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NEW BOUNDS FOR DISTANCE ENERGY OF A GRAPH

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Abstract. For any connected graph G, the distance energy, $\mathcal{E}_D(G)$ is defined as the sum of the absolute eigenvalues of its distance matrix. Distance energy was introduced by Indulal *et al.* in the year 2008 [10]. It has significant importance in QSPR analysis of molecular descriptor to study their physico-chemical properties. Our interest in this article is to establish new lower and upper bounds for distance energy.

Key words and Phrases: Distance matrix, Wiener index, Bounds for distance energy of a graph.

1. INTRODUCTION

In chemistry, Huckle molecular Orbital(HMO) theory is used to calculate π-electron energy of conjugated hydrocarbon. Later it was proved this quantity is equivalent to $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are eigenvalues of the respective molecular graph and called it as energy of graph. The studies on

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graph energy can be seen in papers [5, 6]. For detailed survey on applications on graph energy, see papers [2, 3, 4, 7]. The bounds for $\mathcal{E}(G)$ can be found in papers [12, 13, 14, 11].

In what follows in this paper, we take the graph G as simple undirected graph G with n vertices and m edges. For any two vertices v_i and v_j, the distance between them is denoted by d_{ij} and is defined as the shortest path from v_i to v_j. Two parameters that are of interest are Wiener index, $W(G)$ and distance matrix $A_D(G)$. They are respectively defined by $W(G) = \sum_{i<j} d_{ij}$ and $A(G) = A_D(G) = [d_{ij}]$. For the sake of simplicity Wiener index is written as W. Clearly $A_D(G)$ is a symmetric matrix, its eigenvalues are root of equation $\phi(G:\mu)=|\mu I - A(G)| = 0$. These eigenvalues are called $D-$eigenvalues or $D-$spectrum which are generally ordered in the form $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The largest eigenvalue μ_1 is called the distance spectral radius of the graph G. Given a graph G, the distance energy of G is defined by $\mathcal{E}_D(G) = \sum_{i=1}^{n} |\mu_i|$.

For a connected graph G, Koolen and Moulton upper bound [8] for distance energy in terms of W, M and n is

$$\mathcal{E}_D(G) \leq \Big(\frac{2W}{n}\Big) + \sqrt{(n-1)\Big(2M - \Big(\frac{2W}{n}\Big)^2\Big)} \quad for\ 2W \geq n \tag{1}$$

where $M = \sum_{i<j}^{n} d_{ij}^2$. Further results on upper bounds can also seen in the paper [9].

McClelland bounds [8] for distance energy of graph which is true for any connected graph G

$$\sqrt{2M + n(n-1)|det(A)|^{\frac{2}{n}}} \leq \mathcal{E}_D(G) \leq \sqrt{2Mn}. \tag{2}$$

For all studies on distance energy refer papers [1, 10, 15]. We use the following two lemmas, which followed from the properties of distance eigenvalues [8].

Lemma 1.1. *Let G be a graph with $n \geq 3$ vertices and m edges. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be $D-$eigenvalues of G then*

$$\sum_{i=1}^{n} \mu_i = 0$$

and

$$\sum_{i=1}^{n} \mu_i^2 = 2M.$$

Lemma 1.2. *If $\mu_1(G)$ is distance spectral radius of the graph G then $\mu_1(G) \geq \frac{2W}{n}$. Since $2W \geq n$, $\mu_1 \geq 1$.*

Throughout this paper, during proof of the theorems we use notations $M = \sum_{i<j}^{n} d_{ij}^2$ and $A_D(G) = A$. Note that $M = \sum_{i<j}^{n} d_{ij}^2 \geq \sum_{i<j}^{n} d_{ij} = W$ and $\sqrt{M} = \sqrt{\sum_{i<j}^{n} d_{ij}^2} \leq \sum_{i<j}^{n} d_{ij} = W$.

2. MAIN RESULTS

2.1. Lower bound for spectral distance radius.

Lemma 2.1. *If A is adjacency distance matrix of a graph G with n vertices and m edges then*

$$|det(A)| \leq (2M)^{\frac{n}{2}}. \tag{3}$$

Proof. Derivation follows from $|det(A)| = |\mu_1\mu_2...\mu_n| = |\mu_1||\mu_2|...|\mu_n|$. But

$$|det(A)| \leq |\mu_1||\mu_1|...|\mu_1| = |\mu_1|^n \leq (\sqrt{2M})^n.$$

This gives $|det(A)| \leq (2M)^{\frac{n}{2}}$. □

Lemma 2.2. *If G is a connected graph with n vertices and m edges then the largest distance eigenvalue, μ_1 of G satisfies*

$$|\mu_1| \geq |det(A)|^{\frac{1}{n}}. \tag{4}$$

Proof. Using the relation $\mu_1 + \mu_2 + \cdots + \mu_n = 0$ on distance eigenvalues of the graph G gives $\mu_2 + \cdots + \mu_n = -\mu_1$. Since $\mu_1 \geq 1$, the sum $\mu_2 + \cdots + \mu_n$ is negative quantity. Therefore

$$\mu_2 + \cdots + \mu_n \leq |\mu_2\mu_3 \cdots \mu_n|^{\frac{1}{n-1}}.$$

i.e.

$$-\mu_1 \leq \frac{|\mu_1\mu_2 \cdots \mu_n|^{\frac{1}{n-1}}}{\mu_1^{\frac{1}{n-1}}},$$

which implies

$$-\mu_1^{\frac{n}{n-1}} \leq |det(A)|^{\frac{1}{n-1}}.$$

So,

$$|\mu_1|^{\frac{2n}{n-1}} \leq |det(A)|^{\frac{2}{n-1}}$$

if $|\mu_1| \leq 1$ and $|\mu_1|^{\frac{2n}{n-1}} \geq |det(A)|^{\frac{2}{n-1}}$ if $|\mu_1| \geq 1$. But $|\mu_1| \geq 1$. Hence $|\mu_1| \geq |det(A)|^{\frac{1}{n}}$. □

Lemma 2.3. *If G is a graph with n vertices and m edges then the largest distance eigenvalue, μ_1 of G satisfies*

$$|\mu_1| \geq \frac{|det(A)|^{\frac{1}{n}}}{\sqrt{n}}. \tag{5}$$

Proof. Arithmetic and geometric mean of $|\mu_1|, |\mu_2|, \cdots, |\mu_n|$ are respectively are

$$\frac{|\mu_1| + |\mu_2| + \cdots + |\mu_n|}{n}$$

and

$$|\mu_1 \mu_2 \cdots \mu_n|^{\frac{1}{n}}.$$

Since arithmetic mean is greater than or equal to geometric mean it follows that

$$\frac{|\mu_1| + |\mu_2| + \cdots + |\mu_n|}{n} \geq |\mu_1 \mu_2 \cdots \mu_n|^{\frac{1}{n}}.$$

$$\frac{|\mu_1| + |\mu_2| + \cdots + |\mu_n|}{\sqrt{n}} \geq \frac{|\mu_1| + |\mu_2| + \cdots + |\mu_n|}{n} \geq |\mu_1 \mu_2 \cdots \mu_n|^{\frac{1}{n}}.$$

Therefore

$$\frac{|\mu_1| + |\mu_2| + \cdots + |\mu_n|}{\sqrt{n}} \geq |\mu_1 \mu_2 \cdots \mu_n|^{\frac{1}{n}}$$

implies

$$\frac{n|\mu_1|}{\sqrt{n}} \geq |det(A)|^{\frac{1}{n}}.$$

$$|\mu_1| \geq \frac{|det(A)|^{\frac{1}{n}}}{\sqrt{n}}.$$

□

2.2. **Bounds for distance energy of graph.**

Lemma 2.4. *If G is a graph with n vertices and m edges and A is the adjacency distance matrix which is non-singular then*

$$n|det(A)|^{\frac{1}{n}} \leq \mathcal{E}_D(G) \leq \frac{2Mn}{|det(A)|^{\frac{1}{n}}}. \tag{6}$$

Proof. Using inequality of arithmetic and geometric mean of $|\mu_1|, |\mu_2|, \cdots, |\mu_n|$ we have

$$\frac{|\mu_1| + |\mu_2| + \cdots + |\mu_n|}{n} \geq |\mu_1 \mu_2 \cdots \mu_n|^{\frac{1}{n}}.$$

So,

$$\mathcal{E}_D(G) \geq n|det(A)|^{\frac{1}{n}}.$$

From $\frac{|\mu_1|+|\mu_2|+\cdots+|\mu_n|}{n} \geq |det(A)|^{\frac{1}{n}}$ gives $|\mu_1| \geq |det(A)|^{\frac{1}{n}}$. So,

$$|\mu_1| \sum_{i=1}^{n} |\mu_i| \geq |det(A)|^{\frac{1}{n}} \sum_{i=1}^{n} |\mu_i|.$$

Since $|\mu_i| \leq |\mu_1|\ \forall i$, therefore $n|\mu_1|^2 \geq |det(A)|^{\frac{1}{n}}\mathcal{E}(G)$. But $|\mu_1|^2 \leq 2M$ from which we have $\mathcal{E}_D(G) \leq \frac{2Mn}{|det(A)|^{\frac{1}{n}}}$. Thus $n|det(A)|^{\frac{1}{n}} \leq \mathcal{E}_D(G) \leq \frac{2Mn}{|det(A)|^{\frac{1}{n}}}$. □

We use Holder's inequality inequality to get bounds for energy of graphs

Holder's inequality: If $x_{ij}(i = 1, 2, ..., n$ and $j = 1, 2, 3, ..., n)$ is a non-negative real numbers then $\prod_{i=1}^{n}\Big(\sum_{j=1}^{n} x_{ij}\Big)^{\frac{1}{n}} \geq \sum_{j=1}^{n}\Big(\prod_{i=1}^{n} x_{ij}^{\frac{1}{n}}\Big)$

$$i.e., \Big(x_{11}+x_{12}+...+x_{1n}\Big)^{\frac{1}{n}}\Big(x_{21}+x_{22}+...+x_{2n}\Big)^{\frac{1}{n}}...\Big(x_{n1}+x_{n2}+...+x_{nn}\Big)^{\frac{1}{n}} \geq$$
$$\Big(x_{11}^{\frac{1}{n}}x_{21}^{\frac{1}{n}}...x_{n1}^{\frac{1}{n}}\Big) + \Big(x_{12}^{\frac{1}{n}}x_{22}^{\frac{1}{n}}...x_{n2}^{\frac{1}{n}}\Big) + ... + \Big(x_{1n}^{\frac{1}{n}}x_{2n}^{\frac{1}{n}}...x_{nn}^{\frac{1}{n}}\Big).$$

Theorem 2.5. *Let G be a graph with n vertices and m edges with $2M \geq n$. If A is a adjacency distance matrix which is non-singular then*

$$n^{\frac{n-1}{n}}|det(A)|^{\frac{1}{n}} \leq \mathcal{E}_D(G) < \frac{(4M)^{n^2}}{|det(A)|^{(n-1)}}. \tag{7}$$

Proof. Apply Holder's inequality using

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{|\mu_1|} & \cdots & \frac{1}{|\mu_1|} \\ \frac{1}{|\mu_2|} & 1 & \cdots & \frac{1}{|\mu_2|} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{|\mu_n|} & \frac{1}{|\mu_n|} & \cdots & 1 \end{pmatrix}$$

and simplify left hand side and right hand side of inequality separately.

$$LHS = \Big(1+\frac{n-1}{|\mu_1|}\Big)^{\frac{1}{n}}\Big(1+\frac{n-1}{|\mu_2|}\Big)^{\frac{1}{n}}...\Big(1+\frac{n-1}{|\mu_n|}\Big)^{\frac{1}{n}} \leq \Big(1+\frac{n-1}{|\mu_1|}\Big)\Big(1+\frac{n-1}{|\mu_2|}\Big)...\Big(1+\frac{n-1}{|\mu_n|}\Big).$$

Since $2M \geq n > (n-1)$ it follows that

$$LHS < \Big(1+\frac{2M}{|\mu_1|}\Big)\Big(1+\frac{2M}{|\mu_2|}\Big)...\Big(1+\frac{2M}{|\mu_n|}\Big)$$

. But

$$|\mu_i| \leq \sqrt{2M} \leq 2M \Rightarrow 1 \leq \frac{2M}{|\mu_i|}\ \ \forall i.$$

So

$$\begin{aligned} LHS &< \Big(\frac{2M}{|\mu_1|}+\frac{2M}{|\mu_1|}\Big)\Big(\frac{2M}{|\mu_2|}+\frac{2M}{|\mu_2|}\Big)...\Big(\frac{2M}{|\mu_n|}+\frac{2M}{|\mu_n|}\Big) \\ &= \Big(\frac{4M}{|\mu_1|}\Big)\Big(\frac{4M}{|\mu_2|}\Big)...\Big(\frac{4M}{|\mu_n|}\Big) \\ &= \frac{(4M)^n}{|\mu_1\mu_2...\mu_n|} = \frac{(4M)^n}{|det(A)|} \end{aligned}$$

$$
\begin{aligned}
RHS &= \frac{1}{|\mu_2|^{\frac{1}{n}}|\mu_3|^{\frac{1}{n}}...|\mu_n|^{\frac{1}{n}}} + \frac{1}{|\mu_1|^{\frac{1}{n}}|\mu_3|^{\frac{1}{n}}...|\mu_n|^{\frac{1}{n}}} + ... + \frac{1}{|\mu_1|^{\frac{1}{n}}|\mu_2|^{\frac{1}{n}}...|\mu_{n-1}|^{\frac{1}{n}}} \\
&= \frac{|\mu_1|^{\frac{1}{n}}}{|\mu_1\mu_2...\mu_n|^{\frac{1}{n}}} + \frac{|\mu_2|^{\frac{1}{n}}}{|\mu_1\mu_2...\mu_n|^{\frac{1}{n}}} + ... + \frac{|\mu_n|^{\frac{1}{n}}}{|\mu_1\mu_2...\mu_n|^{\frac{1}{n}}} \\
&= \frac{1}{|det(A)|^{\frac{1}{n}}} \sum_{i=1}^{n} |\mu_i|^{\frac{1}{n}}.
\end{aligned}
$$

Therefore

$$\frac{1}{|det(A)|^{\frac{1}{n}}} \sum_{i=1}^{n} |\mu_i|^{\frac{1}{n}} < \frac{(4M)^n}{|det(A)|}$$

and

$$\sum_{i=1}^{n} |\mu_i|^{\frac{1}{n}} < \frac{(4M)^n}{|det(A)|^{(1-\frac{1}{n})}}.$$

But

$$\Big(\sum_{i=1}^{n} |\mu_i|\Big)^{\frac{1}{n}} \leq \sum_{i=1}^{n} |\mu_i|^{\frac{1}{n}}.$$

Hence

$$\Big(\sum_{i=1}^{n} |\mu_i|\Big)^{\frac{1}{n}} < \frac{(4M)^n}{|det(A)|^{(\frac{n-1}{n})}}$$

and

$$\mathcal{E}_D(G) < \frac{(4M)^{n^2}}{|det(A)|^{(n-1)}}.$$

To get lower bound we apply Holder's inequality using the substitution

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \dots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} |\mu_1| & |\mu_1| & \cdots & |\mu_1| \\ |\mu_2| & |\mu_2| & \cdots & |\mu_2| \\ \vdots & \vdots & \dots & \vdots \\ |\mu_n| & |\mu_n| & \cdots & |\mu_n| \end{pmatrix}$$

$$(n|\mu_1|)^{\frac{1}{n}} + (n|\mu_2|)^{\frac{1}{n}} + \cdots + (n|\mu_n|)^{\frac{1}{n}} \geq n(|\mu_1||\mu_2|\cdots|\mu_n|)^{\frac{1}{n}}.$$

$$|\mu_1|^{\frac{1}{n}} + |\mu_2|^{\frac{1}{n}} + \cdots + |\mu_n|^{\frac{1}{n}} \geq n^{\frac{n-1}{n}}(|det(A)|)^{\frac{1}{n}}$$

But $(|\mu_1| + |\mu_2| + \cdots + |\mu_n|) \geq |\mu_1|^{\frac{1}{n}} + |\mu_2|^{\frac{1}{n}} + \cdots + |\mu_n|^{\frac{1}{n}}$. Therefore

$$\mathcal{E}_D(G) \geq n^{\frac{n-1}{n}}|det(A)|^{\frac{1}{n}}.$$

Combining above bounds we have, $n^{\frac{n-1}{n}}|det(A)|^{\frac{1}{n}} \leq \mathcal{E}_D(G) < \dfrac{(4m)^{n^2}}{|det(A)|^{(n-1)}}$.
□

2.3. **Lower and upper bound for distance energy of graph.**

Theorem 2.6. *Let G be a graph with $n(\geq 2)$ vertices and m edges with $2M \geq n$ then*

$$\mathcal{E}_D(G) \geq \frac{2M}{n} + \Big(\frac{|det(A)|}{\frac{2M}{n}}\Big)^{\frac{1}{n-1}}. \tag{8}$$

Proof. Apply arithmetic mean and geometric mean inequality to real numbers $|\mu_2|, |\mu_3|, \cdots, |\mu_n|$ for $(n-1)$ terms,

$$\frac{|\mu_2| + |\mu_3| + \cdots + |\mu_n|}{n-1} \geq |\mu_2\mu_3\cdots\mu_n|^{\frac{1}{n-1}}.$$

$$\Big(|\mu_2| + |\mu_3| + \cdots + |\mu_n|\Big) \geq \frac{|\mu_2| + |\mu_3| + \cdots + |\mu_n|}{n-1} \geq |\mu_2\mu_3\cdots\mu_n|^{\frac{1}{n-1}}.$$

So,

$$\mathcal{E}(G) - |\mu_1| \geq \frac{|\mu_1\mu_2\cdots\mu_n|^{\frac{1}{n-1}}}{|\mu_1|\frac{1}{n-1}}.$$

And

$$\mathcal{E}_D(G) \geq |\mu_1| + \frac{|det(A)|^{\frac{1}{n-1}}}{|\mu_1|\frac{1}{n-1}}.$$

Let $|\mu_1| = x$ and $\Psi(x) = x + \frac{|det(A)|^{\frac{1}{n-1}}}{x^{\frac{1}{n-1}}}$. We shall minimize the function by finding $\Psi'(x)$ and $\Psi''(x)$. At maxima or minima $\Psi'(x) = 0$ which gives $1 - \frac{|det(A)|^{\frac{1}{n-1}}}{n-1}x^{-\frac{n}{n-1}} = 0$. Thus the function $\Psi(x)$ attains maxima or minima at $x = \frac{|det(A)|^{\frac{1}{n}}}{(n-1)^{\frac{n-1}{n}}}$. At this point, $\Psi''(x) = \frac{n}{(n-1)^2}|det(A)|^{\frac{1}{n-1}}x^{\frac{1-2n}{n-1}} \geq 0$. This means the function attains the minimum value at this point. The minimum value is

$$\Psi\Big(\frac{|det(A)|^{\frac{1}{n}}}{(n-1)^{\frac{n-1}{n}}}\Big) = \frac{n|det(A)|^{\frac{1}{n}}}{(n-1)^{\frac{(n-1)}{n}}}.$$

But the function is increasing in the interval $\frac{|det(A)|^{\frac{1}{n}}}{(n-1)^{\frac{n-1}{n}}} \leq |detA|^{\frac{1}{n}} \leq \frac{2M}{n} \leq |\mu_1| \leq \sqrt{2M}$.

$$\mathcal{E}_D(G) \geq \Psi\Big(\frac{2M}{n}\Big).$$

$$\mathcal{E}_D(G) \geq \frac{2M}{n} + \Big(\frac{|det(A)|}{\frac{2M}{n}}\Big)^{\frac{1}{n-1}}.$$

□

Theorem 2.7. *Let G be a graph with $n(\geq 3)$ vertices and m edges with $2M \geq n$ then*

$$\mathcal{E}_D(G) \geq \frac{2M}{n} + \frac{(n-2)^{\frac{1}{n}}|det(A)|^{\frac{n-1}{n(n-2)}}}{(\frac{2M}{n})^{\frac{1}{n-2}}}. \tag{9}$$

Proof. Apply arithmetic mean and geometric mean inquality to real numbers $|\mu_2|, |\mu_3|, \cdots, |\mu_{n-1})|$ for $(n-2)$ terms,

$$\frac{|\mu_2| + |\mu_3| + \cdots + |\mu_{n-1}|}{n-2} \geq |\mu_2\mu_3 \cdots \mu_{n-1}|^{\frac{1}{n-2}}.$$

$$\Big(|\mu_2| + |\mu_3| + \cdots + |\mu_{n-1}|\Big) \geq \frac{|\mu_2| + |\mu_3| + \cdots + |\mu_{n-1}|}{n-2} \geq |\mu_2\mu_3 \cdots \mu_{n-1}|^{\frac{1}{n-2}}.$$

So,

$$\mathcal{E}(G) - |\mu_1| - |\mu_n| \geq \frac{|\mu_1\mu_2 \cdots \mu_n|^{\frac{1}{n-2}}}{|\mu_1\mu_n|\frac{1}{n-2}},$$

$$\mathcal{E}_D(G) \geq |\mu_1| + |\mu_n| + \frac{|det(A)|^{\frac{1}{n-2}}}{|\mu_1\mu_n|\frac{1}{n-2}}.$$

Let $|\mu_1| = x$, $|\mu_n| = y$ and $g(x, y) = x + y + \frac{|det(A)|^{\frac{1}{n-2}}}{(xy)^{\frac{1}{n-2}}}$. Using partial differentiation we minimize the function by finding $g_x(x, y)$, $g_y(x, y)$, $g_{xx}(x, y)$, $g_{yy}(x, y)$, $g_{xy}(x, y)$ and $\Delta = g_{xx}g_{yy} - g_{xy}^2$.

$$g_x = 1 - \frac{|det(A)|^{\frac{1}{n-2}}}{n-2}(xy)^{\frac{1-n}{n-2}}y,$$

$$g_y = 1 - \frac{|det(A)|^{\frac{1}{n-2}}}{n-2}(xy)^{\frac{1-n}{n-2}}x,$$

$$g_{xx} = -\frac{y^2(1-n)|det(A)|^{\frac{1}{n-2}}}{(n-2)^2}(xy)^{\frac{3-2n}{n-2}},$$

$$g_{yy} = -\frac{x^2(1-n)|det(A)|^{\frac{1}{n-2}}}{(n-2)^2}(xy)^{\frac{3-2n}{n-2}},$$

$$g_{xy} = -\frac{|det(A)|^{\frac{1}{n-2}}}{n-2}\Big((xy)^{\frac{1-n}{n-2}} + y\frac{n-1}{n-2}(xy)^{\frac{3-2n}{n-2}}\Big),$$

$$\Delta = \frac{(xy)^2(1-n)^2|det(A)|^{\frac{2}{n-2}}}{(n-2)^4}(xy)^{\frac{6-4n}{n-2}} - \frac{|det(A)|^{\frac{2}{n-2}}}{(n-2)^2}\Big((xy)^{\frac{1-n}{n-2}} + y\frac{n-1}{n-2}(xy)^{\frac{3-2n}{n-2}}\Big)^2.$$

At maxima or minima $g_x = 0$, $g_y = 0$ which gives

$$(xy)^{\frac{1-n}{n-2}}y = \frac{n-2}{|det(A)|^{\frac{1}{n-2}}}$$

and

$$(xy)^{\frac{1-n}{n-2}}x = \frac{n-2}{|det(A)|^{\frac{1}{n-2}}}.$$

Solving these equations gives

$$x = y = \frac{|det(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n}}}.$$

Thus the function $g(x, y)$ attains maxima or minima at

$$x = y = \frac{|det(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n}}},$$

At this point, g_{xx} and g_{yy} are greater than equal to zero. Further $\Delta \leq 0$. This means that the function attains the minimum value at this point. The minimum value is given by,

$$g\Big(\frac{|det(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n}}}, \frac{|det(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n}}}\Big).$$

Since $2M \geq n$, $g(x, y)$ increases in the interval

$$|\det(A)|^{\frac{1}{n}} \leq \frac{2M}{n} \leq x \leq \sqrt{2M}$$

and

$$0 \leq y \leq |\det(A)|^{\frac{1}{n}} \leq \frac{2M}{n} \leq \sqrt{2M}.$$

At

$$y = \frac{|det(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n}}},$$

$$g(x, y) = x + \frac{(n-2)^{\frac{1}{n}}|det(A)|^{\frac{n-1}{n(n-2)}}}{x^{\frac{1}{n-2}}}.$$

Therefore,

$$\mathcal{E}_D(G) \geq g\Big(x, \frac{|det(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n}}}\Big) \geq g\Big(\frac{2M}{n}, \frac{|det(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n}}}\Big).$$

Hence,

$$\mathcal{E}_D(G) \geq \frac{2M}{n} + \frac{(n-2)^{\frac{1}{n}}|det(A)|^{\frac{n-1}{n(n-2)}}}{(\frac{2M}{n})^{\frac{1}{n-2}}}.$$

□

Theorem 2.8. *Let G be a graph with $n \geq 2$ vertices, m edges and G is a non-singular graph then*

$$\mathcal{E}_D(G) \leq \sqrt{2M} + \frac{(n-1)(2M)}{|det(A)|^{\frac{1}{n}}}. \tag{10}$$

Proof. We know that $|\mu_1| \geq |det(A)|^{\frac{1}{n}}$, which implies

$$|\mu_1| \sum_{i=2}^{n} |\mu_i| \geq |det(A)|^{\frac{1}{n}} \sum_{i=2}^{n} |\mu_i|.$$

Since $|\mu_i| \leq |\mu_1|\ \forall i$, therefore

$$(n-1)|\mu_1|^2 \geq |det(A)|^{\frac{1}{n}} \Big(\mathcal{E}(G) - |\mu_1|\Big).$$

Thus

$$\mathcal{E}_D(G) \leq |\mu_1| + \frac{(n-1)|\mu_1|^2}{|det(A)|^{\frac{1}{n}}}.$$

Let $|\mu_1| = x$ and $f(x) = x + \frac{(n-1)x^2}{|det(A)|^{\frac{1}{n}}}$. At maxima or minima $f'(x) = 0$ which gives

$$1 + \frac{(n-1)2x}{|det(A)|^{\frac{1}{n}}} = 0.$$

Hence the function attains maximum or minimum value at

$$x = -\frac{|det(A)|^{\frac{1}{n}}}{2(n-1)}.$$

Since $f''(x) = \frac{2(n-1)}{|det(A)|^{\frac{1}{n}}} > 0$ the function attains minimum value at this point.

The minimum value

$$f\Big(-\frac{|det(A)|^{\frac{1}{n}}}{2(n-1)}\Big) = -\frac{|det(A)|^{\frac{1}{n}}}{2(n-1)} + \frac{|det(A)|^{\frac{1}{n}}}{4(n-1)} = -\frac{|det(A)|^{\frac{1}{n}}}{4(n-1)}.$$

But $f(x)$ is an increasing function in the region $-\frac{|det(A)|^{\frac{1}{n}}}{2(n-1)} \leq x \leq \sqrt{2M}$. Hence $f(x) \leq f(\sqrt{2M})$. Therefore

$$\mathcal{E}_D(G) \leq \sqrt{2M} + \frac{(n-1)(2M)}{|det(A)|^{\frac{1}{n}}}.$$

□

3. CONCLUDING REMARKS

In this paper, an effort has been made to obtain new bounds for distance energy of graph in a simplest way. Are these lower and upper bounds better than Koolen-Moulton and McClelland bounds (1.1 and 1.2)? It is yet to proved and is a scope for further research.

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