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A NOTE ON STATISTICAL LIMIT AND CLUSTER POINTS OF THE ARITHMETICAL FUNCTIONS $(a_p(n))$, $(\gamma(n))$ and $(\tau(n))$

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Abstract. In this paper, the set of statistical limit and cluster points of the arithmetical functions $(a_p(n)), (\gamma(n))$ and $(\tau(n))$ are studied by using natural density of subsets of natural numbers $\mathbb{N}$. In addition to this, statistical limit and cluster points of $r-th$ difference functions $(\Delta^r\gamma(n))$ and $(\Delta^r\tau(n))$ for each fixed $r \in \mathbb{N}$ are also investigated.

Key words and Phrases: Limit point, Statistical cluster and limit point, Statistical convergence, Arithmetical functions.

1. INTRODUCTION

In [3] and [12], Fast and Steinhaus introduced the concept of statistical convergence independently as a generalization of classical convergence. Prior to these studies in [14], Zygmund gave a name almost convergence to this concept and established a relation between statistical convergence and strong summability. In the last two decades, this concept was studied as a nonmatrix summability method by various authors including Connor [2], Fridy [5], Fridy-Miller [6], Schoenberg [11] and many others. Especially, Fridy in [7] viewed the statistical convergence as a sequential limit concept and by extending this concept in a natural way, he defined the statistical analogue of the set of limit points and cluster points of a number sequence. Both concepts are just the natural extension of the ordinary limit points but in the sense of statistical convergence.

An arithmetic function is a function defined on natural numbers which takes values in the real or complex numbers. Although aritmetical functions are actually a sequence, they have some applications in the theory of analytic numbers due to

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their unique properties. A few of the best known arithmetic functions are Euler Phi function $\phi(n)$, number of divisors function $\beta(n)$, the sum of divisors function $\sigma(n)$, etc. Some papers which are directly related this topic [1], [4], [8] and [10], etc.

In this paper, we are going to study the statistical limit and statistical cluster points of the arithmetic functions; $a_p(n)$, $\gamma(n)$ and $\tau(n)$.

Let K be a subset of natural numbers $\mathbb{N}$. For any $n \in \mathbb{N}$, let $K_n := \{k \leq n : k \in K\}$ and $|K_n|$ denotes the number of elements of K_n. The natural density of the set K is given by

$$\delta(K) := \lim_{n\to\infty} (\frac{|K_n|}{n}),$$

if this limit exists.

A real number sequence $x = (x_k)_{k=1}^{\infty}$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has a natural density of zero. In this case, we write $st - limx = L$.

Let (y_k) be a sequence, $(y_{k(j)})$ be a sub-sequence and let $K = \{k(j) : j \in \mathbb{N}\}$.

If $\delta(K) = 0$, then the sub-sequence $(y_{k(j)})$ is called thin sub-sequence of (y_n), otherwise it is called nonthin sub-sequence. That is, we call $(y_{k(j)})$ nonthin sub-sequence if K does not have density zero.

Definition 1.1. *A number λ is a statistical limit point of the number sequence $x = (x_n)$ provided that there is a nonthin sub-sequence of $x = (x_n)$ that converges to λ.*

Definition 1.2. *A number γ is a statistical cluster point of the number sequence $x = (x_n)$ provided that for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - \gamma| \leq \varepsilon\}$ does not have density zero.*

It is also well known that, a number l is an ordinary limit point of a sequence (x_k) if there exists a sub-sequence that converges to l.

Definition 1.3. *Let p be a prime number. The arithmetic function $a_p(n)$ is defined as follows $a_p(1) = 0$ and if $n \geq 1$, then $a_p(n)$ is the unique positive integer $j \geq 0$ satisfying $p^j|n$ but not $p^{j+1} \nmid n$.*

It is possible to express a given number $n \in \mathbb{N}$ as follows:

$$n = {a_1}^{b_1} = {a_2}^{b_2} = {a_3}^{b_3} \cdots = {a_{\gamma(n)}}^{b_{\gamma(n)}}. \tag{1}$$

Definition 1.4. *$\gamma(n)$ is the number of different ways of expressing the number n as in the form in equation (1) and the sequence $\tau(n)$ is the sum of each b_i's in (1). That is;*

$$\tau(n) := b_1 + b_2 + b_3 + \cdots + b_{\gamma(n)},$$

for all $n \geq 1$.

For any $n, m \in \mathbb{N}$; we are going to use a symbol (n, m) for the greatest common factor of n and m throughout the paper. The following two Lemma's are needed in the proof of some of our results.

Lemma 1.5. [9] *For any two real numbers* $\alpha > 1$ *and* β *denote*

$$S = \{n\alpha + \beta : n \in \mathbb{N}\}.$$

Then,

$$\delta(S) = \frac{1}{\alpha}.$$

Lemma 1.6. [9] *If*

$$A_0 = \{n > 1 : n = p_1^{\alpha_1}.p_2^{\alpha_2}...p_n^{\alpha_n} \text{ with } (\alpha_1, ...\alpha_n) = 1\},$$

then $\delta(A_0) = 1$.

2. MAIN RESULTS

The definition and elementary properties of the arithmetical function $(a_p(n))$ were studied by Šalát in [13]. Later, the idea given in [13] were extended to the concept of ideal convergence in [4].

2.1. The statistical limit and cluster point of $(a_p(n))$.

In this subsection, we will study the statistical limit and cluster points (which were studied by Fridy in [7] for real valued number sequences) of the arithmetical function $(a_p(n))$.

Let us consider the following sets:

$$K_0 := \{n \in \mathbb{N} : (n, p) = 1\}, K_1 := \{n \in \mathbb{N} : n = pk_i \text{ where } k_i \in K_0\},$$

$$K_2 := \{n \in \mathbb{N} : n = p^2 k_i \text{ where } k_i \in K_0\}, K_3 := \{n \in \mathbb{N} : n = p^3 k_i \text{ where } k_i \in K_0\},$$

$$\vdots$$

$$K_m := \{n \in \mathbb{N} : n = p^m k_i \text{ where } k_i \in K_0\}$$

and so on. Also, let

$$B := \bigcup_{i=1}^{\infty} K_i. \tag{2}$$

Lemma 2.1. *For each* $i \neq j$*, the sets* K_i *and* K_j *defined above are mutually disjoint subset of natural numbers.*

PROOF. Let $n \in K_i$ be an arbitrary element. Then, $n = p^i k_1$ where $k_1 \in K_0$. For $j \neq i$ suppose that $n \in K_j$, then we have also $n = p^j k_2$ where $k_2 \in K_0$.

Now, if $i < j$, then $n = p^j k_2 = p^i p^{j-i} k_2 = p^i k_1$ holds. This implies that $k_1 = p^{j-i} k_2$ and this also in turn implies that k_1 could not be in K_0 which contradicts to our assumption $n \in K_0$. Hence, $n \notin K_j$.

Similarly, if we consider $i > j$, then $n = p^i k_1 = p^j p^{i-j} k_1 = p^j k_2$. This implies that $k_2 = p^{i-j} k_1$ and hence k_2 could not be in K_0 which contradicts to our assumption $k_2 \in K_0$. Therefore, $n \notin K_j$. This completes the proof.

Lemma 2.2. *The set B defined in (2) is exactly the same as the set*

$$C := \{n : n = pk,\ k \in \mathbb{N}\}.$$

That is, we have $B = C$.

Proof.

Let $n \in B$ be an arbitrary element. By Lemma 2.1, the sets K_i's are mutually disjoint. Then, for a fixed $i \in \mathbb{N}$ we have $n \in K_i$. This implies that $n = p^i k_0$, where $k_0 \in K_0$. Now, if $i = 1$, then $n \in C$ holds. If $i > 1$, then $n = p(p^{i-1}k_0)$. This implies that $n = pk_2$ for $k_2 = p^{i-1}k_0$. So, we have $n \in C$. Hence, we have $B \subseteq C$.

To prove the converse inclusion, let $n \in C$ be an arbitrary element. Then, $n = pk$ where $k \in \mathbb{N}$. Now, if $(p, k) = 1$, then we have $n \in K_1$. If $(p, k) \neq 1$, then $(p, k) = p$ this implies $n = p^2 k_2$. If $(p, k_2) = 1$, then $n \in K_2$ holds. Since n is an arbitrary but fixed element, then if we continue in this way we will stop at some point $i \in \mathbb{N}$ such that $n = p^i k_i$ and $(k_i, p) = 1$ which implies that $n \in K_i$. Therefore, $C \subseteq B$. This completes the proof.

Lemma 2.3. *For the set $K_0 := \{n \in \mathbb{N} : (n, p) = 1\}$ we have*

$$\delta(K_0) = \frac{p-1}{p}. \tag{3}$$

Proof. By Lemma 2.2, we have

$$B = \bigcup_{i=1}^{\infty} K_i = \{n : n = pk,\ k \in \mathbb{N}\}.$$

We also have $\mathbb{N} \setminus \{1\} = K_0 \cup B$ and $K_0 \cap B = \emptyset$. We know that $1 = \delta(\mathbb{N} \setminus \{1\}) = \delta(B) + \delta(K_0)$ and from Lemma 1.5 we have

$$\delta(B) = \delta(C) = \frac{1}{p},$$

and it gives that

$$\delta(K_0) = 1 - \frac{1}{p} = \frac{p-1}{p}.$$

Lemma 2.4. *For each $m \in \mathbb{N}$, the density of K_m defined above is given by the following formula:*

$$\delta(K_m) = \frac{p-1}{p^{m+1}}.$$

Proof. Let $S_n := \{k_1^0 \leq k_2^0 \leq k_3^0 \leq k_4^0 \leq k_5^0 \cdots \leq k_n^0\}$ be a set which represents the first n elements of the set K_0. It is clear from the definition of density and (3) that

$$\delta(K_0) = \lim_{n \to \infty} \frac{K_0(S_n)}{S_n} = \frac{p-1}{p}$$

holds. Now, let

$$K_i := \{n \in \mathbb{N} : n = p^i k \text{ where } i \in \mathbb{N} \text{ and } k \in K_0\}.$$

By considering following equality

$$|K_i(p^i S_n)| = |K_0(S_n)| = n$$

for any $i \in \mathbb{N}$, we have

$$\delta(K_i) = \lim_{n\to\infty}\left(\frac{K_i(p^i S_n)}{p^i S_n}\right) = \lim_{n\to\infty}\left(\frac{n}{p^i S_n}\right) =$$
$$= \frac{1}{p^i}\lim_{n\to\infty}\left(\frac{K_0(S_n)}{S_n}\right) = \frac{1}{p^i}\delta(K_0) =$$
$$= \frac{1}{p^i}\left(\frac{p-1}{p}\right) = \left(\frac{p-1}{p^{i+1}}\right).$$

Hence, the result follows.

Now, we are ready to say something about the set of statistical limit and cluster points of the arithmetical function $(a_p(n))$.

Theorem 2.5. *The set of all statistical cluster and limit points of the (aritmetic function) sequence $(a_p(n))$ is $\mathbb{N} \cup \{0\}$.*

Proof. For each $i \in \mathbb{N}$, it is clear from the definition of K_i that we have:

$$(a_p(n))_{n\in K_i} = (i, i, i, ...)$$

and

$$\lim_{n\to\infty}\{a_p(n)\}_{K_i} = i$$

holds. Furthermore, from Lemma 2.4, the natural density of K_i is not zero.

In other words, for each $i \in \mathbb{N}$ there is a nonthin sub-sequence of $(a_p(n))$ that converges to i.

Similarly for each $n \in K_0$ the value of the sequence is zero and $\delta(K_0) = \frac{p-1}{p}$ which is non zero. Hence the theorem holds.

Corollary 2.6. *For any prime number p, the (arithmetical function) sequence $(a_p(n))$ is not statistically convergent to any real number.*

Proof. Let x be any real number and p be any prime number. Then, for all $\varepsilon > 0$ the natural density of the set

$$K(\varepsilon) = \{k \in \mathbb{N} : |a_p(n) - x| \geq \varepsilon\}$$

can not be zero.

In other words, if $x \notin \mathbb{N}$ then it is clear that $\delta(K(\varepsilon)) = 1$ which is non zero and if $x \in \mathbb{N}$ then for any $m \in \mathbb{N} \setminus \{x\}$ the set K_m is the subset of $K(\varepsilon)$. Because of monotonicity properties of density and Lemma 2.4 the density of $\delta(K(\epsilon)) \neq 0$. Thus, the proof of the corollary is completed.

$\square$

2.2. **The statistical limit and cluster point of $(\gamma(n))$ and $(\tau(n))$.** In [10], Mycielski defined the arithmetic functions $(\gamma(n))$ and $(\tau(n))$. Later, in [4], the ideal convergence of these arithmetical functions was studied. In addition to the result in [4], we are going to study the statistical limit and cluster points of these arithmetical functions.

Now, let us recall the definitions of $(\gamma(n))$ and $(\tau(n))$. For a given number $n \in \mathbb{N}$, it is possible to express it in the following different way:

$$n = {a_1}^{b_1} = {a_2}^{b_2} \; = \; {a_3}^{b_3} = \cdots = {a_{\gamma(n)}}^{b_{\gamma(n)}}. \tag{4}$$

In (4), $\gamma(n)$ represents the number of different ways of expressing the number n, and the sequence $\tau(n)$ is also defined as the sum of b_i's in (4). That is,

$$\tau(n) := b_1 + b_2 + b_3 + \cdots + b_{\gamma(n)},$$

for all $n \geq 1$.

In order to help us see the range of the sequence closely, let us consider the following sets:

$$A_0 = \{n \in \mathbb{N} : \gamma(n) = 1\}, \tag{5}$$

$$A_1 = \{n \in \mathbb{N} : \gamma(n) = 2\},\ A_2 = \{n \in \mathbb{N} : \gamma(n) = 3\},$$
$$A_3 = \{n \in \mathbb{N} : \gamma(n) = 4\},\ A_4 = \{n \in \mathbb{N} : \gamma(n) = 5\},$$
$$\vdots$$
$$A_m = \{n \in \mathbb{N} : \gamma(n) = m+1\},$$

and so on. Let us denote the union of these sets by

$$D := \bigcup_{i=1}^{\infty} A_i. \tag{6}$$

Lemma 2.7. *For all $i \geq 1$, $\delta(A_i) = 0$ holds.*

Proof.

Let $D = \bigcup_{i=1}^{\infty} A_i$. Then, we have $\mathbb{N} \setminus \{1\} = A_0 \cup D$ and it is clear from (5)-(6) that $A_0 \cap D = \emptyset$ holds. Since $\delta(N) = 1 = \delta(D) + \delta(A_0)$, then we have $\delta(A_0) = 1$ from Lemma 1.6. This implies that $\delta(D) = 0$. Hence, the desired result follows immediately.

Theorem 2.8. *The only statistical cluster and statistical limit point for both (arithmetical functions) sequences $(\gamma(n))$ and $(\tau(n))$ is 1.*

Proof. For any $\epsilon > 0$ it is clear that

$$A_0 = \{n \in N : |\gamma(n) - 1| \leq \epsilon\}$$

and from Lemma 2.7 we know that $\delta(A_0) = 1$ which is nonzero. This implies that 1 is a cluster point of the sequence $\gamma(n)$. But for each $n \in N$ and $n \neq 1$ together with the result of Lemma 2.7, we can not find a nonthin sub-sequence of $\gamma(n)$ that converges to n. In other words, for each n the set

$$\{n \in N : |\gamma(n) - n| \leq \epsilon\} = A_n$$

has zero density. Therefore, the only statistical cluster and limit point of the sequence $(\gamma(n))$ is 1.

Since we can easily verify that the following two sets are actually the same:

$$H_0 = \{n \in N : \tau(n) = 1\} = \{n \in N : \gamma(n) = 1\} = A_0.$$

As a result of this fact the proof for this part is exactly the same as that of $(\gamma(n))$.

Corollary 2.9. *Both sequences $(\gamma(n))$ and $(\tau(n))$ are statistically convergent to* 1.

Proof. Here, we will prove only for $(\gamma(n))$ and leave out the proof of $(\tau(n))$ because it is exactly the same way as the proof of $(\gamma(n))$. From the fact that for all $\varepsilon > 0$ the natural density of the set $K(\varepsilon) = \{k \in \mathbb{N} : |\gamma(n) - 1| > \varepsilon\}$ is zero and as a result of Theorem 2.8, the desired result follows immediately.

2.3. **The statistical limit and cluster point of $\Delta\gamma(n)$ and $\Delta\tau(n)$.** In this subsection, we study the statistical limit points and cluster points of the difference sequences $(\Delta\gamma(n))$ and $(\Delta\tau(n))$.

For each $n \in \mathbb{N}$ it can be defined as follows:

$$(\Delta\gamma(n)) = (\gamma(n+1)) - (\gamma(n)); (\Delta\tau(n)) = (\tau(n+1) - \tau(n)).$$

Now, let us consider the following sets:

$$B = \{n \in \mathbb{N} : \gamma(n) \neq 1\},\ K_0 = \{n \in \mathbb{N} : \gamma(n) = 1\},$$

$$H = \{n \in \mathbb{N} : \Delta\gamma(n) = 0\},\ D = \{n \in \mathbb{N} : \gamma(n) = 1 = \gamma(n+1)\},$$

and

$$F = \{n \in \mathbb{N} : n \in H \text{ but } n \notin D\},\ L = \{n \in \mathbb{N} : n \in K_0 \text{ but } n \notin K\}.$$

It is clear to see that $F \subset B$. This implies $\delta(F) = 0$. Furthermore, we have $K_0 = D \cup L$.

Now, $n \in L$ if $n+1 \in B$. This also implies that $\delta(L) \leq \delta(B) = 0$. Therefore, $(\Delta\gamma(n))_D$ is nonthin sub-sequence of the sequence $(\Delta\gamma(n))$ because of $\delta(D) = 1$.

Theorem 2.10. *Statistical limit and cluster point of each function $(\Delta\gamma(n))$ and $(\Delta\tau(n))$ is only* 0.

Proof. From the above discussion we can clearly see that the only non-thin sub sequence of the sequence $(\Delta\gamma(n))$ is $(\Delta\gamma(n))_D$ and

$$(\Delta\gamma(n))_D \to 0,$$

when $n \to \infty$. Furthermore, we showed that

$$\delta(D) = 1$$

This clearly tells us that there is no other nonthin sub-sequence other than ΔK_0.

Since $D = \{n \in \mathbb{N} : \gamma(n) = 1 = \gamma(n+1)\} = \{n \in \mathbb{N} : \tau(n) = 1 = \tau(n+1)\}$, then the proof for this part is exactly the same as $\Delta\gamma(n)$. Hence, it is omitted here.

2.4. **The statistical limit and cluster points of** $(\Delta^r\gamma(n)$ **)and** $(\Delta^r\tau(n))$**.** Here, we will investigate the statistical limit and cluster points of $(\Delta^r\gamma(n))$ where r is any natural number with $r \geq 2$.

$$\Delta^r\gamma(n) = \Delta^{r-1}\gamma(n+1) - \Delta^{r-1}\gamma(n) = \sum_{j=0}^{r}(-1)^j\binom{r}{j}\gamma(n+r-j)$$

Now, consider the following sets

$$B = \{n \in \mathbb{N} : \gamma(n) \neq 1\}; K_0 = \{n \in \mathbb{N} : \gamma(n) = 1\}; H = \{n : \Delta^r\gamma(n) = 0\};$$

$$D = \{n \in \mathbb{N} : \gamma(n) = \gamma(n+1) = \gamma(n+2) = \cdots = \gamma(n+r) = 1\}$$

and

$$F = \{n \in H \ but \ n \notin D\}.$$

Here, let us calculate the statistical density of the set F by dividing it in to simple and mutually disjoint sets; $F_1,\ F_1,\ F_1,\ \ldots F_r$ such that

$$F = F_1 \ \cup \ F_1 \ \cup \ F_1 \ \cup \ \cdots \cup F_r$$

holds, where

$$F_0 = \{n : \gamma(n) \neq 1 \ and \ \Delta^r\gamma(n) = 0\},$$

$$F_1 = \{n : \gamma(n) = 1,\ \gamma(n+1) \neq 1 \ and \ \Delta^r\gamma(n) = 0\},$$

$$F_2 = \{n : \gamma(n) = 1 = \gamma(n+1), \gamma(n+2) \neq 1 \ and \ \Delta^r\gamma(n) = 0\},$$

$$F_3 = \{n : \gamma(n) = 1 = \gamma(n+1) = \gamma(n+2),\ \gamma(n+3) \neq 1 \ and \ \Delta^r\gamma(n) = 0\},$$

$$\vdots$$

$$F_r = \{n : \gamma(n) = \gamma(n+1) = \cdots = \gamma(n+r-1) = 1,\ \gamma(n+r) \neq 1 \ and \ \Delta^r\gamma(n) = 0\}.$$

It is clear that $F_1 \subset B$ and from Theorem 2.7 we have $\delta(B) = 0$. Hence $\delta(F_1) = 0$.

Now, we have also $n \in F_2$ if and only if $n+2 \in B$. This also implies that

$$\delta(B) = \lim_{n\to\infty}\frac{|s_n^B|}{s_n^B} = \lim_{n\to\infty}\frac{n}{s_n^B} = 0 = \lim_{n\to\infty}\frac{n}{s_n^B - 2} = \lim_{n\to\infty}\frac{n}{s_n^{F_2}} = \lim_{n\to\infty}\frac{|s_n^{F_2}|}{s_n^{F_2}} = \delta(F_2)$$

Thus, we have also $\delta(F_2) = 0$. Similarly, we also have $n \in F_r$ if and only if $n + r \in B$. Hence, this gives that

$$\delta(F_r) = \lim_{n\to\infty}\frac{|s_n^{F_r}|}{s_n^{F_r}} = \lim_{n\to\infty}\frac{n}{s_n^{F_r}} = \lim_{n\to\infty}\frac{n}{s_n^B - r} = \lim_{n\to\infty}\frac{n}{s_n^B} = \delta(B) = 0.$$

Therefore, $\delta(F_r) = 0$. From this we obtain that $\delta(F_r) = 0$. This in turn implies that $\delta(D) = 1$.

Hence, $(\Delta^r\gamma(n))_D$ is a non-thin sub-sequence of $(\Delta^r\gamma(n))$.

Theorem 2.11. *The statistical cluster and limit point of the sequences* $(\Delta^r\gamma(n))$ *and* $(\Delta^r\tau(n))$ *is* 0.

Proof. From the above discussion we can clearly see that the only non-thin sub-sequence of the sequence $(\Delta^r\gamma(n))$ is $(\Delta^r\gamma(n))_D$ and $(\Delta^r\gamma(n))_D \to 0$, as $n \to \infty$.

Furthermore, we have also showed that $\delta(D) = 1$. This clearly implies the fact that there is no other nonthin sub-sequence other than $(\Delta^r\gamma(n))_D$. Hence, the result follows

The discussion above as well as the proof for $(\Delta^r\tau(n))$ is exactly the same to that of $(\Delta^r\gamma(n))$. Therefore, we leave the proof here.

3. Conclusion

In this study, it was especially focused on calculating the statistical limit and cluster points of some arithmetic functions. Because of the definition of each arithmetic function, they are handled separately and the results are given in four subsections. As a continuation of this study, the same problem can be made by considering different densities such as logarithmic density, uniform density, etc. For example, it may be interesting to find the $T-$statistical limit and cluster points of the arithmetic functions handled by the density produced by any summation matrix $T = (t_{n,k})$.

It is know from Theorem 2.5 that the statistical limit and cluster points of the arithmetical function $(a_p(n))$ is the set $\mathbb{N} \cup \{0\}$.

In [13], it is shown that the sequence $(a_p(n).g_p(n))$ is statistical convergent to zero when $g_p(n) = \frac{logp}{logn}$.

Naturally, following problem is worth investigating: Let $x \in \mathbb{N}$ be an arbitrary statistical limit or cluster point of the $(a_p(n))$. What should be the $g_p(n)$ for the sequence $(a_p(n).g_p(n))$ converge to statistically to the point x?

REFERENCES

[1] Balaz, V., Gogola,J. and Visnyai, T., I_c^q-convergence of arithmetical functions, *J. Number Theory*, **183** (2008), 74-83.

[2] Connor, J. S., The statistical and strong p-Cesaro convergence of sequences, *Analysis*, **8** (1988), 47-63.

[3] Fast, H., Sur la convergence statistique., *Colloq.Math*, **2** (1951), 241-244.

[4] Fehéra, Z., Lászlóa, B., Mačajb, M. and Šalátb, T., Remarks on arithmetical functions $a_p(n)$, $\gamma(n)$, $\tau(n)$, *Annales Mathematicae et Informaticae*, **33** (2006), 35-43.

[5] Fridy, J. A., On statistical convergence, *Analysis*, **5** (1985), 301-313.

[6] Fridy J. A., Miller H. I., A matrix characterization of statistical convergence, *Analysis*, **11** (1991), 59-66.

[7] Fridy, J. A., Statistical limit points, *Proceedings of the American Mathematical Society*, **118 No. 4** (1993), 1187-1192.

[8] Janos T. T., Ferdinand F., Jozsef B. and Laszlo Z., On $I_{<_q}$ and $I_{\leq_q}$ convergence of arithmetic functions, *Periodica Mathematica Hungarica*, (2020), 1-12.

[9] Milan, P., *Density and Related Topics*, (2017), 8-10.

[10] Mycielski, J., Sur les représentations des nombres naturels par des puissances à base et exposant naturels, *Colloquium Mathematicum*, **2** (1951), 254-260.

[11] Schoenberg, I. J., The integrability of certain functions and related summability methods, matrix characterization of statistical convergence, *Amer. Math.*, **66** (1959), 361-375.

[12] Steinhaus H., Sur la convergence ordinaire at la convergence asymptotique, *Colloc. Math.* **2.1** (1951), 73-74.

[13] Šalát, T., On the function a_p, $p^{a_p(n)} \backslash\backslash\ n$ $(n > 1)$, *Mathematica Slovaca*, **44 No. 2** (1994), 143-151.

[14] Zygmund, A., *Trigonometric series*, 2nd Ed. Vol. II, Cambridge Univ. press, London and New York, 1979.