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COMPLETE CONVERGENCES AND STRONG LAWS OF LARGE NUMBERS FOR WEIGHTED SUMS PAIRWISE NQD RANDOM VARIABLES SEQUENCE

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Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables. Some complete convergences and strong laws of large numbers for a weighted sums sequence of pairwise NQD random variables are obtained. The results obtainted generalize the results of Cabrera and Volodin (see [3]).

1. INTRODUCTION

Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d) random variables. The Marcinkiewicz-Zygmund strong laws of large numbers (SLLN) provides that

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} (X_i - EX_i) \to 0 a.s. for 1 \leq \alpha < 2$$

and

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} X_i \to 0 a.s. for 0 < \alpha < 1$$

if and only if $E|X|^\alpha < \infty$. The case $\alpha = 1$ is due to Kolmogorov.

As for negatively associated (NA) random variables, Joag and Proschan [8] gave the following definition.

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Definition 1.1. [8] *A finite family of random variables* $\{X_i, 1 \leq i \leq n\}$ *is said to be negatively associated (NA) if for every pair of disjoint subsets* T_1 *and* T_2 *of* $\{1, 2, ..., n\}$, *we have*

$$Cov(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \leq 0,$$

whenever f_1 *and* f_2 *are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.*

Let $\{X, X_n, n \geq 1\}$ be an identically distributed NA sequence, Matula [10] proved the Kolmogorov strong laws of large numbers. Two random variables X and Y are negative quadrant dependent (NQD), if for all $x, y \in R$, we have

$$P(X > x, Y > y) \leq P(X > x)P(Y > y).$$

A random variables sequence $\{X_k, k \in N\}$ is said to be pairwise NQD, if for all $i \neq j$, X_i and X_j are NQD.

The concept of NQD was given by Lehmann [9]. We known that NQD is more general than NA. So NQD is very general. As for pairwise NQD random variables sequences, Wu [13] Proved Kolmogorov strong law of large numbers and complete convergence for pairwise NQD random sequences. Cabrera and Volodin [3] obtained mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability. The result of Cabrera and Volodin [3] sees the following Theorem.

Theorem A *Let* $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *be an array of rowwise pairwise NQD random variables and*$\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *be an array of constants with* $\sum_{k=u_n}^{v_n} |a_{nk}| \leq C$ *for all* $n \in N$ *and some constant* $C > 0$. *Let moreover* $\{h(n), n \geq 1\}$ *be an increasing sequence of positive constants with* $h_n \uparrow \infty$ *as* $n \uparrow \infty$. *Suppose that*
(a) $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *is h- integrable concerning the array of constants* $\{a_{nk}\}$,
(b) $h^2(n) \sum_{k=u_n}^{v_n} a_{nk}^2 \to 0$ *as* $n \to \infty$.
Let $T_n = \sum_{k=u_n}^{v_n} a_{nk} X_{nk}$, *for all* $n \geq 1$. *Then* $\lim_{n\to\infty} |T_n| = 0$ *in Probability.*

The main purpose of this paper is to establish some complete convergences and strong laws of large numbers for a weighted sums sequence of pairwise NQD random variables. The results obtainted generalize the results of Cabrera and Volodin [3].

2. MAIN RESULTS

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leq Cb_n$. In order to prove our results, we need the concept of the Hsu-Robbins-Erdös law of large numbers (see [6], [7]) and the following lemma. Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d) random variables and denote $S_n = \sum_{i=1}^{n} X_i$. The Hsu-Robbins-Erdös law of large numbers (see [6], [7]) states that

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} P(|S_n| > \varepsilon n) < \infty$$

is equivalent to $EX = 0$, $EX^2 < \infty$.

This is a foundamental theorm in probability theory and has been intensively investigated by many authors in the past decades. One of the most important results is Baum and Katz [1] law of large numbers, which states that, for $p < 2$ and $r \geq p$,

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P(|S_n| > \varepsilon n^{\frac{1}{p}}) < \infty$$

if and only if $E|X|^r < \infty, r \geq 1$, and $EX = 0$. There have been many extensions in various directions. Two of them are Chow and Lai ([4], [5]) proposed a two sided estimate and Petrov [11].

Lemma 2.1. [9] *Let X and Y are NQD, then*
(i) $EXY \leq EXEY$;
(ii) $P(X > x, Y > y) \leq P(X > x)P(Y > y)$, for all $x, y \in R$;
(iii) if $f(x), g(x)$ are non-decreasing (or non-increasing) functions, then $f(X), g(X)$ are also NQD.

Lemma 2.2. *Let $\{Y_i, i \geq 1\}$ be a sequence of centered pairwise NQD random variables and $E|Y_i|^2 < \infty$ for every $i \geq 1$. Then there exists C, such that*

$$E|\sum_{i=1}^{n} Y_i|^2 \leq \sum_{i=1}^{n} E|Y_i|^2$$

$$E \max_{1\leq k\leq n} |\sum_{i=1}^{k} Y_i|^2 \leq C \log^2 n \sum_{i=1}^{n} E|Y_i|^2$$

.

Proof. By Lemma 2.1 and $EY_i = 0$, then

$$E|\sum_{i=1}^{n} Y_i|^2 \leq \sum_{i=1}^{n} E|Y_i|^2 + 2 \sum_{1\leq i<j\leq n} EY_iEY_j \leq \sum_{i=1}^{n} E|Y_i|^2.$$

By $E|\sum_{i=1}^n Y_i|^2 \leq \sum_{i=1}^n E|Y_i|^2$ and Theorem 2.4.1 in Stout [12], we have

$$E\max_{1\leq k\leq n}|\sum_{i=1}^k Y_i|^2 \leq C(\frac{\log(2n)}{\log 2})^2\sum_{i=1}^n E|Y_i|^2 \leq C\log^2 n\sum_{i=1}^n E|Y_i|^2.$$

Now we give the concept of $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ is h- integrable concerning the array of constants $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$.

Definition 2.3. [3] *Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of random variables and $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ an array of constants with $\sum_{k=u_n}^{v_n}|a_{nk}| \leq C$ for all $n \in N$ and some constant $C > 0$. Let moreover $\{h(n), n \geq 1\}$ be an increasing sequence of positive constants with $h_n \uparrow \infty$ as $n \uparrow \infty$. The array $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ is said to be h- integrable concerning the array of constants $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ if the following two conditions hold:*

$$\lim_{n\to\infty}\sup_{n\geq 1}\sum_{k=u_n}^{v_n}|a_{nk}|E|X_{nk}| < \infty$$

and

$$\lim_{n\to\infty}\sum_{k=u_n}^{v_n}|a_{nk}|E|X_{nk}|I(|X_{nk}| > h(n)) = 0.$$

We inspired by the concept of $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ is h- integrable concerning the array of constants $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ that we get the following Theorem.

Theorem 2.4 *Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of rowwise pairwise NQD random variables and $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of constants with $\sum_{k=u_n}^{v_n}|a_{nk}| \leq C$ for all $n \in N$ and some constant $C > 0$. Let moreover $\{h(n), n \geq 1\}$ be an increasing sequence of positive constants with $h_n \uparrow \infty$ as $n \uparrow \infty$. Suppose that*
(a) $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ is h- integrable concerning the array of constants $\{a_{nk}\}$,
(b) $h^2(n)\sum_{k=u_n}^{v_n}a_{nk}^2 \to 0$ as $n \to \infty$,
(c) $h(n) \geq C(\log n)^{1+\delta}$ for some $\delta > 0$.
Let $T_n = \sum_{k=u_n}^{v_n}a_{nk}X_{nk}$, for all $n \geq 1$ and $EX_{nk} = 0$, for all $n \geq 1, u_n \leq k \leq v_n$. Then

$$\forall\varepsilon > 0, \sum_{n=1}^{\infty}n^{-1}P(\max_{1\leq j\leq n}|T_j| > \varepsilon) < \infty. \tag{2.1}$$

Proof. $\forall i \geq 1$, define $X_i^{(n)} = X_{ni}I(|a_{ni}X_{ni}| \leq h(n)) + h(n)I(|a_{ni}X_{ni}| > h(n)) - h(n)I(|a_{ni}X_i| < -h(n))$, $T_j^{(n)} = \sum\limits_{i=1}^{j}(X_i^{(n)} - EX_i^{(n)})$, then $\forall\varepsilon > 0$,

$$P(\max_{1\le j\le n}|T_j| > \varepsilon) \le P(\max_{1\le j\le n}|a_{nj}X_{nj}| > h(n)) + P(\max_{1\le j\le n}|T_j^{(n)}| > \varepsilon - \max_{1\le j\le n}|\sum_{i=1}^{j} EX_i^{(n)}|). \quad (2.2)$$

First we show that

$$\max_{1\le j\le n}|\sum_{i=1}^{j} EX_i^{(n)}| \to 0, as n \to \infty. \quad (2.3)$$

In fact, by (a) and (b), then

$$\begin{aligned}
&\max_{1\le j\le n}|\sum_{i=1}^{j} EX_i^{(n)}| \\
&= \max_{1\le j\le n}|\sum_{i=1}^{j} E[X_{ni}I(|a_{ni}X_{ni}| \le h(n)) + h(n)I(|a_{ni}X_{ni}| > h(n)) - h(n)I(|a_{ni}X_{ni}| < -h(n))]| \\
&\le \max_{1\le j\le n}|\sum_{i=1}^{j} EX_{ni}I(|X_{ni}| > h(n))| + h(n)\sum_{j=1}^{n} P(|X_{nj}| > h(n)) \to 0, as n \to \infty. \quad (2.4)
\end{aligned}$$

From (2.4), hence (2.3) is true. From (2.2) and (2.3), it follows that for n large enough

$$P(\max_{1\le j\le n}|T_j| > \varepsilon) \le P(\max_{1\le j\le n}|a_{nj}X_{nj}| > h(n)) + P(\max_{1\le j\le n}|T_j^{(n)}| > \frac{\varepsilon}{2}).$$

Hence we need only to prove that

$$\begin{aligned}
I &=: \sum_{n=1}^{\infty} n^{-1}\sum_{j=1}^{n} P(|a_{nj}X_{nj}| > h(n)) < \infty, \\
II &=: \sum_{n=1}^{\infty} n^{-1}P(\max_{1\le j\le n}|T_j^{(n)}| > \frac{\varepsilon}{2}) < \infty. \quad (2.5)
\end{aligned}$$

By Markov inequality, (a), (b) and (c), it follows easily that

$$\begin{aligned}
I &= \sum_{n=1}^{\infty} n^{-1}\sum_{j=1}^{n} P(|a_{nj}X_{nj}| > h(n)) \\
&\le \sum_{n=1}^{\infty} n^{-1}\sum_{j=u_n}^{v_n} P(|a_{nj}X_{nj}| > h(n)) \\
&\le \sum_{n=1}^{\infty} n^{-1}\sum_{j=u_n}^{v_n} \frac{E|a_{nj}X_{nj}|}{h(n)} \\
&\le C\sum_{n=1}^{\infty} n^{-1}(\log n)^{-1-\delta} < \infty. \quad (2.6)
\end{aligned}$$

By Markov inequality, Lemma 2.2, (a), (b) and (c), then

$$\begin{aligned}
II &= \sum_{n=1}^{\infty} n^{-1} P(\max_{1\le j\le n} |T_j^{(n)}| > \frac{\varepsilon}{2}) \\
&\le \sum_{n=1}^{\infty} n^{-1} (\varepsilon/2)^{-2} E \max_{1\le j\le n} |T_j^{(n)}|^2 \\
&\le C \sum_{n=1}^{\infty} n^{-1} (\log n)^2 \sum_{j=1}^{n} E|X_j^{(n)}|^2 \\
&\le C \sum_{n=1}^{\infty} n^{-1} (\log n)^2 \sum_{i=1}^{n} E|X_{ni} I(|a_{ni}X_{ni}| \le h(n)) + h(n) I(|a_{ni}X_{ni}| > h(n))|^2 \\
&< \infty. \qquad (2.7)
\end{aligned}$$

Now we complete the proof of Theorem 2.4.

Corollary 2.5. *Under the conditions of Theorem* 2.1*, then*

$$\lim_{n\to\infty} |T_n| = 0 \quad a.s.$$

Proof. By (2.1), we have

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P(\max_{1\le j\le n} |T_j| > \varepsilon) \\
&= \sum_{i=0}^{\infty} \sum_{n=2^i}^{2^{i+1}-1} n^{-1} P(\max_{1\le j\le n} |T_j| > \varepsilon) \\
&\ge \frac{1}{2} \sum_{i=1}^{\infty} P(\max_{1\le j\le 2^i} |T_j| > \varepsilon).
\end{aligned}$$

By Borel-Cantelli Lemma, we have

$$P(\max_{1\le j\le 2^i} |T_j| > \varepsilon i.o.) = 0.$$

Hence

$$\lim_{i\to\infty} \max_{1\le j\le 2^i} |T_j| = 0 \quad a.s.$$

and using

$$\max_{2^{i-1}\le n<2^i} |T_n| \le \max_{1\le j\le 2^i} |T_j|,$$

We have

$$\lim_{n\to\infty} |T_n| = 0 \quad a.s.$$

Now we complete the proof of Corollary 2.5.

Remark Corollary 2.5. generalizes the results of Cabrera and Volodin [3].

3. CONCLUDING REMARKS

In this paper, some complete convergences and strong laws of large numbers for a weighted sums sequence of pairwise NQD random variables are obtained.

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