ISOMORPHISM BETWEEN ENDOMORPHISM RINGS OF MODULES OVER A SEMI SIMPLE RING

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Abstract. Our question is what ring R which all modules over R are determined, up to isomorphism, by their endomorphism rings? Examples of this ring are division ring and simple Artinian ring. Any semi simple ring does not satisfy this property. We construct a semi simple ring R but R is not a simple Artinian ring which all modules over R are determined, up to isomorphism, by their endomorphism rings.

Key words and Phrases: Modules, division ring, simple Artinian ring, semi simple ring

1. Introduction

Let M, N be modules over a ring R and set of all R-homomorphisms from M to N is written $\mathrm{Hom}_R(M, N)$. Then $\mathrm{Hom}_R(M, N)$ is Abelian group over addition of mapping. Moreover, $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$ is a ring over addition and composition of mapping, and called endomorphism ring of M. In general if two modules are isomorphic then their endomorphism rings are isomorphic. The converse is not true. The Baer-Kaplansky theorem states that two Abelian torsion groups are isomorphic if and only if their endomorphism rings are isomorphic (see [2] and [4]). Ivanov got same result with the Baer-Kaplansky theorem. Let R be the upper triangular ring over a division ring and $\mathcal{F}$ be the category of R-modules which have a summand isomorphic to R. Then modules in $\mathcal{F}$ are determined, up isomorphism, by their endomorphism rings (see [3]). Our question is what ring

which modules in R-MOD are determined, up to isomorphism, by their endomorphism rings? Examples of this ring are division ring and simple Artinian ring. Any semi simple ring does not satisfy this property. We construct a semi simple ring R but R is not a simple Artinian ring which modules in R-MOD are determined, up to isomorphism, by their endomorphism rings.

In this paper a ring will be a ring with unity and all modules will be nonzero unital right module except if in special case. Definition of ring, module, and others which are used in this paper refer to [1].

2. IP-isomorphism

In Baer-Kaplansky theorem and Ivanov result, isomorphism between endomorphism rings of two modules always "preserve indecomposable direct summand", which called an *IP-isomorphism*. For a class of modules which have a decomposition $M = \oplus_{i \in \mathcal{I}} M_i$ with property that every indecomposable direct summand of M is contained in the sum of finite number of the M_i (we say that M has *the finite embedding property*), an IP-isomorphism between endomorphism rings of modules give an isomorphism between the modules.

An element e in a ring R is called *idempotent* if $e^2 = e$. Two idempotents e_1 and e_2 in R are called *orthogonal* if $e_1 e_2 = 0 = e_2 e_1$. An idempotent $e \neq 0$ in R is called *primitive idempotent* if e can not be a sum of two nonzero orthogonal idempotent.

Let M be a module over a ring R and e an idempotent in $End_R(M)$. Then $1 - e$ is also idempotent in $\mathrm{End}_R(M)$. Moreover, e and $1 - e$ orthogonal and M has decomposition $M = eM \oplus (1 - e)M$. Then the direct summand eM is indecomposable if and only if e is a primitive idempotent in $\mathrm{End}_R(M)$ ([1], Lemma 5.11).

Let given M and N be modules over a ring R, and $\varphi : M \longrightarrow N$ any R-isomorphism. The mapping α which defined

$$\alpha : \mathrm{End}_R(M) \longrightarrow End_R(N), \ \alpha(\psi) = \varphi \psi \varphi^{-1}$$

is a ring isomorphism. Furthermore, for any idempotent e in $\mathrm{End}_R(M)$ satisfies

$$\alpha(e)N = \varphi e \varphi^{-1} N = (\varphi e)(\varphi^{-1} N) = (\varphi e)M = \varphi(eM).$$

Therefore

$$\varphi|_{eM} : eM \longrightarrow \alpha(e)N$$

is a R-isomorphism. So $eM \cong \alpha(e)N$, for all e idempotent in $\mathrm{End}_R(M)$.

In general, if $\beta : \mathrm{End}_R(M) \longrightarrow \mathrm{End}_R(N)$ is any ring isomorphism and e is any primitive idempotent in $\mathrm{End}_R(M)$ then $\beta(e)$ is also a primitive idempotent in $\mathrm{End}_R(N)$ but it must not $eM \cong \beta(e)N$.

Let M and N be modules over a ring R. A ring isomorphism

$$\phi : \mathrm{End}_R(M) \longrightarrow \mathrm{End}_R(N)$$

is called *IP-isomorphism* if $\phi(e)N \cong eM$, for all primitive idempotents e in $\mathrm{End}_R(M)$. The following proposition gives property that an IP-isomorphism between endomorphism rings will give module isomorphism.

Proposition 2.1. *Let M and N be modules over a ring R where M has the finite imbedding property with respect to a decomposition into indecomposable direct summands and N be generated by indecomposable direct summands. Then M and N are isomorphic if and only if there is an IP-isomorphism between $\mathrm{End}_R(M)$ and $\mathrm{End}_R(N)$.*

PROOF. See [3] Proposition 1.

3. MAIN RESULTS

Let $\mathbb{Z}_p$ and $\mathbb{Z}_q$ be fields of integer number modulo p and q, respectively, where p and q are different prime numbers. Let

$$R = \begin{bmatrix} \mathbb{Z}_p & 0 & 0 \\ 0 & \mathbb{Z}_q & \mathbb{Z}_q \\ 0 & \mathbb{Z}_q & \mathbb{Z}_q \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{bmatrix} \ \middle| \ a \in \mathbb{Z}_p, \quad b,c,d,e \in \mathbb{Z}_q \right\} \quad (1)$$

Then R is a semi simple ring R but R is not a simple Artinian ring. As a left modul over itself, R has decomposition over simples modules as follows.

$$R \cong \begin{bmatrix} \mathbb{Z}_p \\ 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathbb{Z}_q \\ \mathbb{Z}_q \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathbb{Z}_q \\ \mathbb{Z}_q \end{bmatrix} = \begin{bmatrix} \mathbb{Z}_p \\ 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathbb{Z}_q \\ \mathbb{Z}_q \end{bmatrix}^{(2)}. \quad (2)$$

Proposition 3.1. *Let M and N be modules over the ring R in (1). Then every ring isomorphism between $\mathrm{End}_R(M)$ and $\mathrm{End}_R(N)$ is an IP-isomorphism.*

PROOF. Let $\phi : \mathrm{End}_R(M) \to \mathrm{End}_R(N)$ be any ring isomorphism and e be any primitive idempotent in $\mathrm{End}_R(M)$. We will be shown that $eM \cong \phi(e)N$. Because e primitive idempotent in $\mathrm{End}_R(M)$ then

$$eM \cong \begin{bmatrix} \mathbb{Z}_p \\ 0 \\ 0 \end{bmatrix} \text{ or } eM \cong \begin{bmatrix} 0 \\ \mathbb{Z}_q \\ \mathbb{Z}_q \end{bmatrix}$$

and $\phi(e)$ is idempotent primitive in $\mathrm{End}_R(N)$. Suppose that $eM \cong \begin{bmatrix} \mathbb{Z}_p \\ 0 \\ 0 \end{bmatrix}$ then

$$\mathbb{Z}_p \cong \mathrm{End}_R(eM) \cong e\mathrm{End}_R(M)e \cong \phi(e)\mathrm{End}_R(N)\phi(e) \cong \mathrm{End}_R(\phi(e)N).$$

Suppose that $\phi(e)N \ncong \begin{bmatrix} \mathbb{Z}_p \\ 0 \\ 0 \end{bmatrix}$ then $\phi(e)N \cong \begin{bmatrix} 0 \\ \mathbb{Z}_q \\ \mathbb{Z}_q \end{bmatrix}$. So that $\operatorname{End}_R(\phi(e)N) \cong$

$\mathbb{Z}_q \ncong \mathbb{Z}_p$. So $\phi(e)N \cong \begin{bmatrix} \mathbb{Z}_p \\ 0 \\ 0 \end{bmatrix} \cong eM$. In the case of $eM \cong \begin{bmatrix} 0 \\ \mathbb{Z}_q \\ \mathbb{Z}_q \end{bmatrix}$, with proof as before, we have the same results. So ϕ is IP-isomorphism.

Corollary 3.2. *Let M and N be modules over the ring R in (1). Then $M \cong N$ if and only if $\operatorname{End}_R(M) \cong \operatorname{End}_R(N)$.*

PROOF. Because ring isomorphism between $\operatorname{End}_R(M)$ and $\operatorname{End}_R(N)$ always in the form of IP-isomorphism and decomposition of semi simple modules fulfills the insertion properties so that according to the Proposition 2.1 obtained $M \cong N$.

As a generalization of the results above, the following propositions are obtained.

Proposition 3.3. *Let R be a semi simple ring and has decomposition*

$$R \cong I_1^{(n_1)} \oplus \cdots \oplus I_s^{(n_s)} \tag{3}$$

where I_k is a simple ideal, $I_k \cong I_l$ if and only if $k = l$, and

$$R \cong \mathbb{M}_{n_1}(D_1) \oplus \cdots \oplus \mathbb{M}_{n_s}(D_s) \tag{4}$$

where D_k is a division ring. Then the following properties are equivalent.

  (a) Every ring isomorphism $\phi : \operatorname{End}_R(M) \longrightarrow \operatorname{End}_R(N)$, where M and N are modules over R, is a IP-isomorphism.
  (b) $\operatorname{End}_R(I_k) \ncong \operatorname{End}_R(I_l)$, $k \neq l$.
  (c) $D_k \ncong D_l$, $k \neq l$.

Corollary 3.4. *Let R be a semi simple ring which has decomposition (3) and $\operatorname{End}_R(I_k) \ncong \operatorname{End}_R(I_l)$, $k \neq l$. Then for M and N modules over R, we have $M \cong N$ if and only if $\operatorname{End}_R(M) \cong \operatorname{End}_R(N)$.*

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