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COMPLEMENTARY RAMSEY NUMBERS AND RAMSEY GRAPHS

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Abstract. In this paper, we consider a variant of Ramsey numbers which we call complementary Ramsey numbers $\bar{R}(m,t,s)$. We first establish their connections to pairs of Ramsey (s,t)-graphs. Using the classification of Ramsey (s,t)-graphs for small s,t, we determine the complementary Ramsey numbers $\bar{R}(m,t,s)$ for $(s,t) = (4,4)$ and $(3,6)$.

Key words and Phrases: Ramsey number, Ramsey graph

Abstrak. Dalam paper ini dikaji suatu varian dari bilangan Ramsey $\bar{R}(m,t,s)$. Pertama, dikaji hubungan antara varian ini dengan *Ramsey* (s,t)*-graph* untuk s,t yang nilainya kecil. Kemudian, dengan menggunakan klasifikasi dari *Ramsey* (s,t)*-graph* untuk s,t yang nilainya kecil, ditentukan bilangan Ramsey $\bar{R}(m,t,s)$ komplementer untuk $(s,t) = (4,4)$ dan $(s,t) = (3,6)$.

Kata kunci: Bilangan Ramsey, Graf Ramsey

1. Introduction

For any given positive integers $n_1,\dots,n_c$, there is an integer, $\bar{R}$, such that if the edges of a complete graph of order at least $\bar{R}$ are colored with c different colors, then for some i between 1 and c, there exists a complete subgraph of order n_i all of whose edges have colors different from i. The smallest such integer $\bar{R}$ is denoted by $\bar{R}(n_1,\dots,n_c)$. Note that $\bar{R}(n_1,n_2) = R(n_2,n_1)$, an ordinary Ramsey number. These numbers can be traced back (at least) to a paper of Erdős, Hajnal and Rado [5]. Erdős and Szemeredi [6] proved that the diagonal complementary Ramsey number $\bar{R}(n,\dots,n)$ (with c colors), is at most $c^{Cn/c}$, where C is some constant.

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m	3	4	5	6	7	8	9–13	14–
$\bar{R}(m,3,3)$	5	5	5	6	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\bar{R}(m,4,3)$	-	7	8	8	9	$\cdots$	$\cdots$	$\cdots$
$\bar{R}(m,5,3)$	-	-	9	11	12	12	13	14

TABLE 1. $\bar{R}(m, m_2, 3)$ for $m_2 = 3, 4, 5$

m	4	5–6	7	8–10	11–16	17	18–
$\bar{R}(m,4,4)$	10	13	14	15	16	17	18

TABLE 2. $\bar{R}(m, 4, 4)$

m	6	7	8	9–15	16–
$\bar{R}(m,6,3)$	13	14	16	17	18

TABLE 3. $\bar{R}(m, 6, 3)$

Related concepts were considered later. Chung and Liu [4] considered more general Ramsey numbers, and the complementary Ramsey number $\bar{R}(m_1, \dots, m_t)$ is denoted by $R^t_{t-1}(K_{m_1}, \dots, K_{m_t})$ in their notation. Harborth and Möller [8] also considered what they call weakened Ramsey numbers $R_{s,t}(G)$. The relationship is

$$R_{t-1,t}(K_m) = \bar{R}(\underbrace{m, \dots, m}_{t}).$$

Xu, Shao, Su, and Li [12] considered multigraph Ramsey numbers, which are also regarded as more general than complementary Ramsey numbers. In fact, the complementary Ramsey number $\bar{R}(m_1, \dots, m_t)$ is denoted by $f^{(t-1)}(m_1, \dots, m_t)$ in their notation.

For some small values of m_1, m_2, m_3, the complementary Ramsey numbers $R(m_1, m_2, m_3)$ have been determined. We summarize these results in Table 1. Note that we may assume without loss of generality, $m_1 \geq m_2 \geq m_3$. The numbers $R(m, 3, 3)$, $R(m, 4, 3)$, and $R(m, 5, 3)$ have been determined by Theorems 3.3, 3.4, and 3.5, respectively, in [4]. Moreover, $\bar{R}(4, 4, 4) = 10$ by [4, Theorem 3.6].

The purpose of this paper is to determine $\bar{R}(m, 4, 4)$ and $\bar{R}(m, 6, 3)$ for all positive integers m, using the classification of Ramsey $(4, 4)$-graphs and Ramsey $(3, 6)$-graphs, respectively, available from [10]. Our results are tabulated in Tables 2 and 3. The computation needed to verify the entries of these tables was done with the help of Magma [2].

This paper is organized as follows. In Sect. 2, we give definitions and derive immediate consequences. In Sect. 3, we prove some inequalities needed to derive an upper bound for $\bar{R}(5, 5, 5)$. In Sect. 4, we show how to determine the complementary Ramsey number $\bar{R}(m, t, s)$ from the knowledge of Ramsey (s, t)-graphs. In Sect. 5, we determine $\bar{R}(m,4,4)$ and in Sect. 6, we determine $\bar{R}(m,6,3)$, for all positive integers m. We end the article with concluding remarks as Sect. 7.

2. Definitions and notation

For a positive integer n, we denote the set $\{1,\dots,n\}$ by $[n]$, and the set of all n-element subsets of a set X by $\binom{X}{n}$. For positive integers k and n, we denote the set of all edge-coloring of the complete graph K_n by k colors, by $C(n,k)$:

$$C(n,k) = \{f \mid f : \binom{[n]}{2} \to [k]\}.$$

If G is a graph, then we denote by $\alpha(G)$ the independence number of G, and by $\omega(G)$ the clique number of G. We identify a graph whose vertex set is $[n]$, with its set of edges. In particular, for $f \in C(n,k)$ and $i \in [k]$, $f^{-1}(i)$ is regarded as the graph $([n], f^{-1}(i))$. We use the abbreviations $\alpha_i(f) = \alpha(f^{-1}(i))$, $\omega_i(f) = \omega(f^{-1}(i))$. The Ramsey number can be defined as follows:

$$R(m_1,\dots,m_k) = \min\{n \in \mathbb{N} \mid \forall f \in C(n,k),\ \exists i \in [k],\ \omega_i(f) \ge m_i\},$$

where $m_1,\dots,m_k$ are positive integers. The complementary Ramsey number is defined by replacing ω by α in the above definition of the Ramsey number:

$$\bar{R}(m_1,\dots,m_k) = \min\{n \in \mathbb{N} \mid \forall f \in C(n,k),\ \exists i \in [k],\ \alpha_i(f) \ge m_i\}.$$

The following properties are immediate from the definition. In the following, $m_1, m_2, \dots,$ denote positive integers.

$$\begin{aligned} R(m_1,m_2) &= \bar{R}(m_2,m_1), \\ \bar{R}(m_1,\dots,m_k) &= \bar{R}(m_{\sigma(1)},\dots,m_{\sigma(k)}) \qquad (1) \\ &\qquad \text{for any permutation } \sigma \text{ on } [k], \\ \bar{R}(m_1,\dots,m_k,2) &= \min\{m_1,\dots,m_k\}. \qquad (2) \end{aligned}$$

When considering $\bar{R}(m_1,m_2,m_3)$, we may assume $m_1 \ge m_2 \ge m_3 \ge 3$ without loss of generality, by (1) and (2). We shall use an upper bound given in [4, Theorem 2.1]:

$$\bar{R}(m_1,m_2,m_3) \le R(m_2,m_3). \qquad (3)$$

3. Some inequalities

In this section, we establish some inequalities among complementary Ramsey numbers. These inequalities will not be needed for our main results, but it will be used to derive bounds for $\bar{R}(5,5,5)$ in the final section.

Let n and k be positive integers. For $f \in C(n,k)$ and $x \in [n]$, set

$$f_i(x) = |\{y \in [n] \mid f(\{x,y\}) = i\}|.$$

Lemma 3.1. *Let $n, k, m_1, \dots, m_k$ and t be positive integers with $1 \le t \le k$, and let $f \in C(n,k)$. If*

$$\sum_{j=1}^{t} f_j(x) \ge \bar{R}(m_1, m_2, \dots, m_t, m_{t+1}-1, \dots, m_k - 1)$$

for some $x \in [n]$*, then* $\alpha_i(f) \geq m_i$ *for some* $i \in [k]$.

Proof. Set $Y = \{y \in [n] \mid f(\{x, y\}) \in [t]\}$, so that

$$|Y| = \sum_{j=1}^{t} f_j(x).$$

Let $g = f|_{\binom{Y}{2}}$. By the assumption, either there exists i with $1 \leq i \leq t$ such that $\alpha_i(g) \geq m_i$, or there exists i with $t < i \leq k$ such that $\alpha_i(g) \geq m_i - 1$.

If $1 \leq i \leq t$, then $\alpha_i(f) \geq \alpha_i(g) \geq m_i$. If $t < i \leq k$, then there exists an independent set Z in $g^{-1}(i)$ with $|Z| = m_i - 1$. Then $Z \cup \{x\}$ is an independent set in $f^{-1}(i)$. This implies $\alpha_i(f) \geq m_i$. □

Lemma 3.2. *Let* $k, m_1, \ldots, m_k$ *and* t *be positive integers. Let* $[k] = \bigcup_{j=1}^{t} M_j$ *be a nontrivial partition. For each* $i \in [k]$ *and* $j \in [t]$*, define*

$$m_i^{(j)} = \begin{cases} m_i & \text{if } i \in M_j, \\ m_i - 1 & \text{otherwise.} \end{cases}$$

Then

$$\bar{R}(m_1, \ldots, m_k) \leq \sum_{j=1}^{t} \bar{R}(m_1^{(j)}, \ldots, m_k^{(j)}) - t + 2.$$

In particular,

$$\bar{R}(m_1, m_2, m_3) \leq \bar{R}(m_1, m_2, m_3 - 1) + \bar{R}(m_1 - 1, m_2 - 1, m_3). \tag{4}$$

Proof. Let n denote the right-hand side of the inequality. If $f \in C(n, k)$ and $x \in [n]$, then

$$\begin{aligned} \sum_{j=1}^{t} \sum_{i \in M_j} f_i(x) &= \sum_{i=1}^{k} f_i(x) \\ &= n - 1 \\ &= \sum_{j=1}^{t} \bar{R}(m_1^{(j)}, \ldots, m_k^{(j)}) - t + 1. \\ &> \sum_{j=1}^{t} (\bar{R}(m_1^{(j)}, \ldots, m_k^{(j)}) - 1). \end{aligned}$$

Thus, there exists $j \in [t]$ such that

$$\sum_{i \in M_j} f_i(x) \geq \bar{R}(m_1^{(j)}, \ldots, m_k^{(j)}).$$

By Lemma 3.1, there exists $i \in [k]$ such that $\alpha_i(f) \geq m_i$. This implies $\bar{R}(m_1, \ldots, m_k) \leq n$. □

4. Ramsey (s,t)-graphs

A graph G is said to be a Ramsey (s,t)-graph if $\omega(G) < s$ and $\alpha(G) < t$. We denote by $\mathcal{R}_n(s,t)$ the set of Ramsey (s,t)-graphs on the vertex-set $[n]$. Database of Ramsey graphs can be found in [10].

We write $G \supseteq H$ if H is a subgraph of G. Recall that, since we identify a graph with its set of edges, this means that G and H have the same set of vertices, and the set of edges of H is a subset of that of G. For a graph G and its subgraph H, we denote by $G - H$ whose edge set consists of edges of G which are not an edge of H.

Lemma 4.1. *Let m_1, m_2, m_3 be positive integers greater than 2. For a positive integer n, define*

$$a_n(m_3, m_2) = \min\{\alpha(G - H) \mid G, H \in \mathcal{R}_n(m_3, m_2),\ G \supseteq H\}.$$

Then

$$\bar{R}(m_1, m_2, m_3) = 1 + \max\{n \in \mathbb{N} \mid a_n(m_3, m_2) < m_1\}.$$

Proof. Define

$$\mathcal{F}_n = \{f \mid f : \binom{[n]}{2} \to [3],\ \alpha(f^{-1}(i)) < m_i\ (\forall i \in [3])\},$$
$$\mathcal{G}_n = \{(G, H) \mid G, H \in \mathcal{R}_n(m_3, m_2),\ G \supseteq H,\ \alpha(G - H) < m_1\}.$$

Then there is a bijection $\Phi : \mathcal{F}_n \to \mathcal{G}_n$ defined by

$$\Phi : f \mapsto (f^{-1}(\{1,2\}), f^{-1}(2)) \quad (f \in \mathcal{F}_n).$$

Indeed, for $f \in \mathcal{F}_n$, write $G = f^{-1}(\{1,2\})$, $H = f^{-1}(2)$. Then

$$\begin{aligned} m_1 &> \alpha(f^{-1}(1)) = \alpha(G - H), \\ m_2 &> \alpha(f^{-1}(2)) = \alpha(H), \\ m_3 &> \alpha(f^{-1}(3)) = \omega(G). \end{aligned}$$

Since $G \supseteq H$, we have $\omega(H) \le \omega(G)$ and $\alpha(G) \le \alpha(H)$. Thus, $G, H \in \mathcal{R}_n(m_3, m_2)$, and Φ is well defined. It is clear that Φ is a bijection. Now

$$\begin{aligned} \bar{R}(m_1, m_2, m_3) - 1 &= \max\{n \in \mathbb{N} \mid \mathcal{F}_n \ne \emptyset\} \\ &= \max\{n \in \mathbb{N} \mid \mathcal{G}_n \ne \emptyset\} \\ &= \max\{n \in \mathbb{N} \mid a_n(m_3, m_2) < m_1\}. \end{aligned}$$

$\square$

5. The complementary Ramsey numbers $\bar{R}(m,4,4)$

Greenwood and Gleason [7] proved $R(4,4) = 18$, which implies $\bar{R}(m,4,4) \le 18$ for all $m \in \mathbb{N}$ by (3). The list of all Ramsey $(4,4)$-graphs can be found in [10]. For example, $\mathcal{R}_{17}(4,4)$ consists of a single graph, while $\mathcal{R}_{16}(4,4)$ consists of two graphs with the same size. Thus $a_n(4,4) = n$ for $n = 16, 17$. For $9 \le n \le 15$, we can use computer to determine $a_n(4,4)$. Let us briefly describe how to perform this computation. Observe that

$$a_n(4,4) = \min\{\alpha(G-H) \mid G, H \in \mathcal{R}_n(4,4),\ G \supseteq H,\\ G \text{ is maximal in } \mathcal{R}_n(4,4)\}.$$

Note that, by the same reason, we may further assume that H is a minimal Ramsey $(4,4)$-graph on n vertices in the above definition of $a_n(4,4)$. However, enumerating all containment relations between maximal and minimal Ramsey $(4,4)$-graphs on n vertices could be difficult. So we use a different approach. Fix a maximal Ramsey $(4,4)$-graph G on n vertices, and we try to find a subgraph H of G such that $H \in \mathcal{R}_n(4,4)$ and $\alpha(G-H) \le m$. This will guarantee $a_n(4,4) \le m$.

If an edge $e \in G$ satisfies $\alpha(G-e) \ge 4$, then we must have $e \in H$. Indeed, if $e \notin H$, then $G - e \supseteq H$, so $\alpha(G-e) \le \alpha(H) \le 3$. This is a contradiction.

If another edge $e' \in G$ satisfies $\alpha(G-e-e') > m$, then we must have $e' \notin H$. Indeed, if $e' \in H$, then $G-e-e' \supseteq G-H$, so $\alpha(G-e-e') \le \alpha(G-H) \le m$. This is a contradiction.

These criteria reduces the number of edges e'' for which we need to determine whether $e'' \in H$ or $e'' \notin H$. For such edges, we need to consider both possibilities, and eventually, either we have a desired subgraph H, or the conclusion that G has no such subgraph H.

Even though the number of Ramsey $(4,4)$-graphs is quite large, the number of maximal Ramsey $(4,4)$-graphs of a given number of vertices is small. We can quickly perform the above process for each of maximal Ramsey $(4,4)$-graphs to decide the truth of the inequality $a_n(4,4) \le m$. This leads to the determination of the values $a_n(4,4)$ for all $n \in \{9, 10, \ldots, 15\}$. Our results are given in Table 4.

n	9	10–12	13	14	15	16	17	18–
$a_n(4,4)$	3	4	6	7	10	16	17	∞

Table 4. $a_n(4,4)$

Theorem 5.1. *We have*

$$\bar{R}(m,4,4) = \begin{cases} 10 & \text{if } m = 4, \\ 13 & \text{if } m = 5, 6, \\ 14 & \text{if } m = 7, \\ 15 & \text{if } 8 \le m \le 10, \\ 16 & \text{if } 11 \le m \le 16, \\ 17 & \text{if } m = 17, \\ 18 & \text{if } m \ge 18. \end{cases}$$

Proof. Immediate from Table 4 and Lemma 4.1. □

6. The complementary Ramsey numbers $\bar{R}(m,6,3)$

Kéry [9] proved $R(3,6) = 18$ (see also [3]), which implies $\bar{R}(m,6,3) \le 18$ for all $m \in \mathbb{N}$ by (3). The list of all Ramsey $(3,6)$-graphs can be found in [10]. For $12 \le n \le 17$, we can use computer to determine $a_n(3,6)$, exactly in the same manner as described in Sect. 5. Our results are given in Table 5.

n	12	13	14–15	16	17	18–
$a_n(3,6)$	5	6	7	8	15	∞

Table 5. $a_n(3,6)$

Theorem 6.1. *We have*

$$\bar{R}(m,6,3) = \begin{cases} 13 & \text{if } m = 6, \\ 14 & \text{if } m = 7, \\ 16 & \text{if } m = 8, \\ 17 & \text{if } 9 \le m \le 15, \\ 18 & \text{if } m \ge 16. \end{cases}$$

Proof. Immediate from Table 5 and Lemma 4.1. □

7. Concluding remarks

If we could show $\bar{R}(5,5,5) = 12$, then this would have contributed to the proof of a conjecture of Einhorn–Schoenberg [11]. This was our motivation to study complementary Ramsey numbers. However, Theorem 2.6 in [4] already gives $\bar{R}(5,5,5) \ge 17$, which is improved in [12, Table 2] as $\bar{R}(5,5,5) \ge 20$.

As for an upper bound, observe first, by (4) and Tables 1, 2, we have

$$\begin{aligned} \bar{R}(5,5,4) &\le \bar{R}(5,5,3) + \bar{R}(4,4,4) \\ &= 19. \end{aligned}$$

Thus, again by (4) and Table 2, we have

$$\begin{aligned} \bar{R}(5,5,5) &\le \bar{R}(5,4,4) + \bar{R}(4,5,5) \\ &\le 32. \end{aligned}$$

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