NEW BOUNDS FOR DISTANCE ENERGY OF A GRAPH

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Abstract. For any connected graph \( G \), the distance energy, \( E_D(G) \) is defined as the 
sum of the absolute eigenvalues of its distance matrix. Distance energy was intro-
duced by Indulal et al. in the year 2008 [10]. It has significant importance in QSPR 
analysis of molecular descriptor to study their physico-chemical properties. Our 
interest in this article is to establish new lower and upper bounds for distance energy.

Key words and Phrases: Distance matrix, Wiener index, Bounds for distance energy 
of a graph.

1. INTRODUCTION

In chemistry, Huckle molecular Orbital(HMO) theory is used to calculate 
\( \pi \)-electron energy of conjugated hydrocarbon. Later it was proved this quantity 
is equivalent to \( E(G) = \sum_{i=1}^{n} |\lambda_i|, \) where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) are eigenvalues of 
the respective molecular graph and called it as energy of graph. The studies on

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In what follows in this paper, we take the graph $G$ as simple undirected graph $G$ with $n$ vertices and $m$ edges. For any two vertices $v_i$ and $v_j$, the distance between them is denoted by $d_{ij}$ and is defined as the shortest path from $v_i$ to $v_j$. Two parameters that are of interest are Wiener index, $W(G)$ and distance matrix $A_D(G)$. They are respectively defined by $W(G) = \sum_{i<j} d_{ij}$ and $A_D(G) = [d_{ij}]$. For the sake of simplicity Wiener index is written as $W$. Clearly $A_D(G)$ is a symmetric matrix, its eigenvalues are root of equation $\phi(G; \mu) = |\mu I - A(G)| = 0$. These eigenvalues are called $D-$eigenvalues or $D-$spectrum which are generally ordered in the form $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The largest eigenvalue $\mu_1$ is called the distance spectral radius of the graph $G$. Given a graph $G$, the distance energy of $G$ is defined by $E_D(G) = \sum_{i=1}^{n} |\mu_i|$. For a connected graph $G$, Koolen and Moulton upper bound [8] for distance energy in terms of $W$, $M$ and $n$ is

$$E_D(G) \leq \left(\frac{2W}{n}\right) + \sqrt{(n-1)(2M - \left(\frac{2W}{n}\right)^2)} \quad \text{for} \quad 2W \geq n$$

(1)

where $M = \sum_{i<j} d_{ij}^2$. Further results on upper bounds can also seen in the paper [9].

McClelland bounds [8] for distance energy of graph which is true for any connected graph $G$

$$\sqrt{2M + n(n-1)|\det(A)|^2} \leq E_D(G) \leq \sqrt{2Mn}. \quad (2)$$

For all studies on distance energy refer papers [1, 10, 15]. We use the following two lemmas, which followed from the properties of distance eigenvalues [8].

**Lemma 1.1.** Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be $D-$eigenvalues of $G$ then

$$\sum_{i=1}^{n} \mu_i = 0$$

and

$$\sum_{i=1}^{n} \mu_i^2 = 2M.$$
Lemma 1.2. If $\mu_1(G)$ is distance spectral radius of the graph $G$ then $\mu_1(G) \geq \frac{2W}{n}$. Since $2W \geq n$, $\mu_1 \geq 1$.

Throughout this paper, during proof of the theorems we use notations $M = \sum_{i<j} d_{ij}^2$ and $A_D(G) = A$. Note that $M = \sum_{i<j} d_{ij}^2 \geq \sum_{i<j} d_{ij} = W$ and $\sqrt{M} = \sqrt{\sum_{i<j} d_{ij}^2} \leq \sum_{i<j} d_{ij} = W$.

2. MAIN RESULTS

2.1. Lower bound for spectral distance radius.

Lemma 2.1. If $A$ is adjacency distance matrix of a graph $G$ with $n$ vertices and $m$ edges then

$$|\det(A)| \leq (2M)^\frac{n}{2}.$$  \hspace{1cm} (3)

Proof. Derivation follows from $|\det(A)| = |\mu_1 \mu_2 ... \mu_n| = |\mu_1| |\mu_2| ... |\mu_n|$. But $|\det(A)| \leq |\mu_1| |\mu_2| ... |\mu_n| = |\mu_1|^n \leq (\sqrt{2M})^n$. This gives $|\det(A)| \leq (2M)^\frac{n}{2}$. \hspace{1cm} $\Box$

Lemma 2.2. If $G$ is a connected graph with $n$ vertices and $m$ edges then the largest distance eigenvalue, $\mu_1$ of $G$ satisfies

$$|\mu_1| \geq |\det(A)|^{\frac{1}{n}}.$$  \hspace{1cm} (4)

Proof. Using the relation $\mu_1 + \mu_2 + \cdots + \mu_n = 0$ on distance eigenvalues of the graph $G$ gives $\mu_2 + \cdots + \mu_n = -\mu_1$. Since $\mu_1 \geq 1$, the sum $\mu_2 + \cdots + \mu_n$ is negative quantity. Therefore

$$\mu_2 + \cdots + \mu_n \leq |\mu_2 \mu_3 \cdots \mu_n|^{\frac{1}{n-1}},$$

i.e.

$$-\mu_1 \leq \frac{|\mu_2 \mu_3 \cdots \mu_n|^{\frac{1}{n-1}}}{\mu_1^{\frac{n-2}{n-1}}},$$

which implies

$$-\mu_1^{\frac{n}{n-1}} \leq |\det(A)|^{\frac{1}{n-1}}.$$  \hspace{1cm} (5)

So,

$$|\mu_1|^{\frac{n}{n-1}} \leq |\det(A)|^{\frac{n}{n-1}}$$

if $|\mu_1| \leq 1$ and $|\mu_1|^{\frac{2n}{n-1}} \geq |\det(A)|^{\frac{2}{n-1}}$ if $|\mu_1| \geq 1$. But $|\mu_1| \geq 1$. Hence $|\mu_1| \geq |\det(A)|^{\frac{1}{n}}$. \hspace{1cm} $\Box$. 
Lemma 2.3. If $G$ is a graph with $n$ vertices and $m$ edges then the largest distance eigenvalue, $\mu_1$ of $G$ satisfies
\[
|\mu_1| \geq \frac{|\det(A)|^{\frac{1}{n}}}{\sqrt{n}}. 
\]  
(5)

Proof. Arithmetic and geometric mean of $|\mu_1|, |\mu_2|, \ldots, |\mu_n|$ are respectively are
\[
\frac{|\mu_1| + |\mu_2| + \cdots + |\mu_n|}{n}
\]
and
\[
|\mu_1\mu_2\cdots\mu_n|^{\frac{1}{n}}.
\]
Since arithmetic mean is greater than or equal to geometric mean it follows that
\[
|\mu_1| + |\mu_2| + \cdots + |\mu_n| \geq |\mu_1\mu_2\cdots\mu_n|^\frac{1}{n}.
\]
Therefore
\[
\frac{|\mu_1| + |\mu_2| + \cdots + |\mu_n|}{\sqrt{n}} \geq \frac{|\mu_1\mu_2\cdots\mu_n|^{\frac{1}{n}}}{\sqrt{n}}.
\]
implies
\[
\frac{n|\mu_1|}{\sqrt{n}} \geq |\det(A)|^{\frac{1}{n}}.
\]
\[
|\mu_1| \geq \frac{|\det(A)|^{\frac{1}{n}}}{\sqrt{n}}.
\]

2.2. Bounds for distance energy of graph.

Lemma 2.4. If $G$ is a graph with $n$ vertices and $m$ edges and $A$ is the adjacency distance matrix which is non-singular then
\[
n|\det(A)|^{\frac{1}{n}} \leq E_D(G) \leq \frac{2Mn}{|\det(A)|^{\frac{1}{n}}}. 
\]  
(6)

Proof. Using inequality of arithmetic and geometric mean of $|\mu_1|, |\mu_2|, \ldots, |\mu_n|$ we have
\[
\frac{|\mu_1| + |\mu_2| + \cdots + |\mu_n|}{n} \geq |\mu_1\mu_2\cdots\mu_n|^\frac{1}{n}.
\]
So,
\[
E_D(G) \geq n|\det(A)|^{\frac{1}{n}}.
\]
From \[
\frac{|\mu_1| + |\mu_2| + \cdots + |\mu_n|}{n} \geq |\det(A)|^{\frac{1}{n}} \]
gives $|\mu_1| \geq |\det(A)|^{\frac{1}{n}}$. So,
\[
|\mu_1| \sum_{i=1}^{n} |\mu_i| \geq |\det(A)|^{\frac{1}{n}} \sum_{i=1}^{n} |\mu_i|.
\]
New bounds for energy of graphs

Since $|\mu_i| \leq |\mu| \forall i$, therefore $n|\mu|^2 \geq |\det(A)|^{1/2}\mathcal{E}(G)$. But $|\mu|^2 \leq 2M$ from which we have $\mathcal{E}_D(G) \leq \frac{2Mn}{|\det(A)|^{1/2}}$. Thus $n|\det(A)|^{1/2} \leq \mathcal{E}_D(G) \leq \frac{2Mn}{|\det(A)|^{1/2}}$. □

We use Holder’s inequality inequality to get bounds for energy of graphs

**Holder’s inequality:** If $x_{ij}(i = 1, 2, ..., n$ and $j = 1, 2, 3, ..., n$) is a non-negative real numbers then $\prod_{i=1}^{n} \left( \sum_{j=1}^{n} x_{ij} \right)^{\frac{1}{p}} \geq \sum_{j=1}^{n} \left( \prod_{i=1}^{n} x_{ij} \right)^{\frac{1}{q}}$

i.e., $\left( x_{11} + x_{12} + ... + x_{1n} \right)^{\frac{1}{p}} \left( x_{21} + x_{22} + ... + x_{2n} \right)^{\frac{1}{p}} \left( x_{n1} + x_{n2} + ... + x_{nn} \right)^{\frac{1}{p}} \geq \left( x_{11}^p + x_{21}^p + ... + x_{n1}^p \right)^{\frac{1}{q}} \left( x_{12}^p + x_{22}^p + ... + x_{n2}^p \right)^{\frac{1}{q}} ... \left( x_{n1}^p + x_{n2}^p + ... + x_{nn}^p \right)^{\frac{1}{q}}$

**Theorem 2.5.** Let $G$ be a graph with $n$ vertices and $m$ edges with $2M \geq n$. If $A$ is a adjacency distance matrix which is non-singular then

$$n \frac{n-1}{2} |\det(A)|^{1/2} \leq \mathcal{E}_D(G) < \frac{(4M)^n}{|\det(A)|^{(n-1)}}. \quad (7)$$

**Proof.** Apply Holder’s inequality using

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \geq \begin{pmatrix} \frac{1}{|\mu_1|} & 1 & \cdots & \frac{1}{|\mu_1|} \\ \frac{1}{|\mu_2|} & 1 & \cdots & \frac{1}{|\mu_2|} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{|\mu_n|} & \frac{1}{|\mu_n|} & \cdots & 1 \end{pmatrix}$$

and simplify left hand side and right hand side of inequality separately.

$$LHS = \left( 1 + \frac{n-1}{|\mu_1|} \right)^{\frac{1}{p}} \left( 1 + \frac{n-1}{|\mu_2|} \right)^{\frac{1}{q}} ... \left( 1 + \frac{n-1}{|\mu_n|} \right)^{\frac{1}{q}} \leq \left( 1 + \frac{n-1}{|\mu_1|} \right) \left( 1 + \frac{n-1}{|\mu_2|} \right) ... \left( 1 + \frac{n-1}{|\mu_n|} \right).$$

Since $2M \geq n > (n-1)$ it follows that

$$LHS < \left( 1 + \frac{2M}{|\mu_1|} \right) \left( 1 + \frac{2M}{|\mu_2|} \right) ... \left( 1 + \frac{2M}{|\mu_n|} \right)$$

But

$$|\mu_i| \leq \sqrt{2M} \leq 2M \Rightarrow 1 \leq \frac{2M}{|\mu_i|} \forall i.$$

So

$$LHS < \left( \frac{2M}{|\mu_1|} + \frac{2M}{|\mu_1|} \right) \left( \frac{2M}{|\mu_2|} + \frac{2M}{|\mu_2|} \right) ... \left( \frac{2M}{|\mu_n|} + \frac{2M}{|\mu_n|} \right)$$

$$= \left( \frac{4M}{|\mu_1|} \right) \left( \frac{4M}{|\mu_2|} \right) ... \left( \frac{4M}{|\mu_n|} \right)$$

$$= \frac{\left( 4M \right)^n}{|\det(A)|}$$
\[ \text{RHS} = \frac{1}{|\mu_2|^{\frac{1}{n}}|\mu_3|^{\frac{1}{n}} \cdots |\mu_n|^{\frac{1}{n}}} + \frac{1}{|\mu_1|^{\frac{1}{n}}|\mu_3|^{\frac{1}{n}} \cdots |\mu_n|^{\frac{1}{n}}} + \cdots + \frac{1}{|\mu_1|^{\frac{1}{n}}|\mu_2|^{\frac{1}{n}} \cdots |\mu_{n-1}|^{\frac{1}{n}}} = \frac{1}{|\det(A)|^{\frac{1}{n}}} \sum_{i=1}^{n} |\mu_i|^{\frac{1}{n}}. \]

Therefore
\[ \frac{1}{|\det(A)|^{\frac{1}{n}}} \sum_{i=1}^{n} |\mu_i|^{\frac{1}{n}} < \frac{(4M)^n}{|\det(A)|} \]

and
\[ \sum_{i=1}^{n} |\mu_i|^{\frac{1}{n}} < \frac{(4M)^n}{|\det(A)|^{(1 - \frac{1}{n})}}. \]

But
\[ \left( \sum_{i=1}^{n} |\mu_i| \right)^{\frac{1}{n}} \leq \sum_{i=1}^{n} |\mu_i|^{\frac{1}{n}}. \]

Hence
\[ \left( \sum_{i=1}^{n} |\mu_i| \right)^{\frac{1}{n}} < \frac{(4M)^n}{|\det(A)|^{\frac{n}{n(n-1)}}} \]

and
\[ \mathcal{E}_D(G) < \frac{(4M)^n^2}{|\det(A)|^{(n-1)}}. \]

To get lower bound we apply Holder’s inequality using the substitution

\[
\begin{pmatrix}
  x_{11} & x_{12} & \cdots & x_{1n} \\
  x_{21} & x_{22} & \cdots & x_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{n1} & x_{n2} & \cdots & x_{nn}
\end{pmatrix} =
\begin{pmatrix}
  |\mu_1| & |\mu_1| & \cdots & |\mu_1| \\
  |\mu_2| & |\mu_2| & \cdots & |\mu_2| \\
  \vdots & \vdots & \ddots & \vdots \\
  |\mu_n| & |\mu_n| & \cdots & |\mu_n|
\end{pmatrix}.
\]

\[(n|\mu_1|)^{\frac{1}{n}} + (n|\mu_2|)^{\frac{1}{n}} + \cdots + (n|\mu_n|)^{\frac{1}{n}} \geq n(|\mu_1||\mu_2| \cdots |\mu_n|)^{\frac{1}{n}}.\]

\[|\mu_1|^{\frac{1}{n}} + |\mu_2|^{\frac{1}{n}} + \cdots + |\mu_n|^{\frac{1}{n}} \geq n^{\frac{n-1}{n}} (|\det(A)|)^{\frac{1}{n}} \]

But \(|\mu_1| + |\mu_2| + \cdots + |\mu_n| | \geq |\mu_1|^{\frac{1}{n}} + |\mu_2|^{\frac{1}{n}} + \cdots + |\mu_n|^{\frac{1}{n}}. \) Therefore
\[ \mathcal{E}_D(G) \geq n^{\frac{n-1}{n}} |\det(A)|^{\frac{1}{n}}. \]

Combining above bounds we have, \( n^{\frac{n-1}{n}} |\det(A)|^{\frac{1}{n}} \leq \mathcal{E}_D(G) < \frac{(4m)^n^2}{|\det(A)|^{(n-1)}}. \) \qed
2.3. Lower and upper bound for distance energy of graph.

**Theorem 2.6.** Let $G$ be a graph with $n \geq 2$ vertices and $m$ edges with $2M \geq n$, then

$$
E_D(G) \geq \frac{2M}{n} + \left( \frac{|\text{det}(A)|}{2M} \right)^{\frac{1}{n-1}}. 
$$

**Proof.** Apply arithmetic mean and geometric mean inequality to real numbers $|\mu_2|, |\mu_3|, \ldots, |\mu_n|$ for $(n-1)$ terms,

$$
\frac{|\mu_2| + |\mu_3| + \cdots + |\mu_n|}{n-1} \geq |\mu_2 \mu_3 \cdots \mu_n|^{\frac{1}{n-1}}.
$$

So,

$$
E(G) - |\mu_1| \geq \frac{|\mu_1 \mu_2 \cdots \mu_n|^{\frac{1}{n-1}}}{|\mu_1|^{\frac{1}{n-1}}}. 
$$

And

$$
E_D(G) \geq |\mu_1| + \left( \frac{|\text{det}(A)|}{|\mu_1|} \right)^{\frac{1}{n-1}}. 
$$

Let $|\mu_1| = x$ and $\Psi(x) = x + \left( \frac{|\text{det}(A)|}{x} \right)^{\frac{1}{n-1}}$. We shall minimize the function by finding $\Psi'(x)$ and $\Psi''(x)$. At maxima or minima $\Psi'(x) = 0$ which gives

$$
1 - \frac{|\text{det}(A)|}{(n-1)x^{\frac{n-2}{n}}} = 0. 
$$

Thus the function $\Psi(x)$ attains maxima or minima at $x = \frac{|\text{det}(A)|^{\frac{1}{n-1}}}{(n-1)^{\frac{n-2}{n}}}$. At this point, $\Psi''(x) = \frac{n}{(n-1)^2} |\text{det}(A)|^{\frac{1}{n-1}} x^{\frac{1-2n}{n}} \geq 0$. This means the function attains the minimum value at this point. The minimum value is

$$
\Psi\left( \frac{|\text{det}(A)|^{\frac{1}{n-1}}}{(n-1)^{\frac{n-2}{n}}} \right) = \frac{n|\text{det}(A)|^{\frac{1}{n}}}{(n-1)^{\frac{n-1}{n}}}. 
$$

But the function is increasing in the interval $\left[ \frac{|\text{det}(A)|^{\frac{1}{n}}}{(n-1)^{\frac{n-2}{n}}} \right] \leq |\text{det}(A)|^{\frac{1}{n}} \leq \frac{2M}{n} \leq |\mu_1| \leq \sqrt{2M}$.

$$
E_D(G) \geq \Psi\left( \frac{2M}{n} \right). 
$$

$$
E_D(G) \geq \frac{2M}{n} + \left( \frac{|\text{det}(A)|}{2M} \right)^{\frac{1}{n-1}}. 
$$

□
Theorem 2.7. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges with $2M \geq n$ then

$$E_D(G) \geq \frac{2M}{n} + \frac{(n-2)^{\frac{3}{2}}|\text{det}(A)|^{\frac{n-1}{n}}}{}{n^{\frac{1}{2}}}.$$ (9)

**Proof.** Apply arithmetic mean and geometric mean inequality to real numbers $|\mu_2|, |\mu_3|, \ldots, |\mu_{n-1}|$ for $(n-2)$ terms,

$$\frac{|\mu_2| + |\mu_3| + \cdots + |\mu_{n-1}|}{n-2} \geq |\mu_2\mu_3 \cdots \mu_{n-1}|^{\frac{1}{n-2}}.$$

So,

$$E(G) - |\mu_1| - |\mu_n| \geq \frac{|\mu_1\mu_2 \cdots \mu_n|^{\frac{1}{n}}}{}{|\mu_1\mu_n|^{\frac{1}{n-2}}}.$$

$$E_D(G) \geq |\mu_1| + |\mu_n| + \frac{|\text{det}(A)|^{\frac{1}{n}}}{}{|\mu_1\mu_n|^{\frac{1}{n-2}}}.$$ (9)

Let $|\mu_1| = x, |\mu_n| = y$ and $g(x, y) = x + y + \frac{|\text{det}(A)|^{\frac{1}{n}}}{}{(xy)^{\frac{1}{2}}}$. Using partial differentiation we minimize the function by finding $g_x(x, y), g_y(x, y), g_{xx}(x, y), g_{yy}(x, y), g_{xy}(x, y)$ and $\Delta = g_{xx}g_{yy} - g_{xy}^2$.

$$g_x = 1 - \frac{|\text{det}(A)|^{\frac{1}{n}}}{}{n-2} (xy)^{\frac{1-n}{2}} y,$$

$$g_y = 1 - \frac{|\text{det}(A)|^{\frac{1}{n}}}{}{n-2} (xy)^{\frac{1-n}{2}} x,$$

$$g_{xx} = \frac{y^2(1-n)|\text{det}(A)|^{\frac{1}{n}}}{}{(n-2)^2} (xy)^{\frac{3-2n}{2}},$$

$$g_{yy} = \frac{x^2(1-n)|\text{det}(A)|^{\frac{1}{n}}}{}{(n-2)^2} (xy)^{\frac{3-2n}{2}},$$

$$g_{xy} = -\frac{|\text{det}(A)|^{\frac{1}{n}}}{}{n-2} \left((xy)^{\frac{1-n}{2}} + y \frac{n-1}{n-2} (xy)^{\frac{3-2n}{2}}\right).$$

$$\Delta = \frac{(xy)^2(1-n)^2|\text{det}(A)|^{\frac{2}{n}}}{}{(n-2)^4} (xy)^{\frac{6-4n}{2}} - \frac{|\text{det}(A)|^{\frac{2}{n}}}{}{(n-2)^2} \left((xy)^{\frac{1-n}{2}} + y \frac{n-1}{n-2} (xy)^{\frac{3-2n}{2}}\right)^2.$$

At maxima or minima $g_x = 0, g_y = 0$ which gives

$$(xy)^{\frac{1-n}{2}} y = \frac{n-2}{|\text{det}(A)|^{\frac{1}{n}}}.$$
and

\[(xy)^{\frac{n-2}{n^2}} x = \frac{n-2}{n \cdot |\text{det}(A)|^{\frac{n-2}{n^2}}}.\]

Solving these equations gives

\[x = y = \frac{|\text{det}(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n^2}}}.\]

Thus the function \(g(x, y)\) attains maxima or minima at

\[x = y = \frac{|\text{det}(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n^2}}}.\]

At this point, \(g_{xx}\) and \(g_{yy}\) are greater than equal to zero. Further \(\Delta \leq 0\). This means that the function attains the minimum value at this point. The minimum value is given by,

\[g\left(\frac{|\text{det}(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n^2}}}, \frac{|\text{det}(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n^2}}}\right).\]

Since \(2M \geq n\), \(g(x, y)\) increases in the interval

\[|\text{det}(A)|^{\frac{1}{n}} \leq \frac{2M}{n} \leq x \leq \sqrt{2M}\]

and

\[0 \leq y \leq |\text{det}(A)|^{\frac{1}{n}} \leq \frac{2M}{n} \leq \sqrt{2M}.\]

At

\[y = \frac{|\text{det}(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n^2}}},\]

\[g(x, y) = x + \frac{(n-2)^{\frac{n-2}{n}} |\text{det}(A)|^{\frac{n-1}{(n-2)(n-2)}}}{x^{\frac{n-2}{n^2}}}.\]

Therefore,

\[\mathcal{E}_D(G) \geq g\left(x, \frac{|\text{det}(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n^2}}}\right) \geq g\left(\frac{2M}{n}, \frac{|\text{det}(A)|^{\frac{1}{n}}}{(n-2)^{\frac{n-2}{n^2}}}\right).\]

Hence,

\[\mathcal{E}_D(G) \geq \frac{2M}{n} + \frac{(n-2)^{\frac{n-2}{n}} |\text{det}(A)|^{\frac{n-1}{(n-2)(n-2)}}}{(2M/n)^{\frac{n-2}{n^2}}}.\]

\[\square\]

**Theorem 2.8.** Let \(G\) be a graph with \(n \geq 2\) vertices, \(m\) edges and \(G\) is a non-singular graph then

\[\mathcal{E}_D(G) \leq \sqrt{2M} + \frac{(n-1)(2M)}{|\text{det}(A)|^{\frac{1}{n}}}.\] (10)
Proof. We know that $|\mu_1| \geq |\det(A)|^{\frac{1}{n}}$, which implies

$$|\mu_1| \sum_{i=2}^{n} |\mu_i| \geq |\det(A)|^{\frac{1}{n}} \sum_{i=2}^{n} |\mu_i|.$$ 

Since $|\mu_i| \leq |\mu_1| \forall i$, therefore

$$(n-1)|\mu_1|^2 \geq |\det(A)|^{\frac{1}{n}} \left( \mathcal{E}(G) - |\mu_1| \right).$$

Thus

$$\mathcal{E}_D(G) \leq |\mu_1| + \frac{(n-1)|\mu_1|^2}{|\det(A)|^{\frac{1}{n}}}. $$

Let $|\mu_1| = x$ and $f(x) = x + \frac{(n-1)x^2}{|\det(A)|^{\frac{1}{n}}}$. At maxima or minima $f'(x) = 0$ which gives

$$1 + \frac{(n-1)2x}{|\det(A)|^{\frac{1}{n}}} = 0.$$ 

Hence the function attains maximum or minimum value at

$$x = -\frac{|\det(A)|^{\frac{1}{n}}}{2(n-1)}.$$ 

Since $f''(x) = \frac{2(n-1)}{|\det(A)|^{\frac{1}{n}}} > 0$ the function attains minimum value at this point.

The minimum value

$$f\left( -\frac{|\det(A)|^{\frac{1}{n}}}{2(n-1)} \right) = -\frac{|\det(A)|^{\frac{1}{n}}}{2(n-1)} + \frac{|\det(A)|^{\frac{1}{n}}}{4(n-1)} = -\frac{|\det(A)|^{\frac{1}{n}}}{4(n-1)}.$$ 

But $f(x)$ is an increasing function in the region $-\frac{|\det(A)|^{\frac{1}{n}}}{2(n-1)} \leq x \leq \sqrt{2M}$. Hence $f(x) \leq f(\sqrt{2M})$. Therefore

$$\mathcal{E}_D(G) \leq \sqrt{2M} + \frac{(n-1)(2M)}{|\det(A)|^{\frac{1}{n}}}.$$ 

\[\square\]

3. CONCLUDING REMARKS

In this paper, an effort has been made to obtain new bounds for distance energy of graph in a simplest way. Are these lower and upper bounds better than Koolen-Moulton and McClelland bounds (1.1 and 1.2)? It is yet to proved and is a scope for further research.
REFERENCES


