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ON THE UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING THREE-POINT SETS IM

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Abstract. We show that if two meromorphic functions on the complex plane sharing some three-point sets IM, then they are indentical under some conditions.

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1. INTRODUCTION AND THE MAIN RESULTS

For nonconstant meromorphic functions f and g on $\mathbb{C}$ and a finite set S in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, we say that f and g share S CM (counting multiplicities) if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ two functions $f - f(z_0)$ and $g - g(z_0)$ have the same multiplicity of zero at z_0, where the notations $f - \infty$ and $g - \infty$ mean $1/f$ and $1/g$, respectively. Also, if $f^{-1}(S) = g^{-1}(S)$, then we say that f and g share S IM (ignoring multiplicities). In particular if S is a one-point set $\{a\}$, then we say also that f and g share a CM or IM. The notion of meromorphic functions sharing sets is introduced by Gross in [1].

In [4] and [5], R. Nevanlinna showed the following two theorems:

Theorem 1.1. *Let f and g be two distinct nonconstant meromorphic functions on $\mathbb{C}$ and let $a_1, \cdots, a_4$ be four distinct points in $\overline{\mathbb{C}}$. If f and g share $a_1, \cdots, a_4$ CM, then f is a Möbius transform of g, i.e. $f = (ag+b)/(cg+d)$ for some complex numbers a, b, c, d with $ad - bc \neq 0$, and there exists a permutation σ of $\{1, 2, 3, 4\}$ such that $a_{\sigma(3)}, a_{\sigma(4)}$ are Picard exceptional values of f and g and the cross ratio $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) = -1$.*

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Theorem 1.2. *Let f and g be two nonconstant meromorphic functions on $\mathbb{C}$ sharing five distinct points in $\overline{\mathbb{C}}$ IM, then $f = g$.*

Since Nevanlinna, many researchers have been studying various versions of uniquness theorem of meromorphic functions sharing finite sets CM or IM (see, *e.g.*, [7], [8], [11] and [12]). In particular, the second author researched meromorphic functions sharing four two-point sets CM, and also he gave the following result in [9]

Theorem 1.3. *Let $S_1, \cdots, S_5$ be pairwise disjoint one-point or two-point sets in $\overline{\mathbb{C}}$. If two nonconstant meromorphic functions f and g on $\mathbb{C}$ share $S_1, \cdots, S_5$ IM, then f is a Möbius transform of g.*

Remark. Let T be the Möbius transformation of the conclusion in Theorem 1.3. By the proof of Theorem 1.3, we see that T is of order 2, *i.e.*, $T^2 = T \circ T$ is the identity if T is not the identity.

Since any Möbius transformation except the indentity has at most two fixed points and any nonconstant meromorphic function has at most two exceptional values, we have, as a corollary of Theorem 1.3,

Corollary 1.4. *Let $S_1, \cdots, S_5$ be pairwise disjoint one-point or two-point sets in $\overline{\mathbb{C}}$. Assume that there is no Möbius transformation T except the identity with at most two points z in $\overline{\mathbb{C}}$ satisifying one of the following conditions: (i) $z \in S_j$ and $T(z) \notin S_j$ for some $j = 1, \cdots, 5$; (ii) $z \notin \cup_{j=1}^5 S_j$ and $T(z) \in \cup_{j=1}^5 S_j$. Then two nonconstant meromorphic functions on $\mathbb{C}$ sharing $S_1, \cdots, S_5$ IM are identical.*

In particular, in the case of five two-point sets we see

Theorem 1.5. *Let $S_1, \cdots, S_5$ be pairwise disjoint two-point sets in $\overline{\mathbb{C}}$. Assume that there is no Möbius transformation which interchanges two elements of S_j for distinct three j. Then, two nonconstant meromorphic functions on $\mathbb{C}$ sharing $S_1, \cdots, S_5$ IM are identical.*

We assume that S_1, S_2, S_3 be pairwise disjoint two-point sets in $\mathbb{C}$. Let $S_j = \{\xi_j, \eta_j\}$ and let $P_j(z) = z^2 + a_j z + b_j$, where $a_j = -(\xi_j + \eta_j), b_j = \xi_j \eta_j$. We see that the following three conditions are equivalent: (i) there exists a Möbius transformation which interchanges ξ_j and η_j for $j = 1, 2, 3$; (ii) $\begin{vmatrix} 1 & \xi_1 + \eta_1 & \xi_1\eta_1 \\ 1 & \xi_2 + \eta_2 & \xi_2\eta_2 \\ 1 & \xi_3 + \eta_3 & \xi_3\eta_3 \end{vmatrix} = 0$; (iii) P_1, P_2, P_3 are linearly dependent over $\mathbb{C}$ (see Lemma 3.2 in [6]).

For a finite set S in $\mathbb{C}$, we define its defining polynomial P as a polynomial without multiple zeros satisfying $S = \{z \in \mathbb{C} : P(z) = 0\}$, then P is called a defining polynomial of S. Therefore, we can restate Theorem 1.5 as follows:

Theorem 1.6. *Let $S_1, \cdots, S_5$ be pairwise disjoint two-point sets in $\mathbb{C}$ and let P_j be a defining polynomial of S_j. If any three of $P_1, \cdots, P_5$ are linearly independent over $\mathbb{C}$, then two nonconstant meromorphic functions on $\mathbb{C}$ sharing $S_1, \cdots, S_5$ IM are identical.*

In this paper, we consider the uniqueness of meromorphic functions sharing three-point sets and one-point sets IM. The above condition (i) is geometrical, but it is hard to treat it in our problem. Therefore we adapt the condition (iii) to our problem, and about the uniqueness of meromorphic functions sharing three-point sets IM, we show

Theorem 1.7. *Let $S_1, \cdots, S_5$ be pairwise disjoint three-point sets in $\mathbb{C}$ and let P_j be a defining polynomial of S_j. If any three of $P_1, \cdots, P_5$ are linearly independent over $\mathbb{C}$, then two nonconstant meromorphic functions on $\mathbb{C}$ sharing $S_1, \cdots, S_5$ IM are identical.*

We assume that the reader is familiar with the standard notations and results of the value distribution theory (see, for example, [2]). In particular, we express by $S(r, f)$ quantities such that $\lim_{r\to\infty, r\notin E} S(r, f)/T(r, f) = 0$, where E is a subset of $(0, \infty)$ with finite linear measure and it is variable in each case.

2. PROOF OF THEOREM 1.7

First, for general polynomials we show the following lemma.

Lemma 2.1. *Let P_1, P_2, P_3 be three polynomials of degree d, where d is a positive integer. Assume that P_j has no multiple zeros and that P_j and P_k have no common zeros for distinct j and k in $\{1, 2, 3\}$. Moreover, assume that P_1, P_2, P_3 are linearly independent over $\mathbb{C}$. Then not all of $P_1(\xi_j)/P_2(\xi_j)$ $(j = 1, \cdots, d)$ are the same, where $\xi_1, \cdots, \xi_d$ are the zeros of P_3.*

Proof. On the contrary, we assume that all of of $P_1(\xi_j)/P_2(\xi_j)$ $(j = 1, \cdots, d)$ are the same value λ:

$$P_1(\xi_j) - \lambda P_2(\xi_j) = 0 \quad (j = 1, \cdots, d).$$

Put $Q(z) = P_1(z) - \lambda P_2(z)$, then Q has zeros $\xi_1, \cdots, \xi_d$, and hence it is a nonzero complex multiple of P_3. This implies that P_1, P_2, P_3 are linearly dependent over $\mathbb{C}$, and hence, we have finished the proof. □

We introduce the following Borel's Lemma, whose proof can be found, for example, on p.186 of [3].

Lemma 2.2. *If entire functions $\alpha_0, \alpha_1, \ldots, \alpha_n$ without zeros satisfy*

$$\alpha_0 + \alpha_1 + \cdots + \alpha_n = 0,$$

then for each $j = 0, 1, \cdots, n$ there exists some $k \neq j$ such that α_j / α_k is constant.

For the proof of Theorem 1.7 we need the following result in [10]

Theorem 2.3. *Let p be a non-negative integer and let q be an integer not less than 2. Let $S_1, \cdots, S_p$ be one-point sets in $\mathbb{C}$ and let $S_{p+1}, \cdots, S_{p+q}$ be d-point sets in $\mathbb{C}$, where d is an integer not less than 2. Assume that $S_1, \cdots, S_{p+q}$ are pairwise disjoint and that $p + q \geq 5$. If two distinct nonconstant meromorphic functions f and g on $\mathbb{C}$ share $S_1, \cdots, S_{p+q}$ IM, then there exists distinct j_1, j_2 in $\{p+1, \cdots, p+q\}$ such that $P_{j_1}(f)/P_{j_2}(f) = P_{j_1}(g)/P_{j_2}(g)$, where P_j are defining polynomials of S_j.*

Proof. We may assume that $p \leq 4$ by Theorem 1.2.

By the second main theorem and the first main theorem we have

$$(p + dq - 2)T(r, f) \leq \sum_{j=1}^{p+q} \sum_{\xi \in S_j} \overline{N}(r, \frac{1}{f - \xi}) + S(r, f)$$

$$= \sum_{j=1}^{p+q} \sum_{\xi \in S_j} \overline{N}(r, \frac{1}{g - \xi}) + S(r, f) \leq (p + dq)T(r, g) + S(r, f) \qquad (1)$$

and, by the same way,

$$(p + dq - 2)T(r, g) \leq (p + dq)T(r, f) + S(r, g). \qquad (2)$$

Hence, by (1) and (2), there is no need to distinguish $S(r, f)$ and $S(r, g)$, and so we denote them by $S(r)$.

By $\overline{N}_E(r, \frac{1}{f-\xi})$ and $\overline{N}_N(r, \frac{1}{f-\xi})$ we denote the counting functions which count the point z such that $f(z) = \xi = g(z)$ and $f(z) = \xi \neq g(z)$ counted once, respectively, and we define $\overline{N}_E(r, \frac{1}{g-\xi})$ and $\overline{N}_N(r, \frac{1}{g-\xi})$ by the same way. It is easy to see that $\overline{N}_N(r, \frac{1}{f-\xi}) = \overline{N}_N(r, \frac{1}{g-\xi}) = 0$ for $\xi \in S_1 \cup \cdots \cup S_p$ and that

$$\sum_{\xi \in S_j} \overline{N}_E(r, \frac{1}{f - \xi}) = \sum_{\xi \in S_j} \overline{N}_E(r, \frac{1}{g - \xi}),$$

$$\sum_{\xi \in S_j} \overline{N}_N(r, \frac{1}{f - \xi}) = \sum_{\xi \in S_j} \overline{N}_N(r, \frac{1}{g - \xi}) \qquad (3)$$

for $j = p+1, \cdots, q$. Since $f - g \not\equiv 0$, we have

$$\sum_{j=1}^{p+q} \sum_{\xi \in S_j} \overline{N}_E(r, \frac{1}{f-\xi}) \le \overline{N}(r, \frac{1}{f-g}) \le T(r,f) + T(r,g) + O(1),$$

and hence

$$\begin{aligned} \sum_{j=p+1}^{p+q} \sum_{\xi \in S_j} \overline{N}_N(r, \frac{1}{f-\xi}) &= \sum_{j=1}^{p+q} \sum_{\xi \in S_j} \overline{N}(r, \frac{1}{f-\xi}) - \sum_{j=1}^{p+q} \sum_{\xi \in S_j} \overline{N}_E(r, \frac{1}{f-\xi}) \\ &\ge (p+dq-2)T(r,f) - T(r,f) - T(r,g) + S(r) \\ &= (p+dq-3)T(r,f) - T(r,g) + S(r) \end{aligned}$$

by using (1). By the same way and (3) we have

$$\sum_{j=p+1}^{p+q} \sum_{\xi \in S_j} \overline{N}_N(r, \frac{1}{f-\xi}) \ge (p+dq-3)T(r,g) - T(r,f) + S(r).$$

Adding these two inequalities we obtain

$$\sum_{j=p+1}^{p+q} \sum_{\xi \in S_j} \overline{N}_N(r, \frac{1}{f-\xi}) \ge \frac{1}{2}(p+dq-4)(T(r,f)+T(r,g)) + S(r). \tag{4}$$

Note that $q \ge 2$. From (4) we see that there exist distinct j_1 and j_2 in $\{p+1, \cdots, q\}$ and a subset I of $(0, +\infty)$ of infinite linear measure such that

$$\frac{1}{q}(p+dq-4)(T(r,f)+T(r,g)) + S(r) \le \sum_{\xi \in S_{j_1} \cup S_{j_2}} \overline{N}_N(r, \frac{1}{f-\xi}) \tag{5}$$

holds for $r \in I$. Put $Q(z,w) = (P_{j_1}(z)P_{j_2}(w) - P_{j_1}(w)P_{j_2}(z))/(z-w)$ and $\Phi = Q(f,g)$. Assume that $\Phi \not\equiv 0$. If $f(z), g(z) \in S_{j_1} \cup S_{j_2}$ and $f(z) \ne g(z)$, then $\Phi(z) = 0$. Therefore we have

$$\sum_{\xi \in S_{j_1} \cup S_{j_2}} \overline{N}_N(r, \frac{1}{f-\xi}) \le N_0(r, \frac{1}{\Phi}) \tag{6}$$

holds for $r \in I$, where $N_0(r, \frac{1}{\Phi})$ denotes the counting functions corresponding to the zeros of Φ that are not the poles of f and g. We see that $Q(z,w)$ is a symmetric polynomial of z and w and it has degree at most $d-1$ with respect to each of z and w. By using the first fundamental theorem and the definition of countiong function and that of proximity function, we have

$$\begin{aligned} N_0(r, \frac{1}{\Phi}) &\le N(r, Q(f,g)) + m(r, Q(f,g)) \\ &\le (d-1)(N(r,f) + N(r,g) + m(r,f) + m(r,g)) + O(1) \\ &= (d-1)(T(r,f) + T(r,g)) + O(1). \end{aligned}$$

By connecting (5), (6) and this,

$$\frac{1}{q}(p+dq-4)(T(r,f)+T(r,g)) + S(r) \le (d-1)(T(r,f)+T(r,g)) + O(1)$$

holds for $r \in I$. Here I may be different from that in (5). We obtain $p + dq - 4 \leq q(d-1)$, which contradicts hypothesis $p+q \geq 5$. Therefore we conclude that $\Phi \equiv 0$, which induces that $P_{j_2}(f)/P_{j_1}(f) = P_{j_2}(g)/P_{j_1}(g)$. We have completed the proof. □

Now, we start the proof of Theorem 1.7. By Theorem 2.3, we may assume that $P_1(f)/P_2(f) = P_1(g)/P_2(g)$ by rearranging indices if necessary.

Remark. Let $P_1, \cdots, P_q$ be polynomials of degree d, where q and d are integers such that $q \geq 3, d \geq 2$. Assume that each of $P_1, \cdots, P_q$ has no multiple zeros and that any two of them have no common zeros. Moreover, we assume that $P_1(f)/P_2(f) = P_1(g)/P_2(g)$ for nonconstant meromorphic functions f and g on $\mathbb{C}$ and that f and g share $S_3, \cdots, S_q$ IM. If for each $j = 3, \cdots, q$ the d values $P_1(\xi_{jk})/P_2(\xi_{jk})$ $(k = 1, \cdots, d)$ are distinct for each zero ξ_{jk} of P_j, then f and g share ξ_{jk} IM. In this case, by Theorem 1.2, we see that $f = g$ if $d(q-2) \geq 5$. However, the hypothesis that for each $j = 3, \cdots, q$ the d values $P_1(\xi_{jk})/P_2(\xi_{jk})$ $(k = 1, \cdots, d)$ are distinct is too strong, and we seek another condition about P_j for the uniqueness of meromorphic functions.

Again, we return to the proof. Let $S_j = \{\xi_j, \eta_j, \zeta_j\}$. For $\xi, \eta \in \overline{\mathbb{C}}$, we put $E(\xi, \eta) = \{z \in \mathbb{C} : (f(z), g(z)) = (\xi, \eta) \text{ or } (\eta, \xi)\}$. We separate S_3, S_4, S_5 into two types: (A) some of $E(\xi_j, \eta_j), E(\xi_j, \zeta_j), E(\eta_j, \zeta_j)$ are not empty; (B) all of $E(\xi_j, \eta_j), E(\xi_j, \zeta_j), E(\eta_j, \zeta_j)$ are empty. Then we consider three cases: (I) at least two of S_3, S_4, S_5 are of type (B); (II) one of S_3, S_4, S_5 is of type (B) and the others are of type (A); (III) all of S_3, S_4, S_5 are of type (A).

The case (I). At least two of S_3, S_4, S_5 are of type (B). We may assume that S_4, S_5 are of type (B). Then f and g share ξ_j, η_j, ζ_j IM for $j = 4, 5$. By Theorem 1.2, we get $f = g$.

The case (II). One of S_3, S_4, S_5 is of type (B) and the others are of type (A). We may assume that S_3 is of type (B) and S_4 and S_5 are of type (A). Moreover, we may assume that $E(\xi_j, \eta_j) \neq \emptyset$ for $j = 4, 5$. Then, for $j = 4, 5$, we have $P_1(\xi_j)/P_2(\xi_j) = P_1(\eta_j)/P_2(\eta_j)$, and by assumption and Lemma 2.1 we have $E(\xi_j, \zeta_j) = E(\eta_j, \zeta_j) = \emptyset$, which implies that f and g share $\{\xi_j, \eta_j\}$ and $\{\zeta_j\}$ IM. Since f and g share also ξ_3, η_3, ζ_3 IM, by using Theorem 1.2, we get $f = g$.

The case (III). All of S_3, S_4, S_5 are of type (A). We may assume that $E(\xi_j, \eta_j) \neq \emptyset$ for $j = 3, 4, 5$. Then, for $j = 3, 4, 5$, we have $P_1(\xi_j)/P_2(\xi_j) = P_1(\eta_j)/P_2(\eta_j)$, and by assumption and Lemma 2.1 we have $E(\xi_j, \zeta_j) = E(\eta_j, \zeta_j) = \emptyset$, which implies that f and g share $\{\xi_j, \eta_j\}$ and $\{\zeta_j\}$ IM. By using Theorem 1.3, there exists a Möbius transformation T such that $f = T \circ g$.

Assume that $f \not\equiv g$, and hence T is not the identity. Then T has at most two fixed points, and hence, we may assume that $E(\zeta_5, \zeta_5) = \emptyset$. In this case, ζ_5 is an exceptioanl value of f and g. Furthermore, T interchanges ξ_j and η_j for at least one of $j \in \{3, 4, 5\}$, which are also fixed points of $T^2 = T \circ T$. Since the fixed points of T are also those of T^2, it has at least three fixed points. Therefore, T^2 is the identity. This result follows also from Remark of Theorem 1.3.

(i) The case where $E(\zeta_4,\zeta_4)=\emptyset$ or $E(\zeta_3,\zeta_3)=\emptyset$. It is enough to consider the case where $E(\zeta_4,\zeta_4)=\emptyset$. In this case ζ_4 and ζ_5 are exceptional value of f and g and they have no more exceptional values. Therefore $E(\zeta_3,\zeta_3)\neq\emptyset$, and hence ζ_3 is a fixed point T. Since T has at most two fixed points we may assume that all $E(\xi_1,\xi_1),E(\eta_1,\eta_1),E(\zeta_1,\zeta_1)$ are empty. Furthermore, we may assume that $E(\xi_1,\eta_1)\neq\emptyset$ since $f^{-1}(S_1)\neq\emptyset$. Then $T(\xi_1)=\eta_1$ and $T(\eta_1)=\xi_1$. Since T is a one-to-one mapping of $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$, $E(\xi_1,\zeta_1)=\emptyset, E(\eta_1,\zeta_1)=\emptyset$. However, this implies that ζ_1 is an exceptional value of f and g, which is a contradiction.

(ii) The case where $E(\zeta_3,\zeta_3)\neq\emptyset$ and $E(\zeta_4,\zeta_4)\neq\emptyset$. In this case ζ_3 and ζ_4 are fixed points of T, and it has no more fixed points, in particular, $E(\xi_j,\xi_j)=E(\eta_j,\eta_j)=E(\zeta_j,\zeta_j)=\emptyset$ for $j=1,2$. Since f and g have an exceptional value ζ_5, we may assume that they have no exceptional value in S_1. Hence we may assume that $E(\xi_1,\eta_1)\neq\emptyset$, and we can get a contradiction as in the case (i).

After all, we have $f=g$, and the proof is completed.

3. MEROMORPHIC FUNCTIONS SHARING TWO THREE-POINT SETS AND THREE VALUES IM

Let f and g be two meromorphic functions on $\mathbb{C}$ sharing $S_1,\cdots,S_5$ IM, where S_1,S_2 are three-point sets in $\mathbb{C}$ and S_3,S_4,S_5 are one-point sets in $\mathbb{C}$, and $S_1,\cdots,S_5$ are pairwise disjoint. Suppose that $f\not\equiv g$. Then, by using Theorem 2.3, we have $P_1(f)/P_2(f)=P_1(g)/P_2(g)$, where $P_j(z)=z^3+a_jz^2+b_jz+c_j$ is defining polynoimal of S_j for $j=1,2$, and hence

$$\begin{aligned}&(a_2-a_1)f^2g^2+(b_2-b_1)fg(f+g)+(c_2-c_1)(f^2+fg+g^2)\\&+(a_1b_2-a_2b_1)fg+(a_1c_2-a_2c_1)(f+g)+(b_1c_2-b_2c_1)=0.\end{aligned}\tag{7}$$

Let $S_j=\{\xi_j\}$ for $j=3,4,5$. Now, assume that

$$\begin{aligned}&(a_2-a_1){\xi_j}^4+2(b_2-b_1){\xi_j}^3+3(c_2-c_1){\xi_j}^2\\&+(a_1b_2-a_2b_1){\xi_j}^2+2(a_1c_2-a_2c_1)\xi_j+(b_1c_2-b_2c_1)\neq 0,\end{aligned}\tag{8}$$

for $j=4,5$. Then ξ_4,ξ_5 are exceptional values of f and g, and hence there exist entire functions α,β without zeros such that

$$\frac{f-\xi_4}{f-\xi_5}=\alpha,\quad \frac{g-\xi_4}{g-\xi_5}=\beta.$$

From these we get

$$f=\frac{\xi_4-\xi_5\alpha}{1-\alpha},\quad g=\frac{\xi_4-\xi_5\beta}{1-\beta}.$$

By substituting these into (7), we obtain

$$\sum_{0\le j,k\le 2}A_{jk}\alpha^j\beta^k=0,$$

where A_{jk} are constants and, in particular, $A_{00} = P(\xi_4, \xi_4) \neq 0$ and $A_{22} = P(\xi_5, \xi_5) \neq 0$. Here,

$$P(z,w) = (a_2 - a_1)z^2w^2 + (b_2 - b_1)zw(z+w) + (c_2 - c_1)(z^2 + zw + w^2) \\ + (a_1b_2 - a_2b_1)zw + (a_1c_2 - a_2c_1)(z+w) + (b_1c_2 - b_2c_1).$$

Since f and g are not constant, neither α nor β is constant. By applying Lemma 2.2 to this equation, we can induce that one of $\alpha\beta, \alpha^2\beta$ and $\alpha\beta^2$ is constant.

If $\alpha^2\beta$ is constant, then

$$\Big(\frac{f-\xi_4}{f-\xi_5}\Big)^2 = c_0\frac{g-\xi_5}{g-\xi_4},$$

where c_0 is a nonzero constant, and it follows from this that $T(r,g) = 2T(r,f) + S(r,f)$. On the other hand, by assumption, we have

$$\begin{aligned} 4T(r,g) &\leq \sum_{j=1,2}\{\overline{N}(r,1/(g-\xi_j)) + \overline{N}(r,1/(g-\eta_j)) + \overline{N}(r,1/(g-\zeta_j))\} + S(r,g) \\ &= \sum_{j=1,2}\{\overline{N}(r,1/(f-\xi_j)) + \overline{N}(r,1/(f-\eta_j)) + \overline{N}(r,1/(f-\zeta_j))\} + S(r,g) \\ &\leq 6T(r,f) + S(r,g) \end{aligned}$$

by the second fundamental theorem. However, these are not compatible. Hence $\alpha^2\beta$ is not constant, and similary neither is $\alpha\beta^2$.

Therefore $\alpha\beta$ is a constant, *i.e.,*

$$\frac{f-\xi_4}{f-\xi_5}\cdot\frac{g-\xi_4}{g-\xi_5} = c_0,$$

where c_0 is a nonzero constant. This induces that there is a relation $f = T(g)$, where T is a Möbius transformation interchanging ξ_4 and ξ_5 and it is of order 2.

Since f and g have two exceptional values ξ_4 and ξ_5, they have no more exceptional values, and hence ξ_3 is a fixed point of T. Since Möbius transformation T has at most one fixed point in $S_1 \cup S_2$, we may assume that it has no fixed point in S_1, and hence $E(\xi_1, \xi_1) = E(\eta_1, \eta_1) = E(\zeta_1, \zeta_1) = \emptyset$. If $E(\xi_1, \eta_1) \neq \emptyset$, then $\eta_1 = T(\xi_1)$. Since T is one-to-one and of order 2, we see that $E(\xi_1, \zeta_1) = E(\eta_1, \zeta_1) = \emptyset$. Therefore ζ_1 is an exceptional value of f and g, which is a contradiction. By the same way, we can get a contradiction in the case that $E(\xi_1, \zeta_1) \neq \emptyset$. However, these imply another contradction that ξ_1 is an exceptional value of f and g. Hence, we conclude that $f = g$.

Theorem 3.1. *Let $S_1, \cdots, S_5$ be pairwise disjoint sets in $\mathbb{C}$ with $\sharp S_1 = \sharp S_2 = 3$ and $\sharp S_3 = \sharp S_4 = \sharp S_5 = 1$. Let $P_j(z) = z^3 + a_jz^2 + b_jz + c_j$ be defining polynomial of S_j for $j = 1, 2$ and let $S_j = \{\xi_j\}$ for $j = 3, 4, 5$. If (8) holds for at least two j in $\{3, 4, 5\}$, then two nonconstant meromorphic functions on $\mathbb{C}$ sharing $S_1, \cdots, S_5$ IM are identical.*

4. MEROMORPHIC FUNCTIONS SHARING THREE THREE-POINT SETS AND TWO VALUES IM

Let f and g be two meromorphic functions on $\mathbb{C}$ sharing $S_1, \cdots, S_5$ IM, where S_1, S_2, S_3 are three-point sets in $\mathbb{C}$ and S_4, S_5 are one-point sets in $\mathbb{C}$, and $S_1, \cdots, S_5$ are pairwise disjoint. Suppose that $f \not\equiv g$. Then, by using Theorem 2.3, we may assume that $P_1(f)/P_2(f) = P_1(g)/P_2(g)$, where $P_j(z) = z^3 + a_j z^2 + b_j z + c_j$ is defining polynoimal of S_j for $j = 1, 2, 3$, and hence we have (7). Let $S_j = \{\xi_j\}$ for $j = 4, 5$. If we suppose that

$$(a_l - a_k){\xi_j}^4 + 2(b_l - b_k){\xi_j}^3 + 3(c_l - c_k){\xi_j}^2 + (a_k b_l - a_l b_k){\xi_j}^2 + 2(a_k c_l - a_l c_k)\xi_j + (b_k c_l - b_l c_k) \neq 0, \tag{9}$$

holds for $j = 4, 5$ and distinct k, l in $\{1, 2, 3\}$, then ξ_4, ξ_5 are exceptional values of f and g, and hence, as in the §3, we can obtain a relation $f = T(g)$, where T is a Möbius transformation interchanging ξ_4 and ξ_5 and it is of order 2.

Since T has at most two fixed points, we may assume that there is no fixed point in S_1. Again, by the same way in §3, we can get a contradiction, and we obatain the following theorem:

Theorem 4.1. *Let $S_1, \cdots, S_5$ be pairwise disjoint sets in $\mathbb{C}$ with $\sharp S_1 = \sharp S_2 = \sharp S_3 = 3$ and $\sharp S_4 = \sharp S_5 = 1$. Let $P_j(z) = z^3 + a_j z^2 + b_j z + c_j$ be defining polynomial of S_j for $j = 1, 2, 3$ and let $S_j = \{\xi_j\}$ for $j = 4, 5$. Assume that (9) holds for distinct k, l in $\{1, 2, 3\}$ and $j = 4, 5$. Then, two nonconstant meromorphic functions on $\mathbb{C}$ sharing $S_1, \cdots, S_5$ IM are identical.*

5. MEROMORPHIC FUNCTIONS SHARING FOUR THREE-POINT SETS AND A VALUE IM

Let f and g be two meromorphic functions on $\mathbb{C}$ sharing $S_1, \cdots, S_5$ IM, where S_1, S_2, S_3, S_4 are three-point sets in $\mathbb{C}$ and S_5 is a one-point set in $\mathbb{C}$, and $S_1, \cdots, S_5$ are pairwise disjoint. Suppose that $f \not\equiv g$. Let $P_j(z) = z^3 + a_j z^2 + b_j z + c_j$ be defining polynomial of S_j for $j = 1, 2, 3, 4$. Then, by using Theorem 2.3, we may assume that $P_1(f)/P_2(f) = P_1(g)/P_2(g)$, , and hence we have (7). Let $S_5 = \{\xi_5\}$. If we suppose that

$$(a_k - a_j){\xi_5}^4 + 2(b_k - b_j){\xi_5}^3 + 3(c_k - c_j){\xi_5}^2 + (a_j b_k - a_k b_j){\xi_5}^2 + 2(a_j c_k - a_k c_j)\xi_5 + (b_j c_k - b_k c_j) \neq 0, \tag{10}$$

for distinct j, k in $\{1, 2, 3, 4\}$, then ξ_5 is an exceptional values of f and g.

Let $S_j = \{\xi_j, \eta_j, \zeta_j\}$ for $j = 3, 4$. We separate S_3, S_4 into two types: (A) some of $E(\xi_j, \eta_j), E(\xi_j, \zeta_j), E(\eta_j, \zeta_j)$ are not empty; (B) all of $E(\xi_j, \eta_j), E(\xi_j, \zeta_j)$, $E(\eta_j, \zeta_j)$ are empty. Then we consider three cases: (I) both of S_3 and S_4 are of type (B); (II) one of S_3 and S_4 is of type (B) and the other is of type (A); (III) both of S_3 and S_4 are of type (A).

The case (I). Both of S_3 and S_4 are of type (B). Then f and g share ξ_j, η_j, ζ_j IM for $j = 3, 4$. By Theorem 1.2, we get $f = g$.

The case (II). One of S_3 and S_4 is of type (B) and the other is of type (A). We may assume that S_3 is of type (B) and S_4 is of type (A). Moreover, we may assume that $E(\xi_4, \eta_4) \neq \emptyset$. Then, we have $P_1(\xi_4)/P_2(\xi_4) = P_1(\eta_4)/P_2(\eta_4)$, and by assumption and Lemma 2.1 we have $E(\xi_4, \zeta_4) = E(\eta_4, \zeta_4) = \emptyset$, which implies that f and g share $\{\xi_4, \eta_4\}$ and $\{\zeta_4\}$ IM. Since f and g share also ξ_3, η_3, ζ_3 and ξ_5 IM, by using Theorem 1.2, we get $f = g$.

The case (III). Both of S_3 and S_4 are of type (A). We may assume that $E(\xi_j, \eta_j) \neq \emptyset$ for $j = 3, 4$. Then, we have $P_1(\xi_j)/P_2(\xi_j) = P_1(\eta_j)/P_2(\eta_j)$, and by assumption and Lemma 2.1 we have $E(\xi_j, \zeta_j) = E(\eta_j, \zeta_j) = \emptyset$, which implies that f and g share $\{\xi_j, \eta_j\}$ and $\{\zeta_j\}$ IM for $j = 3, 4$. By using Theorem 1.3, there exists a Möbius transformation T such that $f = T \circ g$.

Assume that $f \neq g$, and hence T is not the identity. Then T has at most two fixed points, and hence, we may assume that $E(\xi_5, \xi_5) = \emptyset$. In this case, ξ_5 is an exceptional value of f and g. Furthermore, T interchanges ξ_j and η_j for at least one of $j \in \{3, 4\}$, which are also fixed points of $T^2 = T \circ T$. Since the fixed points of T are also those of T^2, T^2 has at least three fixed points. Therefore, T^2 is the identity, as mentioned in Remark of Theorem 1.3.

(i) The case where $E(\zeta_4, \zeta_4) = \emptyset$ or $E(\zeta_3, \zeta_3) = \emptyset$. It is enough to consider the case where $E(\zeta_4, \zeta_4) = \emptyset$. In this case ζ_4 and ξ_5 are exceptional values of f and g and they have no more exceptional values. Therefore $E(\zeta_3, \zeta_3) \neq \emptyset$, and hence ζ_3 is a fixed point T. Since T has at most two fixed points, we may assume that all $E(\xi_1, \xi_1), E(\eta_1, \eta_1), E(\zeta_1, \zeta_1)$ are empty. Furthermore, we may assume that $E(\xi_1, \eta_1) \neq \emptyset$ since $f^{-1}(S_1) \neq \emptyset$. Then $T(\xi_1) = \eta_1$ and $T(\eta_1) = \xi_1$. Since T is a one-to-one mapping of $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$, $E(\xi_1, \zeta_1) = \emptyset, E(\eta_1, \zeta_1) = \emptyset$. However, this implies that ζ_1 is an exceptional value of f and g, which is a contradiction.

(ii) The case where $E(\zeta_3, \zeta_3) \neq \emptyset$ and $E(\zeta_4, \zeta_4) \neq \emptyset$. In this case ζ_3 and ζ_4 are fixed points of T, and it has no more fixed points, in particular, $E(\xi_j, \xi_j) = E(\eta_j, \eta_j) = E(\zeta_j, \zeta_j) = \emptyset$ for $j = 1, 2$. Since f and g have an exceptional value ξ_5, we may assume that they have no exceptional value in S_1. Hence we may assume that $E(\xi_1, \eta_1) \neq \emptyset$, and we can get a contradiction as in the case (i).

After all, we have $f = g$.

Theorem 5.1. *Let $S_1, \cdots, S_5$ be pairwise disjoint sets in $\mathbb{C}$ with $\sharp S_1 = \sharp S_2 = \sharp S_3 = \sharp S_4 = 3$ and $\sharp S_5 = 1$. Let P_j be defining polynomial of S_j for $j = 1, 2, 3, 4$ and let $S_5 = \{\xi_5\}$. Assume that P_1, P_2, P_3, P_4 are linearly independent over $\mathbb{C}$, and (10) holds for distinct for distinct $1 \leq j, k \leq 4$. Then, two nonconstant meromorphic functions on $\mathbb{C}$ sharing $S_1, \cdots, S_5$ IM are identical.*

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