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SOME WEIGHTED INEQUALITIES FOR HIGHER-ORDER PARTIAL DERIVATIVES IN TWO DIMENSIONS AND ITS APPLICATIONS

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Abstract. We establish some Ostrowski type inequalities involving higher-order partial derivatives for two dimensional integrals on Lebesgue spaces (L_∞, L_p and L_1). Some applications in Numerical Analysis in connection with cubature formula are given. Finally, with the help of obtained inequality, we establish applications for the kth moment of random variables.

Key words and Phrases: Ostrowski inequality, cubature formula, random variable.

1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty = \sup\limits_{t\in(a,b)} |f'(t)| < \infty$. Then, the inequality holds:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty , \tag{1}$$

for all $x \in [a, b]$ [14]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as *Ostrowski inequality.*

Recently in [2], Barnett and Dragomir proved the following Ostrowski type inequality for double integrals:

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Theorem 1.1. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be continuous on* $[a,b] \times [c,d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ *exists on* $(a,b) \times (c,d)$ *and is bounded, i.e.,*

$$\left\| f''_{x,y} \right\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty.$$

Then, we have the inequality:

$$\left| \int_a^b \int_c^d f(s,t)dtds - (d-c)(b-a)f(x,y) \right. \tag{2}$$
$$\left. - \left[(b-a) \int_c^d f(x,t)dt + (d-c) \int_a^b f(s,y)ds \right] \right|$$
$$\leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{d+c}{2} \right)^2 \right] \left\| f''_{x,y} \right\|_\infty$$

for all $(x,y) \in [a,b] \times [c,d]$.

In [2], the inequality (2) is established by the use of integral identity involving Peano kernels. In [17], Pecarić and Vukelić gave weighted Montgomery's identities for two variables functions. Recently, many authors have worked on the Ostrowski type inequalities for double integrals. For example, Pachpatte obtained a new inequality in the view (2) by using elementary analysis in [15] and [16]. In [7], [8] and [9], some Ostrowski type inequalities for double integrals and applications in numerical analysis in connection with cubature formula are given by researchers. Authors deduced weighted inequality of Ostrowski type for two dimensional integrals in [19] and [20]. Some researchers established some Ostrowski type inequalities for n-times differentiable mappings in [1], [6] and [11]. In [10], weighted integral inequalities for one variable mappings which are $n-$times differentiable are obtained by Erden and Sarıkaya. The researchers established some Ostrowski type inequalities involving higher-order partial derivatives for double integrals in [4], [12] and [21].

In this study, we first establish new integral equality involving higher-order partial derivatives. Then, some inequalities of Ostrowski type for two-dimensional integrals are attained by using this identity. Finally, some applications of the Ostrowski type inequality developed in this work for cubature formula and the kth moment of random variables are given.

2. Integral identity

In order to prove generalized weighted integral inequalities for double integrals, we need the following lemma:

Lemma 2.1. *Let* $f : [a,b] \times [c,d] =: \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function such that the partial derivatives* $\frac{\partial^{k+l} f(t,s)}{\partial t^k \partial s^l}$, $k = 0,1,2,...,n-1$, $l = 0,1,2,...,m-1$

exists and are continuous on Δ, *and assume that the functions* $g : [a, b] \to [0, \infty)$ *and* $h : [c, d] \to [0, \infty)$ *are integrable. In addition,* $P_{n-1}(x, t)$ *and* $Q_{m-1}(y, s)$ *are defined by*

$$P_{n-1}(x,t) := \begin{cases} \frac{1}{(n-1)!} \int\limits_a^t (u-t)^{n-1} g(u)\, du, & a \le t < x \\ \\ \frac{1}{(n-1)!} \int\limits_b^t (u-t)^{n-1} g(u)\, du, & x \le t \le b \end{cases}$$

and

$$Q_{m-1}(y,s) := \begin{cases} \frac{1}{(m-1)!} \int\limits_c^s (u-s)^{m-1} h(u)\, dv, & c \le s < y \\ \\ \frac{1}{(m-1)!} \int\limits_d^s (u-s)^{m-1} h(u)\, dv, & y \le s \le d, \end{cases}$$

where $n, m \in \mathbb{N} \setminus \{0\}$. *Then, for all* $(x, y) \in [a, b] \times [c, d]$, *we have the identity*

$$\int\limits_a^b \int\limits_c^d P_{n-1}(x,t)\, Q_{m-1}(y,s)\, \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} dsdt \tag{3}$$

$$= \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{M_k(x)}{k!} \frac{M_l(y)}{l!} \frac{\partial^{k+l} f(x,y)}{\partial x^k \partial y^l} - \sum_{l=0}^{m-1} \frac{M_l(y)}{l!} \int\limits_a^b g(t) \frac{\partial^l f(t,y)}{\partial y^l} dt$$

$$- \sum_{k=0}^{n-1} \frac{M_k(x)}{k!} \int\limits_c^d h(s) \frac{\partial^k f(x,s)}{\partial x^k} ds + \int\limits_a^b \int\limits_c^d h(s) g(t) f(t,s) dsdt,$$

where $M_k(x)$ *and* $M_l(y)$ *are defined by*

$$M_k(x) = \int\limits_a^b (u-x)^k g(u)\, du, \quad k = 0, 1, 2, ...$$

$$M_l(y) = \int\limits_c^d (u-y)^l h(u)\, du, \quad l = 0, 1, 2, ...$$

Proof. We have the equality

$$\int\limits_a^b \int\limits_c^d P_{n-1}(x,t)\, Q_{m-1}(y,s)\, \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} dsdt$$

$$= \int\limits_a^b P_{n-1}(x,t) \left\{ \int\limits_c^d Q_{m-1}(y,s)\, \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} ds \right\} dt.$$

Applying integration by parts for partial derivatives $\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}$ on $[c,d]$, we obtain

$$\begin{aligned}
&\int_c^d Q_{m-1}(y,s)\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}ds \\
&= \frac{1}{(m-1)!}\int_c^y\int_c^s (u-s)^{m-1}h(u)\,du\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}ds \\
&\quad +\frac{1}{(m-1)!}\int_y^d\int_d^s (u-s)^{m-1}h(u)\,du\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}ds \\
&= \frac{M_{m-1}(y)}{(m-1)!}\frac{\partial^{n+m-1}f(t,y)}{\partial t^n \partial y^{m-1}}+\int_c^d Q_{m-2}(y,s)\frac{\partial^{n+m-1}f(t,s)}{\partial t^n \partial s^{m-1}}ds.
\end{aligned}$$

As we progress by this method, we get

$$\int_c^d Q_{m-1}(y,s)\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}ds=\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\frac{\partial^{n+l}f(t,y)}{\partial t^n \partial y^l}-\int_c^d h(s)\frac{\partial^n f(t,s)}{\partial t^n}ds.$$

Then, we possess

$$\begin{aligned}
&\int_a^b\int_c^d P_{n-1}(x,t)\,Q_{m-1}(y,s)\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}dsdt \qquad (4)\\
&= \sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b P_{n-1}(x,t)\frac{\partial^{n+l}f(t,y)}{\partial t^n \partial y^l}dt-\int_c^d h(s)\int_a^b P_{n-1}(x,t)\frac{\partial^n f(t,s)}{\partial t^n}dtds.
\end{aligned}$$

Similarly, applying integration by parts for partial derivatives $\frac{\partial^{n+l}f(t,y)}{\partial t^n \partial y^l}$ and $\frac{\partial^n f(t,s)}{\partial t^n}$ on $[a,b]$, we can write

$$\begin{aligned}
&\int_a^b P_{n-1}(x,t)\frac{\partial^{n+l}f(t,y)}{\partial t^n \partial y^l}dt \qquad (5)\\
&= \sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\frac{\partial^{k+l}f(x,y)}{\partial x^k \partial y^l}-\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt
\end{aligned}$$

and

$$\int_a^b P_{n-1}(x,t)\frac{\partial^n f(t,s)}{\partial t^n}dt=\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\frac{\partial^k f(x,s)}{\partial x^k}-\int_a^b g(t)f(t,s)dt. \qquad (6)$$

Substituting the identity (5) and (6) in (4), we deduce desired identity (3), and thus the theorem is proved. □

3. Some inequalities for $\frac{\partial^{n+m}f}{\partial t^n \partial s^m}$ belongs to Lebesgue space

We give some results for functions whose $n+m$.th partial derivatives are bounded. We start with the following result.

Theorem 3.1. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a continuous on Δ such that $\frac{\partial^{n+m}f}{\partial t^n \partial s^m}$ exist on $(a,b)\times(c,d)$ and assume that the functions $g:[a,b]\to[0,\infty)$ and $h:[c,d]\to[0,\infty)$ are integrable. If $\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\in L_\infty(\Delta)$, then we have the inequality*

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right.$$
$$\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|$$

$$\leq\frac{1}{n!m!}\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty\begin{cases}M_n(x)M_m(y) & \textit{if } m \textit{ and } n \textit{ are even numbers}\\ M_n(x)\left[M_m(y)-2\int_c^y(u-y)^m h(u)du\right] & \textit{if } m \textit{ is odd number and } n \textit{ is even numbers}\\ M_m(y)\left[M_n(x)-2\int_a^x(u-x)^n g(u)du\right] & \textit{if } m \textit{ is even number and } n \textit{ is odd number}\\ \left[M_n(x)-2\int_a^x(u-x)^n g(u)du\right]\times\left[M_m(y)-2\int_c^y(u-y)^m h(u)du\right] & \textit{if } m \textit{ and } n \textit{ are odd numbers}\end{cases}$$

for all $(x,y)\in[a,b]\times[c,d]$, where

$$\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty=\sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|<\infty.$$

Proof. If we take absolute value of both sides of the equality (3), because $\frac{\partial^{n+m}f}{\partial t^n \partial s^m}$ is a bounded mapping, we can write

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k \partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt \right. \tag{7}$$

$$\left. -\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|$$

$$\leq \int_a^b\int_c^d \left|P_{n-1}(x,t)\right|\left|Q_{m-1}(y,s)\right|\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}\right|dsdt$$

$$\leq \left\|\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\right\|_\infty \int_a^b\int_c^d \left|P_{n-1}(x,t)\right|\left|Q_{m-1}(y,s)\right|dsdt.$$

By definitions of $P_{n-1}(x,t)$ and $Q_{m-1}(y,s)$, we get

$$\int_a^b\int_c^d \left|P_{n-1}(x,t)\right|\left|Q_{m-1}(y,s)\right|dsdt$$

$$= \left[\int_a^x\left|\int_a^t \frac{(u-t)^{n-1}}{(n-1)!}g(u)du\right|dt+\int_x^b\left|\int_b^t \frac{(u-t)^{n-1}}{(n-1)!}g(u)du\right|dt\right]$$

$$\times\left[\int_c^y\left|\int_c^s \frac{(u-s)^{m-1}}{(m-1)!}h(u)du\right|ds+\int_y^d\left|\int_d^s \frac{(u-s)^{m-1}}{(m-1)!}h(u)du\right|ds\right].$$

By using the change of order of integration, we obtain

$$\int_a^b\int_c^d \left|P_{n-1}(x,t)\right|\left|Q_{m-1}(y,s)\right|dsdt$$

$$= \left[\int_a^x \frac{(x-u)^n}{n!}g(u)du+\int_x^b \frac{(u-x)^n}{n!}g(u)du\right]$$

$$\times\left[\int_c^y \frac{(y-u)^m}{m!}h(u)du+\int_y^d \frac{(u-y)^m}{m!}h(u)du\right],$$

which completes the proof. □

Remark 3.2. *Under the same assumptions of Theorem 3.1 with $n = m = 1$, then the following inequality holds:*

$$\left| M_0(x)M_0(y)f(x,y) - M_0(y)\int_a^b g(t)f(t,y)dt \right. \tag{8}$$

$$\left. -M_0(x)\int_c^d h(s)f(x,s)ds + \int_a^b\int_c^d g(t)h(s)f(t,s)dsdt \right|$$

$$\leq \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \left[M_1(x) - 2\int_a^x (x-u)\, g(u)du \right] \left[M_1(y) - 2\int_c^y (y-u)\, h(u)du \right],$$

*which is "**weighted Ostrowski**" type inequality for $\|\cdot\|_\infty$ −norm. This inequality was deduced by Sarikaya and Ogunmez in* [19].

Remark 3.3. *If we take $g(u) = h(u) = 1$ in (8), then the inequality (8) reduce to the inequality (2).*

Remark 3.4. *Taking $g(u) = h(u) = 1$, $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (8), then we have the inequality*

$$\left| (b-a)(d-c) f(\frac{a+b}{2}, \frac{c+d}{2}) - (d-c)\int_a^b f(t, \frac{c+d}{2})dt \right. \tag{9}$$

$$\left. -(b-a)\int_c^d f(\frac{a+b}{2}, s)ds + \int_a^b\int_c^d f(t,s)dsdt \right|$$

$$\leq \frac{(b-a)^2 (d-c)^2}{16} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty,$$

which was given by Barnett and Dragomir in [2].

Remark 3.5. *Under the same assumptions of Theorem 3.1 with $g(u) = h(u) = 1$, then we have the inequality*

$$\left| \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \frac{X_k(x)}{k!}\frac{Y_l(y)}{l!}\frac{\partial^{k+l} f(x,y)}{\partial x^k \partial y^l} - \sum_{l=0}^{m-1}\frac{Y_l(y)}{l!}\int_a^b \frac{\partial^l f(t,y)}{\partial y^l}dt \right. \tag{10}$$

$$\left. -\sum_{k=0}^{n-1}\frac{X_k(x)}{k!}\int_c^d \frac{\partial^k f(x,s)}{\partial x^k}ds + \int_a^b\int_c^d f(t,s)dsdt \right|$$

$$\leq \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty \left[\frac{(b-x)^{n+1} + (x-a)^{n+1}}{(n+1)!} \right] \left[\frac{(d-y)^{m+1} + (y-c)^{m+1}}{(m+1)!} \right]$$

where

$$X_k(x) = \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)} \tag{11}$$

and

$$Y_l(y) = \frac{(d-y)^{l+1} + (-1)^l (y-c)^{l+1}}{(l+1)}. \tag{12}$$

This inequality (10) was proved by Hanna et al. in [12].

Theorem 3.6. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous on Δ such that $\frac{\partial^{n+m} f}{\partial t^n \partial s^m}$ exist on $(a,b) \times (c,d)$ and assume that the functions $g : [a,b] \to [0,\infty)$ and $h : [c,d] \to [0,\infty)$ are integrable. If $\frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty(\Delta)$, then we have the inequality*

$$\begin{aligned} &\left| \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{M_k(x)}{k!} \frac{M_l(y)}{l!} \frac{\partial^{k+l} f(x,y)}{\partial x^k \partial y^l} - \sum_{l=0}^{m-1} \frac{M_l(y)}{l!} \int_a^b g(t) \frac{\partial^l f(t,y)}{\partial y^l} dt \right. \\ &\left. - \sum_{k=0}^{n-1} \frac{M_k(x)}{k!} \int_c^d h(s) \frac{\partial^k f(x,s)}{\partial x^k} ds + \int_a^b \int_c^d h(s) g(t) f(t,s) ds dt \right| \\ \leq \;& \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty \frac{\|g\|_{[a,b],\infty}}{(n+1)!} \frac{\|h\|_{[c,d]\infty}}{(m+1)!} \\ &\times \left[(b-x)^{n+1} + (x-a)^{n+1} \right] \left[(d-y)^{m+1} + (y-c)^{m+1} \right], \end{aligned} \tag{13}$$

for all $(x,y) \in [a,b] \times [c,d]$, where $\|g\|_{[a,b],\infty} = \sup_{u \in [a,b]} |g(u)|$, $\|h\|_{[c,d]\infty} = \sup_{u \in [c,d]} |h(u)|$ and

$$\left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty = \sup_{(t,s) \in (a,b) \times (c,d)} \left| \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} \right| < \infty.$$

Proof. Taking modulus of both sides of the equality (3), because $\frac{\partial^{n+m} f}{\partial t^n \partial s^m}$ is a bounded mapping, we have the inequality (7). Because of boundedness g and

h, and by definitions of $P_{n-1}(x,t)$ and $Q_{m-1}(y,s)$, it follows that

$$\int_a^b \int_c^d |P_{n-1}(x,t)|\,|Q_{m-1}(y,s)|\,dsdt \tag{14}$$

$$\leq \frac{\|g\|_{[a,b],\infty}}{(n-1)!}\frac{\|h\|_{[c,d]\infty}}{(m-1)!}\left\{\int_a^x \int_c^y \left|\int_a^t (u-t)^{n-1}\,du\right|\left|\int_c^s (u-s)^{m-1}\,du\right| dsdt\right.$$

$$+\int_a^x \int_y^d \left|\int_a^t (u-t)^{n-1}\,du\right|\left|\int_d^s (u-s)^{m-1}\,du\right| dsdt$$

$$+\int_x^b \int_c^y \left|\int_b^t (u-t)^{n}\,du\right|\left|\int_c^s (u-s)^{m-1}\,du\right| dsdt$$

$$\left.+\int_x^b \int_y^d \left|\int_b^t (u-t)^{n}\,du\right|\left|\int_d^s (u-s)^{m-1}\,du\right| dsdt\right\}.$$

If we calculate the above four integrals and also substitute the results in (14), we obtain desired inequality (13) which completes the proof. □

Corollary 3.7. *Under the same assumptions of Theorem 3.6 with $n = m = 1$, then the following inequality holds:*

$$\left|M_0(x)M_0(y)f(x,y) - M_0(y)\int_a^b g(t)f(t,y)dt\right. \tag{15}$$

$$\left.-M_0(x)\int_c^d h(s)f(x,s)ds + \int_a^b \int_c^d g(t)h(s)f(t,s)dsdt\right|$$

$$\leq \left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_\infty \|g\|_{[a,b],\infty}\|h\|_{[c,d]\infty}$$

$$\times\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2 + \left(y - \frac{d+c}{2}\right)^2\right],$$

*which is "**weighted Ostrowski**" type inequality for $\|\cdot\|_\infty$ −norm.*

Remark 3.8. *If we take $g(u) = h(u) = 1$ in (15), then the inequality (15) reduce to the inequality (2).*

Remark 3.9. *If we choose $g(u) = h(u) = 1$ in theorem 3.6, then the inequality (13) becomes (10).*

Corollary 3.10. *Under the same assumptions of Theorem 3.6 with $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\frac{a+b}{2})}{k!}\frac{M_l(\frac{c+d}{2})}{l!}\frac{\partial^{k+l}f(\frac{a+b}{2},\frac{c+d}{2})}{\partial x^k\partial y^l}\right. \qquad (16)\\
&-\sum_{l=0}^{m-1}\frac{M_l(\frac{c+d}{2})}{l!}\int_a^b g(t)\frac{\partial^l f(t,\frac{c+d}{2})}{\partial y^l}dt\\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(\frac{a+b}{2})}{k!}\int_c^d h(s)\frac{\partial^k f(\frac{a+b}{2},s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
\leq\ &\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty\frac{\|g\|_{[a,b],\infty}}{(n+1)!}\frac{\|h\|_{[c,d]\infty}}{(m+1)!}\frac{(b-a)^{n+1}}{2^n}\frac{(d-c)^{m+1}}{2^m},
\end{aligned}$$

*which is Ostrowski type inequality for double integrals. Thus, (16) is a higher degree "**weighted mid-point**" inequality for $\|\cdot\|_\infty$ −norm.*

Corollary 3.11. *Choosing $n=m=1$ in (16), we obtain*

$$\begin{aligned}
&\left|M_0(x)M_0(y)f(\frac{a+b}{2},\frac{c+d}{2})-M_0(y)\int_a^b g(t)f(t,\frac{c+d}{2})dt\right.\\
&\left.-M_0(x)\int_c^d h(s)f(\frac{a+b}{2},s)ds+\int_a^b\int_c^d g(t)h(s)f(t,s)dsdt\right|\\
\leq\ &\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty\|g\|_{[a,b],\infty}\|h\|_{[c,d]\infty}\frac{(b-a)^2(d-c)^2}{16},
\end{aligned}$$

*which is "**weighted mid-point**" inequality for double integrals.*

Now, we deduce some inequalities for mappings whose higher-order partial derivatives belongs to either $L_p(\Delta)$ or $L_1(\Delta)$.

Theorem 3.12. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a continuous on Δ such that $\frac{\partial^{n+m}f}{\partial t^n\partial s^m}$ exist on $(a,b)\times(c,d)$ and assume that the functions $g:[a,b]\to[0,\infty)$ and $h:[c,d]\to[0,\infty)$ are integrable. If $\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_p(\Delta)$, $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$, then we have the* *inequality*

$$\begin{aligned} &\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right.\\ &\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\ \leq\ &\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p\frac{\|g\|_{[a,b],\infty}}{n!\,(nq+1)^{\frac{1}{q}}}\frac{\|h\|_{[c,d]\infty}}{m!\,(mq+1)^{\frac{1}{q}}}\\ &\times\left[(x-a)^{nq+1}+(b-x)^{nq+1}\right]^{\frac{1}{q}}\left[(y-c)^{mq+1}+(d-y)^{mq+1}\right]^{\frac{1}{q}}, \end{aligned}$$

for all $(x,y)\in[a,b]\times[c,d]$, *where* $\|g\|_{[a,b],\infty}=\sup_{u\in[a,b]}|g(u)|$, $\|h\|_{[c,d]\infty}=\sup_{u\in[c,d]}|h(u)|$ *and*

$$\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p=\left(\int_a^b\int_c^d\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|^p dsdt\right)^{\frac{1}{p}}.$$

Proof. Using the properties of modulus and from Hölder's inequality, from (3), we find that

$$\begin{aligned} &\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right.\\ &\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\ \leq\ &\left[\int_a^b\int_c^d|P_{n-1}(x,t)|^q\,|Q_{m-1}(y,s)|^q\,dsdt\right]^{\frac{1}{q}}\left[\int_a^b\int_c^d\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|^p dsdt\right]^{\frac{1}{p}}. \end{aligned}$$

Owing to boundedness of g and h, and by definitions of $P_{n-1}(x,t)$ and $Q_{m-1}(y,s)$, we can write

$$\left[\int_a^b\int_c^d |P_{n-1}(x,t)|^q |Q_{m-1}(y,s)|^q\, dsdt\right]^{\frac{1}{q}}$$

$$\leq \frac{\|g\|_{[a,b],\infty}}{(n-1)!}\frac{\|h\|_{[c,d]\infty}}{(m-1)!}\left(\int_a^x\left|\int_a^t (u-t)^{n-1}du\right|^q dt+\int_x^b\left|\int_b^t (u-t)^n du\right|^q dt\right)^{\frac{1}{q}}$$

$$\times\left(\int_c^y\left|\int_c^s (u-s)^{m-1}du\right|^q ds+\int_y^d\left|\int_d^s (u-s)^{m-1}du\right|^q ds\right)^{\frac{1}{q}}.$$

By simple calculations, we easily deduced required inequality, and thus the theorem is proved. □

Corollary 3.13. *Under the same assumptions of Theorem 3.12 with $n=m=1$, then the following inequality holds:*

$$\left|M_0(x)M_0(y)f(x,y)-M_0(y)\int_a^b g(t)f(t,y)dt\right. \tag{17}$$

$$\left.-M_0(x)\int_c^d h(s)f(x,s)ds+\int_a^b\int_c^d g(t)h(s)f(t,s)dsdt\right|$$

$$\leq \left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_p \|g\|_{[a,b],\infty}\|h\|_{[c,d]\infty}$$

$$\times\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left[\frac{(y-c)^{q+1}+(d-y)^{q+1}}{q+1}\right]^{\frac{1}{q}},$$

*which is "**weighted Ostrowski**" type inequality for $\|\cdot\|_p$ −norm.*

Corollary 3.14. *If we choose $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (17), then we have the inequality*

$$\left|M_0(x)M_0(y)f(\frac{a+b}{2},\frac{c+d}{2})-M_0(y)\int_a^b g(t)f(t,\frac{c+d}{2})dt\right.$$

$$\left.-M_0(x)\int_c^d h(s)f(\frac{a+b}{2},s)ds+\int_a^b\int_c^d g(t)h(s)f(t,s)dsdt\right|$$

$$\leq \left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_p \|g\|_{[a,b],\infty}\|h\|_{[c,d]\infty}\frac{(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\frac{(d-c)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}},$$

*which is "**weighted mid-point**" inequality for two dimensional integrals. This inequality is a weighted Ostrowski type inequality for* $\left\| \cdot \right\|_p$ *−norm.*

Remark 3.15. *If we take* $g(u) = h(u) = 1$ *in (17), then we get*

$$\begin{aligned} &\left| (b-a)(d-c) f(x,y) - (d-c) \int_a^b f(t,y)dt \right. \\ &\left. - (b-a) \int_c^d f(x,s)ds + \int_a^b \int_c^d f(t,s)dsdt \right| \\ \leq\ & \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_p \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \\ &\times \left[\frac{(y-c)^{q+1} + (d-y)^{q+1}}{q+1} \right]^{\frac{1}{q}} \end{aligned}$$

which was proved by Dragomir et al. in [7].

Remark 3.16. *Under the same assumptions of Theorem 3.12 with* $g(u) = h(u) = 1$*, then we have the inequality*

$$\begin{aligned} &\left| \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{X_k(x)}{k!} \frac{Y_l(y)}{l!} \frac{\partial^{k+l} f(x,y)}{\partial x^k \partial y^l} - \sum_{l=0}^{m-1} \frac{Y_l(y)}{l!} \int_a^b \frac{\partial^l f(t,y)}{\partial y^l} dt \right. \\ &\left. - \sum_{k=0}^{n-1} \frac{X_k(x)}{k!} \int_c^d \frac{\partial^k f(x,s)}{\partial x^k} ds + \int_a^b \int_c^d f(t,s)dsdt \right| \\ \leq\ & \frac{1}{n!m!} \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p \left[\frac{(x-a)^{nq+1} + (b-x)^{nq+1}}{nq+1} \right]^{\frac{1}{q}} \\ &\times \left[\frac{(y-c)^{mq+1} + (d-y)^{mq+1}}{mq+1} \right]^{\frac{1}{q}} \end{aligned} \tag{18}$$

where $X_k(x)$ *and* $Y_l(y)$ *are defined as in (11) and (12), respectively. The inequality (18) was deduced by Hanna in* [12].

Corollary 3.17. *Under the same assumptions of Theorem 3.12 with* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$, *then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\frac{a+b}{2})}{k!}\frac{M_l(\frac{c+d}{2})}{l!}\frac{\partial^{k+l}f(\frac{a+b}{2},\frac{c+d}{2})}{\partial x^k\partial y^l}\right.\\
&-\sum_{l=0}^{m-1}\frac{M_l(\frac{c+d}{2})}{l!}\int_a^b g(t)\frac{\partial^l f(t,\frac{c+d}{2})}{\partial y^l}dt\\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(\frac{a+b}{2})}{k!}\int_c^d h(s)\frac{\partial^k f(\frac{a+b}{2},s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
\leq\;&\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p\frac{\|g\|_{[a,b],\infty}}{n!\,(nq+1)^{\frac{1}{q}}}\frac{\|h\|_{[c,d]\infty}}{m!\,(mq+1)^{\frac{1}{q}}}\frac{(b-a)^{n+\frac{1}{q}}}{2^n}\frac{(d-c)^{m+\frac{1}{q}}}{2^m},
\end{aligned}$$

*which is "**weighted mid-point**" inequality for double integrals. This inequality is a higher degree weighted Ostrowski type for* $\|\cdot\|_p$ *−norm.*

Theorem 3.18. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a continuous on* Δ *such that* $\frac{\partial^{n+m}f}{\partial t^n\partial s^m}$ *exist on* $(a,b)\times(c,d)$ *and assume that the functions* $g:[a,b]\to[0,\infty)$ *and* $h:[c,d]\to[0,\infty)$ *are integrable. If* $\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_1(\Delta)$, *then we have*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right. \qquad (19)\\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
\leq\;&\frac{\|g\|_{[a,b],\infty}}{n!}\frac{\|h\|_{[c,d]\infty}}{m!}\left[\frac{(x-a)^n+(b-x)^n}{2}+\left|\frac{(b-x)^n-(x-a)^n}{2}\right|\right]\\
&\times\left[\frac{(y-c)^n+(d-y)^n}{2}+\left|\frac{(d-y)^n-(y-c)^n}{2}\right|\right]\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_1,
\end{aligned}$$

for all $(x,y)\in[a,b]\times[c,d]$, *where* $\|g\|_{[a,b],\infty}=\sup_{u\in[a,b]}|g(u)|$, $\|h\|_{[c,d]\infty}=\sup_{u\in[\ c,d]}|h(u)|$ *and*

$$\left\|\frac{\partial^{n+m}}{\partial t^n\partial s^m}\right\|_1=\int_a^b\int_c^d\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|dsdt.$$

Proof. By taking absolute value of (3), we find that

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right.$$
$$\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|$$
$$\leq \int_a^b\int_c^d \left|P_{n-1}(x,t)\right|\left|Q_{m-1}(y,s)\right|\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|dsdt$$
$$\leq \sup_{(t,s)\in(a,b)\times(c,d)}\left|P_{n-1}(x,t)\right|\left|Q_{m-1}(y,s)\right|\int_a^b\int_c^d\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|dsdt.$$

By boundedness g and h, and because of definitions of $P_{n-1}(x,t)$ and $Q_{m-1}(y,s)$, we have

$$\sup_{(t,s)\in(a,b)\times(c,d)}\left|P_{n-1}(x,t)\right|\left|Q_{m-1}(y,s)\right|$$
$$\leq \frac{\|g\|_{[a,b],\infty}}{n!}\frac{\|h\|_{[c,d]\infty}}{m!}\max\left\{(x-a)^n,(b-x)^n\right\}\max\left\{(y-c)^m,(d-y)^m\right\}.$$

We obtain desired inequality (19) using the identity

$$\max\{X,Y\}=\frac{X+Y}{2}+\left|\frac{Y-X}{2}\right|.$$

The proof is thus completed. □

Corollary 3.19. *Under the same assumptions of Theorem 3.18 with $n=m=1$, then the following inequality holds:*

$$\left|M_0(x)M_0(y)f(x,y)-M_0(y)\int_a^b g(t)f(t,y)dt\right. \tag{20}$$
$$\left.-M_0(x)\int_c^d h(s)f(x,s)ds+\int_a^b\int_c^d g(t)h(s)f(t,s)dsdt\right|$$
$$\leq \left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1\|g\|_{[a,b],\infty}\|h\|_{[c,d]\infty}$$
$$\times\left[\frac{(b-a)}{2}+\left|\frac{a+b}{2}-x\right|\right]\left[\frac{(d-c)}{2}+\left|\frac{c+d}{2}-y\right|\right],$$

*which is "**weighted Ostrowski**" inequality for double integrals of the Ostrowski type inequality for $\|\cdot\|_1$ −norm.*

Corollary 3.20. *If we choose* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$ *in (20), then we have the inequality*

$$\begin{aligned} &\left| M_0(x)M_0(y)f(\frac{a+b}{2},\frac{c+d}{2}) - M_0(y)\int\limits_a^b g(t)f(t,\frac{c+d}{2})dt \right. \\ &\left. -M_0(x)\int\limits_c^d h(s)f(\frac{a+b}{2},s)ds + \int\limits_a^b\int\limits_c^d g(t)h(s)f(t,s)dsdt \right| \\ \leq\ & \left\| \frac{\partial^2 f}{\partial t\partial s} \right\|_1 \|g\|_{[a,b],\infty} \|h\|_{[c,d]\infty} \frac{(b-a)(d-c)}{4}, \end{aligned}$$

*which is "**weighted mid-point**" inequality for the two dimensional integrals of the Ostrowski type inequality for* $\|\cdot\|_1$ *−norm.*

Remark 3.21. *If we take* $g(u) = h(u) = 1$ *in (20), then we get*

$$\begin{aligned} &\left| (b-a)(d-c)f(x,y) - (d-c)\int\limits_a^b f(t,y)dt \right. \\ &\left. -(b-a)\int\limits_c^d f(x,s)ds + \int\limits_a^b\int\limits_c^d f(t,s)dsdt \right| \\ \leq\ & \left\| \frac{\partial^2 f}{\partial t\partial s} \right\|_1 \left[\frac{(b-a)}{2} + \left|\frac{a+b}{2} - x\right|\right]\left[\frac{(d-c)}{2} + \left|\frac{c+d}{2} - y\right|\right], \end{aligned} \tag{21}$$

which is Ostrowski type inequality for $\|\cdot\|_1$ *−norm.*

Remark 3.22. *Taking* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$ *in (21), we get*

$$\begin{aligned} &\left| (b-a)(d-c)f(\frac{a+b}{2},\frac{c+d}{2}) - (d-c)\int\limits_a^b f(t,\frac{c+d}{2})dt \right. \\ &\left. -(b-a)\int\limits_c^d f(\frac{a+b}{2},s)ds + \int\limits_a^b\int\limits_c^d f(t,s)dsdt \right| \\ \leq\ & \left\| \frac{\partial^2 f}{\partial t\partial s} \right\|_1 \frac{(b-a)(d-c)}{4}, \end{aligned}$$

*which is "**mid-point**" inequality for double integrals of the Ostrowski type inequality for* $\|\cdot\|_1$ *−norm.*

Remark 3.23. *Under the same assumptions of Theorem 3.18 with $g(u)=h(u)=1$, then we have the inequality*

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{X_k(x)}{k!}\frac{Y_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{Y_l(y)}{l!}\int_a^b\frac{\partial^l f(t,y)}{\partial y^l}dt\right. \tag{22}$$
$$\left.-\sum_{k=0}^{n-1}\frac{X_k(x)}{k!}\int_c^d\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq \frac{1}{n!m!}\left[\frac{(x-a)^n+(b-x)^n}{2}+\left|\frac{(b-x)^n-(x-a)^n}{2}\right|\right]$$
$$\times\left[\frac{(y-c)^n+(d-y)^n}{2}+\left|\frac{(d-y)^n-(y-c)^n}{2}\right|\right]\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_1$$

where $X_k(x)$ and $Y_l(y)$ are defined as in (11) and (12), respectively. The inequality (22) was proved by Hanna et al. in [12].

Corollary 3.24. *Under the same assumptions of Theorem 3.18 with $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, then we have the inequality*

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\frac{a+b}{2})}{k!}\frac{M_l(\frac{c+d}{2})}{l!}\frac{\partial^{k+l}f(\frac{a+b}{2},\frac{c+d}{2})}{\partial x^k\partial y^l}\right. \tag{23}$$
$$-\sum_{l=0}^{m-1}\frac{M_l(\frac{c+d}{2})}{l!}\int_a^b g(t)\frac{\partial^l f(t,\frac{c+d}{2})}{\partial y^l}dt$$
$$\left.-\sum_{k=0}^{n-1}\frac{M_k(\frac{a+b}{2})}{k!}\int_c^d h(s)\frac{\partial^k f(\frac{a+b}{2},s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|$$
$$\leq \left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_1\frac{\|g\|_{[a,b],\infty}}{n!}\frac{\|h\|_{[c,d]\infty}}{m!}\frac{(b-a)^n}{2^n}\frac{(d-c)^m}{2^m},$$

*which is "**weighted mid-point**" inequality for double integrals. Thus, (23) is a heigher degree weighted Ostrowski type inequality for $\|\cdot\|_1$ −norm.*

4. Applications to Cubature Formulae

We now deal with applications of the integral inequalities developed in the previous section, to obtain estimates of cubature formula, which it turns out to have a markedly smaller error than that which may be obtained by the classical results. Thus the following applications in numerical integration are natural to be considered.

Let $I_\nu : a=x_0<x_1<...<x_{\nu-1}<x_\nu=b$ and $J_\mu : c=y_0<y_1<...<y_{\mu-1}<y_\mu=d$ be divisions of the intervals $[a,b]$ and $[c,d]$, $\xi_i\in[x_i,x_{i+1}]$ and

$\eta_j \in [y_j, y_{j+1}]$ with $(i = 0, ..., \nu - 1;\ j = 0, ..., \mu - 1)$. Consider the equivalent

$$\begin{aligned} S(f, I_\nu, J_\mu, \xi, \eta) &= \sum_{l=0}^{m-1}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1} \frac{M_l^{(j)}(\eta_j)}{l!} \int_{x_i}^{x_{i+1}} g(t) \frac{\partial^l f(t, \eta_j)}{\partial y^l} dt \\ &+ \sum_{k=0}^{n-1}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1} \frac{M_k^{(i)}(\xi_i)}{k!} \int_{y_j}^{y_{j+1}} h(s) \frac{\partial^k f(\xi_i, s)}{\partial x^k} ds \\ &- \sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1} \frac{M_k^{(i)}(\xi_i)}{k!} \frac{M_l^{(j)}(\eta_j)}{l!} \frac{\partial^{k+l} f(\xi_i, \eta_j)}{\partial x^k \partial y^l} \end{aligned} \tag{24}$$

where $M_k^{(i)}(\xi_i)$ and $M_l^{(j)}(\eta_j)$ are defined by

$$M_k^{(i)}(\xi_i) = \int_{x_i}^{x_{i+1}} (u - \xi_i)^k g(u) \, du, \quad k = 0, 1, 2, ...;$$

$$M_l^{(j)}(\eta_j) = \int_{y_j}^{y_{j+1}} (u - \eta_j)^l h(u) \, du, \quad l = 0, 1, 2, ...$$

Theorem 4.1. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous on Δ such that $\frac{\partial^{n+m} f}{\partial t^n \partial s^m}$ exist on $(a, b) \times (c, d)$ and assume that the functions $g : [a, b] \to [0, \infty)$ and $h : [c, d] \to [0, \infty)$ are integrable. If $\frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty(\Delta)$, then we have the representation*

$$\int_a^b \int_c^d h(s) g(t) f(t, s) ds dt = S(f, I_\nu, J_\mu, \xi, \eta) + R(f, I_\nu, J_\mu, \xi, \eta)$$

where $S(f, I_\nu, J_\mu, \xi, \eta)$ is defined as in (24) and the remainder term satisfies the estimations:

$$\begin{aligned} &|R(f, I_n, J_m, \xi, \eta)| \\ \leq\ & \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty \frac{\|g\|_{[a,b],\infty}}{(n+1)!} \frac{\|h\|_{[c,d]\infty}}{(m+1)!} \\ &\times \sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1} \left[(x_{i+1} - \xi_i)^{n+1} + (\xi_i - x_i)^{n+1} \right] \left[(y_{j+1} - \eta_j)^{m+1} + (\eta_j - y_j)^{m+1} \right] \end{aligned} \tag{25}$$

for all $(\xi_i, \eta_j) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ with $(i = 0, ..., \nu - 1;\ j = 0, ..., \mu - 1)$, where $\|g\|_{[x_i, x_{i+1}],\infty} = \sup_{u \in [x_i, x_{i+1}]} |g(u)|$, $\|h\|_{[y_j, y_{j+1}],\infty} = \sup_{u \in [y_j, y_{j+1}]} |h(u)|$ and

$$\left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty = \sup_{(t,s) \in (a,b) \times (c,d)} \left| \frac{\partial^{n+m} f(t, s)}{\partial t^n \partial s^m} \right| < \infty.$$

Proof. Applying Theorem 3.6 on the interval $[x_i, x_{i+1}]\times[y_j, y_{j+1}]$, $(i = 0, ..., \nu-1;\ j = 0, ..., \mu-1)$, we obtain

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k^{(i)}(\xi_i)}{k!}\frac{M_l^{(j)}(\eta_j)}{l!}\frac{\partial^{k+l}f(\xi_i,\eta_j)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l^{(j)}(\eta_j)}{l!}\int_{x_i}^{x_{i+1}}g(t)\frac{\partial^l f(t,\eta_j)}{\partial y^l}dt\right.$$
$$\left.-\sum_{k=0}^{n-1}\frac{M_k^{(i)}(\xi_i)}{k!}\int_{y_j}^{y_{j+1}}h(s)\frac{\partial^k f(\xi_i,s)}{\partial x^k}ds+\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}h(s)g(t)f(t,s)dsdt\right|$$
$$\leq \left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty\frac{\|g\|_{[x_i,x_{i+1}],\infty}}{(n+1)!}\frac{\|h\|_{[y_j,y_{j+1}],\infty}}{(m+1)!}$$
$$\times\left[(x_{i+1}-\xi_i)^{n+1}+(\xi_i-x_i)^{n+1}\right]\left[(y_{j+1}-\eta_j)^{m+1}+(\eta_j-y_j)^{m+1}\right]$$

for all $i = 0, ..., \nu-1;\ j = 0, ..., \mu-1$.

Summing over i from 0 to $\nu-1$ and over j from 0 to $\mu-1$ using the generalized triangle inequality we deduce the estimations (25). □

Remark 4.2. *If we take $g(u) = h(u) = 1$ and $m = n = 1$ in Theorem 4.1, then we recapture the cubature formula*

$$\int_a^b\int_c^d f(t,s)dsdt = S(f, I_\nu, J_\mu, \xi, \eta) + R(f, I_\nu, J_\mu, \xi, \eta)$$

where the remainder $R(f, I_\nu, J_\mu, \xi, \eta)$ satisfies the estimation:

$$|R(f,I_n,J_m,\xi,\eta)| \tag{26}$$
$$\leq \left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1}\left\{\left[\frac{(x_{i+1}-x_i)^2}{4}+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\right.$$
$$\left.\times\left[\frac{(y_{i+1}-y_i)^2}{4}+\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)^2\right]\right\}$$

which was given by Barnett and Dragomir in [2].

Remark 4.3. *if we consider the inequality (9), then we recapture the midpoint cubature formula*

$$\int_a^b\int_c^d f(t,s)dsdt = S_M(f, I_\nu, J_\mu) + R_M(f, I_\nu, J_\mu)$$

where the remainder $R_M(f, I_\nu, J_\mu)$ satisfies the estimation:

$$|R_M(f,I_n,J_m)| \leq \frac{1}{16}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\sum_{j=0}^{m-1}(y_{i+1}-y_i)^2$$

which was proved by Barnett and Dragomir in [2].

New composite rules can be produced if the other results given in previous sections are considered, but we omit details.

5. Some applications for the moments

Distribution functions and density functions provide complete descriptions of the distribution of probability for a given random variable. However, they do not allow us to easily make comparisons between two different distributions. The set of moments that uniquely characterizes the distribution under reasonable conditions is useful in making comparisons. Knowing the probability function, we can determine moments if they exist. Applying the mathematical inequalities, some estimations for the moments of random variables were recently studied (see, [3], [5], [13], [18]).

Set X to denote a random variable whose probability density function is $g:[a,b]\to[0,\infty)$ on the interval of real numbers I $(a,b\in I,\ a<b)$ and Y to denote a random variable whose probability density function is $h:[c,d]\to\mathbb{R}$ on the interval of real numbers I $(c,d\in I,\ c<d)$. Denoted by $M_r(x)$ and $M_r(y)$ the r.th central moment of the random variable X and Y, respectively, defined as

$$M_r(x)=\int_a^b (u-E(x))^r\, g(u)\,du,\quad r=0,1,2,...$$

and

$$M_r(y)=\int_c^d (u-E(y))^r\, h(u)\,du,\quad r=0,1,2,...$$

where $E(x)$ and $E(y)$ are the mean of the random variables X and Y, respectively. It may be noted that $M_0(x)=M_0(y)=1$, $M_1(x)=M_1(y)=0$, $M_2(x)=\sigma^2(X)$ and $M_2(y)=\sigma^2(Y)$ where $\sigma^2(X)$ and $\sigma^2(Y)$ are the variance of the random variables X and Y, respectively.

Now, we reconsider the identity (3) by changing conditions given in Lemma 2.1. Herewith, we deduce an identity involving r.th moment.

Lemma 5.1. *Let $f:[a,b]\times[c,d]=:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a continuous function such that the partial derivatives $\frac{\partial^{k+l}f(t,s)}{\partial t^k\partial s^l}$, $k=0,1,2,...,n-1$, $l=0,1,2,...,m-1$ exists and are continuous on Δ, and let X and Y be random variables whose p.d.f. are $g:[a,b]\to[0,\infty)$ and $h:[c,d]\to[0,\infty)$, respectively. Then, for all $(x,y)\in[a,b]\times[c,d]$, we have the identity*

$$\begin{aligned}
&\int_a^b\int_c^d P_{n-1}(x,t)\,Q_{m-1}(y,s)\,\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}dsdt\\
=\ &\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\\
&-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt
\end{aligned}$$

where $n,m \in \mathbb{N}\backslash\{0\}$, $M_k(x)$ *and* $M_l(y)$ *are the* k^{th} *moment, and* $P_{n-1}(x,t)$ *and* $Q_{m-1}(y,s)$ *are defined as in Lemma 2.1.*

Theorem 5.2. *Suppose that all the assumptions of Lemma 5.1 hold. If* $\frac{\partial^{n+m}f}{\partial t^n \partial s^m} \in L_\infty(\Delta)$, *then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k \partial y^l} - \sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt \right. \quad (27)\\
&\left. -\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds + \int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
\leq\ & \left\|\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\right\|_\infty \left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^n \left(\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right)^m
\end{aligned}$$

for all $(x,y) \in [a,b]\times[c,d]$, *where*

$$\left\|\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\right\|_\infty = \sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}\right| < \infty.$$

Proof. By similar methods in the proof of Theorem 3.1, we obtain

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k \partial y^l} - \sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt \right. \quad (28)\\
&\left. -\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds + \int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
\leq\ & \left\|\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\right\|_\infty \left[\int_a^x \frac{(x-u)^n}{n!}g(u)du + \int_x^b \frac{(u-x)^n}{n!}g(u)dudt\right]\\
&\times\left[\int_c^y \frac{(y-u)^m}{m!}h(u)du + \int_y^d \frac{(u-y)^m}{m!}h(u)du\right].
\end{aligned}$$

We observe that

$$\begin{aligned}
& \int_a^x \frac{(x-u)^n}{n!} g(u)du + \int_x^b \frac{(u-x)^n}{n!} g(u)du \\
\leq\ & \frac{1}{n!}\left[\sup_{u\in[a,x]} (x-u)^n \int_a^x g(u)du + \sup_{u\in[x,b]} (u-x)^n \int_x^b g(u)du\right] \\
=\ & \left[(x-a)^n \int_a^x g(u)du + (b-x)^n \int_x^b g(u)du\right] \\
\leq\ & \max\left\{(x-a)^n, (b-x)^n\right\} \int_a^b g(u)du.
\end{aligned}$$

Because g is a p.d.f., $\int_a^b g(u)du = 1$. Using the identity

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{Y-X}{2}\right|,$$

we get

$$\max\left\{(x-a)^n, (b-x)^n\right\} \int_a^b g(u)du = \left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)^n.$$

Similarly, if we examine the other integral in (28), we obtain desired inequality (27). Thus, the proof is completed. □

Remark 5.3. *With the assumptions of theorem 5.2, then we have the represantation*

$$\begin{aligned}
& \left| f(x,y) - \int_a^b g(t)f(t,y)dt \right. \qquad (29) \\
& \left. - \int_c^d h(s)f(x,s)ds + \int_a^b\int_c^d g(t)h(s)f(t,s)dsdt \right| \\
\leq\ & \left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty \left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)\left(\frac{d-c}{2} + \left|y - \frac{c+d}{2}\right|\right).
\end{aligned}$$

Proof. If we take $n = m = 1$ in (27), then we get the inequality (29). □

Similarly, using the other inequalities in section 3, we obtain similar results involving r.th central moment of the random variable X and Y.

Theorem 5.4. *Let $f : [a, b] \times [c, d] =: \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that the partial derivatives $\frac{\partial^{k+l} f(t,s)}{\partial t^k \partial s^l}$, $k = 0, 1$, $l = 0, 1, 2$ exists and are continuous on Δ, and let X and Y be random variables whose p.d.f. are $g : [a, b] \to [0, \infty)$ and $h : [c, d] \to [0, \infty)$, respectively. Then we have*

$$\begin{aligned} & \left| f(x, y) - \int_a^b g(t) f(t, y) dt \right. \\ & \left. - \int_c^d h(s) f(x, s) ds + \int_a^b \int_c^d g(t) h(s) f(t, s) ds dt \right| \\ \leq & \left\| \frac{\partial^4 f}{\partial t^2 \partial s^2} \right\|_\infty \sigma^2(X) \sigma^2(Y) \end{aligned} \tag{30}$$

for all $(x, y) \in [a, b] \times [c, d]$, where

$$\left\| \frac{\partial^4 f}{\partial t^2 \partial s^2} \right\|_\infty = \sup_{(t,s) \in (a,b) \times (c,d)} \left| \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} \right| < \infty.$$

Proof. If we take $n = m = 2$ in (28), we obtain desired inequality (30). $\square$

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