]\eta(X)\eta(Y) \end{aligned}$$

is satisfied on the manifold M, for all $X, Y \in \chi(M)$.

Proof. It can be easily seen from (8) and (9), that

$$P^*(\xi, U)X = 4g(\phi U, \phi X)\xi - \frac{1}{n-1}S^*(U, X)\xi \tag{55}$$

$$P^*(\xi, Y)\xi = 0, P^*(\xi, \xi)Y = 0, P^*(X, Y)\xi = 0 \tag{56}$$

$$W_0^*(X, Y)\xi = \frac{1}{n-1}\eta(X)Q^*Y, W_0^*(\xi, Y)\xi = -\frac{1}{n-1}Q^*Y \tag{57}$$

Let us consider a LP- Sasakian manifold M satisfying the condition

$$(P^*(\xi, U) \circ W_0^*)(X, Y)Z = 0.$$

Then we have

$$\begin{aligned} 0 &= P^*(\xi, U)W_0^*(X, Y)Z - W_0^*(P^*(\xi, U)X, Y)Z \\ &\quad - W_0^*(X, P^*(\xi, U)Y)Z - W_0^*(X, Y)P^*(\xi, U)Z \end{aligned} \tag{58}$$

Replacing Z by ξ in (58), we get

$$\begin{aligned} 0 &= P^*(\xi, U)W_0^*(X, Y)\xi - W_0^*(P^*(\xi, U)X, Y)\xi \\ &\quad - W_0^*(X, P^*(\xi, U)Y)\xi - W_0^*(X, Y)P^*(\xi, U)\xi \end{aligned} \tag{59}$$

Using (55), (56) and (57) in (59), we have

$$\begin{aligned} 0 &= 4S^*(\phi U, \phi Y)\eta(X)\xi - \frac{1}{n-1}S^*(U, Q^*Y)\eta(X)\xi \\ &\quad + 4g(\phi U, \phi X)Q^*Y - \frac{1}{n-1}S^*(U, X)Q^*Y \end{aligned} \tag{60}$$

The inner product of the equation (60) with vector field V gives

$$\begin{aligned} 0 &= 4S^*(\phi U, \phi Y)\eta(X)\eta(V) - \frac{1}{n-1}S^*(U, Q^*Y)\eta(X)\eta(V) \\ &\quad + 4g(\phi U, \phi X)S^*(Y, V) - \frac{1}{n-1}S^*(U, X)S^*(Y, V) \end{aligned} \tag{61}$$

Let $\{e_i\}$ $(1 \le i \le n)$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $U = V = e_i$ and taking summation over $i (1 \le i \le n)$ and using (23), (24), (37), (38) and (39) in (61), we get

$$\begin{aligned} S^2(X,Y) &= 4(n-1)S(X,Y) - 6\psi S(\phi X,Y) + 12(n-1)\psi g(X,\phi Y) \\ &\quad -9\psi^2 g(X,Y) + \left[3(n-1)^2 - 9\psi^2\right]\eta(X)\eta(Y) \end{aligned} \tag{62}$$

Hence the theorem is proved.

□

7. LP-Sasakian manifold satisfying $P^*(\xi,U)\circ W_2^* = 0$

Theorem 7.1. *In an $n-$dimensional LP-Sasakain manifold M of dimension $(n>3)$ if the condition $P^*(\xi,U)\circ W_2^*=0$ holds, then the equation*

$$\begin{aligned} S^2(X,Z) &= 4(n-1)S(X,Z) - 6\psi S(X,\phi Z) + 12(n-1)\psi g(X,\phi Z) \\ &\quad -9\psi^2 g(X,Z) + \left[3(n-1)^2 - 9\psi^2\right]\eta(X)\eta(Z) \end{aligned}$$

is satisfied on M, for all $X,Z\in\chi(M)$.

Proof. Let us consider a LP- Sasakian manifold M satisfying the condition

$$\left(P^*(\xi,U)\circ W_2^*\right)(X,Y)Z = 0 \tag{63}$$

Then we have

$$\begin{aligned} 0 &= P^*(\xi,U)W_2^*(X,Y)Z - W_2^*(P^*(\xi,U)X,Y)Z \\ &\quad -W_2^*(X,P^*(\xi,U)Y)Z - W_2^*(X,Y)P^*(\xi,U)Z \end{aligned} \tag{64}$$

Replacing Y by ξ in (64), we get

$$\begin{aligned} 0 &= P^*(\xi,U)W_2^*(X,\xi)Z - W_2^*(P^*(\xi,U)X,\xi)Z \\ &\quad -W_2^*(X,P^*(\xi,U)\xi)Z - W_2^*(X,\xi)P^*(\xi,U)Z \end{aligned} \tag{65}$$

It is seen that

$$W_2^*(X,\xi)Z = -4g(\phi X,\phi Z)\xi - \frac{1}{n-1}\eta(Z)Q^*X \tag{66}$$

$$W_2^*(\xi,\xi)Z = 0,\ W_2^*(X,\xi)\xi = \frac{1}{n-1}Q^*X \tag{67}$$

Using (55), (56), (66) and (67) in (65), we get

$$\begin{aligned} 0 &= \eta(Z)4g(\phi U,\phi Q^*X)\xi - \frac{1}{n-1}\eta(Z)S^*(U,Q^*X)\xi \\ &\quad 4g(\phi U,\phi Z)Q^*X - \frac{1}{n-1}S^*(U,Z)Q^*X \end{aligned} \tag{68}$$

The inner product of the equation (68) with vector field V gives

$$\begin{aligned} 0 &= \left[4S^*(\phi U,\phi X) - \frac{1}{n-1}S^*(U,Q^*X)\right]\eta(Z)\eta(V) \\ &\quad + \left[4g(\phi U,\phi Z) - \frac{1}{n-1}S^*(U,Z)\right]S^*(X,V) \end{aligned} \tag{69}$$

Let $\{e_i\}\,(1 \leq i \leq n)$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $U = V = e_i$ and taking summation over $i(1 \leq i \leq n)$ and using (23), (24), (37), (38) and (39) in (69), we get

$$\begin{aligned} 0 &= 4\,(n-1)\,S\,(X,Z) - 6\psi S\,(X,\phi Z) + 12\,(n-1)\,\psi g\,(X,\phi Z) \\ &\quad -9\psi^2 g\,(X,Z) + \left[3\,(n-1)^2 - 9\psi^2\right]\eta\,(X)\,\eta\,(Z) - S^2\,(X,Z) \end{aligned} \tag{70}$$

Using (36) in (70), we have

$$\begin{aligned} S^2\,(X,Z) &= 4\,(n-1)\,S\,(X,Z) - 6\psi S\,(X,\phi Z) + 12\,(n-1)\,\psi g\,(X,\phi Z) \\ &\quad -9\psi^2 g\,(X,Z) + \left[3\,(n-1)^2 - 9\psi^2\right]\eta\,(X)\,\eta\,(Z) \end{aligned} \tag{71}$$

Hence the theorem is proved. □

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REFERENCES

[1] Biswas, A. and Baishya, K. K., "Study on generalized pseudo (Ricci) symmetric Sasakian manifold admitting general connection", Bulletin of the Transilvania University of Brasov, 12(2) (2020) 233-246.

[2] Blaga, A. M., "Canonical connection on Para Kenmotso manifold", Novi Sad . J. Math, Vol 45, No.2 (2015) 131-142.

[3] De, U. C. and Matsumoto, K. and Shaikh, A. A., "On Lorentzian para-Sasakian manifolds", Rendiconti del Seminario Matematico di Messina, Serie II, Supplemento al n. 3(1999) , 149-158.

[4] De, U. C. and Sengupta, J., "On a Type of SemiSymmetric Metric Connection on an almost-contact metric connection", Facta Universitatis Ser. Math. Inform. 16(2001) 87-96.

[5] Ghosh, S., "On a class of (k,μ)- contact manifolds", Bull. Cal. Math. Soc. 102 (2010), 219-226.

[6] Mandal, A. and Das, A., "On M-Projective Curvature Tensor of Sasakian Manifolds admitting Zamkovoy Connection", Adv. Math. Sci. J, 9(2020), no.10, 8929-8940.

[7] Matsumoto, K., "On Lorentzian paracontact manifolds", Bull. of Yamagata Univ., Nat. Sci.12 (1989), p151-156.

[8] Mihai, I. and Rosca, R., "On Lorentzian P-Sasakian manifolds, Classical Analysis", World Scientific Publi. (1992), 155-169.

[9] Ozgur, C., "$\varphi-$Conformally flat Lorentzian para Saskian manifolds", Radovi Mathemet-icki,Vol(12), (2003), p-99-106.

[10] Pokhariyal, G. P. and Mishra, R. S., "Curvature tensors and their relativistic signifcance", Yokohama Math. J., 18(1970), 105-108.

[11] Takahashi, T., "Sasakian ϕ-symmetric spaces", Tohoku Mathematical Journal, Second Series, vol. 29, no.1 (1977), pp. 91–113.

[12] Tripathi, M. M. and Gupta. P., "On τ- curvature tensor in K-contact manifold and Sasakian manifold", International Electronic Journal of Mathematics,V-04,(2011), p32-47.

[13] Yano, K. and Bochner, S., "Curvature and Betti numbers", Annals of Mathematics Studies 32, Princeton University Press (1953).

[14] Zamkovoy, S., "Canonical connections on paracontact manifolds", Ann. Global Anal. Geom. 36(1)(2008), 37-60.