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CR-WARPED PRODUCT SUBMANIFOLDS OF SASAKIAN MANIFOLDS ADMITTING CERTAIN CONNECTIONS

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Abstract. The existence or non-existence of a warped product CR-submanifolds of Sasakian manifolds admitting certain connections have been studied. Among others, Ricci solitons of such notions have been investigated.

Key words and Phrases: Sasakian manifold, CR-warped product, semisymmetric metric connection, semisymmetric non-metric connection.

Abstract. Dalam paper ini, dikaji eksistensi atau non-eksistensi dari suatu hasil kali *warped* CR-submanifold dari manifold Sasaki dengan koneksi tertentu. Lebih jauh, dikaji juga *Ricci soliton* dari hasil kali tersebut.

Kata Kunci: Manifold Sasaki, hasil kali *CR-warped*, koneksi metrik semisimetrik, koneksi non- metrik semisimetrik.

1. INTRODUCTION

The semisymmetric connection was introduced in [8]. If such connection is metric compatible then it is called semisymmetric metric connection otherwise semisymmetric non-metric connection. For details we may refer [1], [11],[13], [15], [16], [17], [18].

The concept of CR-submanifold was introduced in [3]. It is the generalization of invariant and anti-invariant submanifold. For detail study, see [2], [6], [10], [12].

Warped product is a generalization of Riemannian product introduced in [4].

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Its existence and non-existence are very much significant. Here, such existence and non-existence are studied in case of CR-submanifolds of Sasakian manifolds.

The Ricci soliton on a Riemannian manifold M with Riemannian metric g is a triplet (g, V, λ) such that

$$\frac{1}{2}\pounds_V g + S + \lambda g = 0. \tag{1}$$

Here S is the Ricci tensor of M. $\pounds_V$ denotes the Lie derivative operator along $V \in \chi(M)$ and $\lambda \in \mathbb{R}$ [9]. Nature of such soliton depends on λ. Such soliton have been studied in section 4 in case of CR-warped product submanifold of Sasakian manifold.

We summarize all the obtained results of this paper in section 5 in tabular form.
Throughout the paper we denote:

(i) Ricci Tensor by "RT",
(ii) Warped product by "WP",
(iii) Semisymmetric metric connection by "SM",
(iv) Semisymmetric non-metric connection by "SNM",
(v) Levi-Civita connection by "LC",
(vi) Almost contact manifold "AC".

2. BACKGROUND

An odd dimensional C^∞ manifold $\bar{M}^{2n+1}(n > 1)$ is called an AC [5] if for all $X \in \chi(\bar{M})$

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{2}$$

where ϕ is a tensor field of type $(1,1)$ and ξ is a vector field (known as characteristic vector field) and η is an 1-form such that $\eta(X) = g(X, \xi)$. From (2) we get

$$\phi\xi = 0 \qquad and \qquad \eta(\phi X) = 0. \tag{3}$$

If M^{2n+1} if it admits a g such that

$$g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y) \tag{4}$$

then it is called almost contact metric manifold [5], denoted by $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$. From (4) we get

$$g(X, \phi Y) = -g(\phi X, Y). \tag{5}$$

Now $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$ is called Sasakian [5] if

$$(\bar{\nabla}_X \phi)(Y) + g(X,Y)\xi = \eta(Y)X, \tag{6}$$

$$\bar{\nabla}_Y \xi = \phi Y \tag{7}$$

$\forall X, Y \in \chi(\bar{M})$ and $\bar{\nabla}$ is the LC. Throughout the paper we denote Sasakian manifold of dimension $(2n+1)$ by $\bar{M}$. The correspondence between SM $\bar{\tilde{\nabla}}$ and SNM $\overset{*}{\bar{\nabla}}$ with $\bar{\nabla}$ are

$$\bar{\tilde{\nabla}}_X Y = \bar{\nabla}_X Y + \eta(Y)X - g(X,Y)\xi, \tag{8}$$

$$\overset{*}{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \eta(Y)X \tag{9}$$

respectively.

Let M be a submanifold of $\bar{M}$ and ∇, $\widetilde{\nabla}$ and $\overset{*}{\nabla}$ are respectively the induced connection of M from the Riemannian connection $\bar{\nabla}$, semisymmetric connection $\widetilde{\bar{\nabla}}$ and $\overset{*}{\bar{\nabla}}$. Then Gauss formula with respect to above three connections are given by

$$\bar{\nabla}_X Y - \nabla_X Y = h(X,Y), \tag{10}$$

$$\widetilde{\bar{\nabla}}_X Y - \widetilde{\nabla}_X Y = \widetilde{h}(X,Y), \tag{11}$$

and

$$\overset{*}{\bar{\nabla}}_X Y - \overset{*}{\nabla}_X Y = \overset{*}{h}(X,Y). \tag{12}$$

Here h(resp. $\widetilde{h}, \overset{*}{h}$) is the second fundamental form admitting ∇(resp. $\widetilde{\nabla}$, $\overset{*}{\nabla}$). Again M tangent to ξ is called [3]

(i) invariant if $\phi(T_pM) \subset T_pM$,
(ii) anti-invariant if $\phi(T_pM) \subset T_p^{\perp}M$,
(iii) CR if $\exists$ two orthogonal distribution D and $D^{\perp}$ so that D is invariant and $D^{\perp}$ is anti-invariant and $TM = D \oplus D^{\perp} \oplus <\xi> \forall\, p \in M$.

The WP [7] of two RM (N_1, g_1) and (N_2, g_2) is the RM $N_1 \times_f N_2 = (N_1 \times N_2, g)$, $g = g_1 + f^2 g_2$; f is the positive definite smooth function on N_1.

If $M = N_1 \times_f N_2$ is a WP submanifold of $\bar{M}$ then we have [4]

$$\nabla_U X = \nabla_X U = X(lnf)U \tag{13}$$

$\forall X \in \Gamma(TN_1)$ and $U \in \Gamma(TN_2)$.

Let M be of the form $N_1 \times_f N_2$ admitting $\widetilde{\nabla}$. Here N_1 and N_2 are orthogonal. If $\xi \in \Gamma(TN_1)$ then

$$\widetilde{\nabla}_X U = X(lnf)U \quad and \quad \widetilde{\nabla}_U X = X(lnf)U + \eta(X)\phi U \tag{14}$$

$\forall X \in \Gamma(TN_1)$ and $U \in \Gamma(TN_2)$. If $\xi \in \Gamma(TN_2)$ then we have [14]

$$\widetilde{\nabla}_X U = X(lnf)U + \eta(U)X \quad and \quad \widetilde{\nabla}_U X = X(lnf)U. \tag{15}$$

Plugging (8) and (10) in (11) yields

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi \tag{16}$$

$$\widetilde{h}(X,Y) = h(X,Y) \tag{17}$$

WP admitting SNM have been studied in [14]. Let $M = N_1 \times_f N_2$ is a WP submanifold of $\bar{M}$ admitting SNM, where N_1 and N_2 are orthogonal.

If $\xi \in \Gamma(TN_1)$ then we have [14]

$$\overset{*}{\nabla}_X U = X(lnf)U \quad and \quad \overset{*}{\nabla}_U X = X(lnf)U + \eta(X)\phi U \tag{18}$$

$\forall X \in \Gamma(TN_1)$ and $U \in \Gamma(TN_2)$. If ξ is tangent to N_2 then we have [14]

$$\overset{*}{\nabla}_X U = X(lnf)U + \eta(U)X \quad and \quad \overset{*}{\nabla}_U X = X(lnf)U. \tag{19}$$

Then by virtue of (9) and (10) we have from (12) that

$$\overset{*}{\nabla}_X Y - \nabla_X Y = \eta(Y)X \tag{20}$$

$$\overset{*}{h} = h. \tag{21}$$

3. CR WARPED PRODUCT SUBMANIFOLD

Let $M = N_1 \times_f N_2$ is a WP CR-submanifold of $\bar{M}$ admitting $\tilde{\tilde{\nabla}}$. It may be noted that such submanifold are also tangent to ξ. So two cases may arise:
case I: $\xi \in \Gamma(TN_1)$
case II: $\xi \in \Gamma(TN_2)$.

Again, case I has two subcases:
(i) N_1 invariant and N_2 anti-invariant submanifold of $\bar{M}$;
(ii) N_1 anti-invariant and N_2 invariant submanifold of $\bar{M}$.

Now for the subcase (i), we state the following:

Theorem 3.1. *There exist WP submanifold $M = N_1 \times_f N_2$ of $\bar{M}$ with respect to $\tilde{\tilde{\nabla}}$ so that N_1 is invariant and N_2 is anti-invariant and $\xi \in \Gamma(TN_1)$.*

Proof. Let $M = N_1 \times_f N_2$ is the given submanifold of $\bar{M}$ with respect to $\tilde{\nabla}$ such that N_1 is invariant and N_2 is anti-invariant and $\xi \in \Gamma(TN_1)$. Then from (14)

$$\tilde{\nabla}_U \xi = \xi(lnf)U + \phi U. \tag{22}$$

Out of (16) one can get

$$\begin{aligned} \tilde{\nabla}_U \xi &= \phi U + U - \eta(U)\xi \\ &= \phi U + U. \end{aligned} \tag{23}$$

From (22) and (23) we get $\xi(lnf)U = U$ i.e., $\xi(lnf) = 1$. Thus such warped product exists and hence the proof.

Now for subcase (ii), we state the following:

Theorem 3.2. *There does not exist any such submanifold $M = N_1 \times_f N_2$ of $\bar{M}$ with respect to $\tilde{\tilde{\nabla}}$ such that N_1 is anti-invariant and N_2 is invariant and $\xi \in \Gamma(TN_1)$.*

Proof. By virtue of (8) from (6) we have

$$(\tilde{\tilde{\nabla}}_X \phi)Y + g(X,Y)\xi + g(X,\phi Y)\xi = \eta(Y)X - \eta(Y)\phi(X) \tag{24}$$

for all $X, Y \in \chi(M)$. Now if $\xi \in \Gamma(TN_1)$ and $U \in \Gamma(TN_2)$ then by virtue of (24) we get

$$(\tilde{\nabla}_U \phi)\xi = U - \phi U \tag{25}$$

and

$$(\widetilde{\nabla}_{\xi}\phi)U = 0. \tag{26}$$

Hence

$$(\widetilde{\nabla}_{U}\phi)\xi + (\widetilde{\nabla}_{\xi}\phi)U = U - \phi U. \tag{27}$$

Again we know

$$(\widetilde{\nabla}_{X}\phi)Y = \widetilde{\nabla}_{X}\phi Y - \phi(\widetilde{\nabla}_{X}Y). \tag{28}$$

On account of (28) and (14) it is found,

$$(\widetilde{\nabla}_{U}\phi)\xi + (\widetilde{\nabla}_{\xi}\phi)U = \xi(lnf)\phi U. \tag{29}$$

Hence from (27) and (29) we get the result.

We now consider case II. In that case two sub cases also arises in similar to case I. For the sub case (i) we have:

Theorem 3.3. *There does not exist any such submanifold $M = N_1 \times_f N_2$ of $\bar{M}$ with respect to $\bar{\widetilde{\nabla}}$ such that N_1 is invariant and N_2 is anti-invariant and $\xi \in \Gamma(TN_2)$.*

Proof. Let us assume that $M = N_1 \times_f N_2$ is a warped product submanifold of $\bar{M}$ with respect to $\bar{\widetilde{\nabla}}$ such that $\xi \in \Gamma(TN_2)$. Then from (15) we get

$$\widetilde{\nabla}_{X}\xi = X(lnf)\xi + X. \tag{30}$$

From (16) we obtain

$$\bar{\widetilde{\nabla}}_{X}\xi = \phi X + X. \tag{31}$$

From (30) and (31) it ensures that

$$X(lnf)\xi = \phi X. \tag{32}$$

Now taking inner product with respect to ξ one can get:

$$X(lnf)g(\xi,\xi) = \eta(\phi X) = 0 \Rightarrow X(lnf) = 0.$$

So f is constant and the case is trivial. Hence the warped product does not exist.

Now for subcase (ii), we state the following:

Theorem 3.4. *There does not exist any such submanifold $M = N_1 \times_f N_2$ of $\bar{M}$ with respect to $\bar{\widetilde{\nabla}}$ such that N_1 is anti-invariant and N_2 is invariant and $\xi \in \Gamma(TN_2)$.*

Proof. If $\xi \in \Gamma(TN_2)$, $X \in \Gamma(TN_1)$, then by virtue of (24) we obtain

$$(\widetilde{\nabla}_{X}\phi)\xi = X - \eta(X)\xi - \phi X \tag{33}$$

$$(\widetilde{\nabla}_{\xi}\phi)X = 0. \tag{34}$$

Hence from (33) and (34) we get

$$(\widetilde{\nabla}_{X}\phi)\xi + (\widetilde{\nabla}_{\xi}\phi)X = X - \phi X - \eta(X)\xi. \tag{35}$$

Again, we know that

$$(\widetilde{\nabla}_X \phi)Y = \widetilde{\nabla}_X \phi Y - \phi(\widetilde{\nabla}_X Y).$$

So, in view of (15) from the above expression we get

$$(\widetilde{\nabla}_\xi \phi)X + (\widetilde{\nabla}_X \phi)\xi = (\phi X lnf)\xi - \phi X. \tag{36}$$

Now, from (35) and (36) we have

$$(\phi X lnf)\xi = X - \eta(X)\xi, \tag{37}$$

which implies:

$$\begin{aligned}(\phi X lnf)g(\xi,\xi) &= g(X,\xi) - \eta(X)g(\xi,\xi)\\ \Rightarrow \phi X lnf &= 0\end{aligned}$$

So f is constant and the warped product does not exist. Which completes the proof.

Remark 1: For the warped product CR-submanifold M of $\bar{M}$ with respect to Levi-Civita connection [10], semisymmetric non-metric connection, all the results remain the same.

4. RICCI SOLITON

Theorem 4.1. *If (g, ξ, λ) is a RS on a warped product submanifold $M = N_1 \times N_2$ of $\bar{M}$ with respect to $\bar{\widetilde{\nabla}}$ such that N_1 invariant, N_2 anti-invariant and $\xi \in \Gamma(TN_1)$ then N_2 is nearly quasi Einstein.*

Proof. Let (g, ξ, λ) be a Ricci soliton on a warped product CR-submanifold M of $\bar{M}$. Then we have:

$$(\widetilde{\mathcal{L}}_\xi g)(X,Y) + 2\widetilde{S}(X,Y) + 2\lambda g(X,Y) = 0 \tag{38}$$

for all $X, Y \in M$, where $\widetilde{S}$ is the Ricci tensor of M admitting $\widetilde{\nabla}$ and $\mathcal{L}_\xi$ is the Lie derivative along ξ on M.
From (23) we get:

$$\widetilde{\nabla}_U \xi = U + \phi U \tag{39}$$

where ξ is tangent to N_1 and $U \in N_2$. Therefore for $U, V \in N_2$ we have

$$(\widetilde{\mathcal{L}}_\xi g)(U,V) = g(U + \phi U, V) + g(U, V + \phi V) = 2g(U,V). \tag{40}$$

Hence from (38) and (40) we get:

$$\widetilde{S}(U,V) + (\lambda + 1)g(U,V) = 0. \tag{41}$$

Again, using (16) we can compute that

$$\widetilde{S}(Y,Z) = S(Y,Z) + (2m-1)[\eta(Y)\eta(Z) - g(Y,Z) - g(\phi Y, Z)] \tag{42}$$

for all $Y, Z \in \Gamma(TM)$.
Hence by virtue of (42) we have from (41) that

$$S(U,V) = (2m-2-\lambda)g(U,V) + (2m-1)g(\phi U, V) \tag{43}$$

for all $U, V \in \Gamma(TN_2)$, so N_2 is nearly quasi Einstein.

Theorem 4.2. *If (g, V, λ) is a RS on a WP submanifold M of $\bar{M}$ with respect to $\overset{*}{\nabla}$ such that $\xi \in \Gamma(TN_1)$ then N_2 is Einstein.*

Proof. For the connection $\overset{*}{\nabla}$ we have from (18) we get

$$\overset{*}{\nabla}_U \xi = U + \phi U. \tag{44}$$

Therefore by virtue of (44) we have

$$\begin{aligned} (\overset{*}{\pounds}_\xi g)(U,V) &= g(U+\phi U, V) + g(U, V+\phi V) \\ &= 2g(U,V) \end{aligned} \tag{45}$$

for all $U, V \in \Gamma(TN_2)$. Since (g, ξ, λ) is a Ricci soliton, we get

$$(\overset{*}{\pounds}_\xi g)(U,V) + 2\overset{*}{S}(U,V) + 2\lambda g(U,V) = 0. \tag{46}$$

Again using (20) we can obtain that

$$\overset{*}{S}(U,V) = S(U,V) + 2m[g(U,V) + \eta(U)\eta(V)]. \tag{47}$$

So, by virtue of (47) from (46) we get

$$S(U,V) = -(2m+\lambda+1)g(U,V) \tag{48}$$

for all $U, V \in \Gamma(TN_2)$, so N_2 is Einstein.

5. CONCLUSION

A comparative study on the existence and non-existence of WP CR-submanifold of Sasakian manifold with respect to SM and SNM is considered. The results with respect to LC in this context was studied in [10]. Here we summarize the facts in next two theorems. From Theorem 3.1- 3.4 and the Remark 1, we can state that:

Theorem 5.1. *Let $M = N_1 \times_f N_2$ be a WP CR-subamnifold of $\bar{M}$. Then we have:*

connection	***nature of*** ξ	***nature of*** N_1 ***and*** N_2	***existance/non-existance***
Levi-Civita	*tangent to* N_1	N_1 *invariant,* N_2 *anti-invariant*	*warped product exist*
	tangent to N_1	N_1 *anti-invariant,* N_2 *invariant*	*warped product does not exist*
	tangent to N_2	N_1 *invariant,* N_2 *anti-invariant*	*warped product does not exist*
	tangent to N_2	N_1 *anti-invariant,* N_2 *invariant*	*warped product does not exist*
semisymmetric metric	*tangent to* N_1	N_1 *invariant,* N_2 *anti-invariant*	*warped product exist*
	tangent to N_1	N_1 *anti-invariant,* N_2 *invariant*	*warped product does not exist*
	tangent to N_2	N_1 *invariant,* N_2 *anti-invariant*	*warped product does not exist*
	tangent to N_2	N_1 *anti-invariant,* N_2 *invariant*	*warped product does not exist*
semisymmetric non-metric	*tangent to* N_1	N_1 *invariant,* N_2 *anti-invariant*	*warped product exist*
	tangent to N_1	N_1 *anti-invariant,* N_2 *invariant*	*warped product does not exist*
	tangent to N_2	N_1 *invariant,* N_2 *anti-invariant*	*warped product does not exist*
	tangent to N_2	N_1 *anti-invariant,* N_2 *invariant*	*warped product does not exist*

From Theorem 4.1 and 4.2 following theorem can be stated:

Theorem 5.2. *If* (g, ξ, λ) *is a RS of WP submanifold* $M = N_1 \times_f N_2$ *of* $\bar{M}$ *such that* N_1 *invariant,* N_2 *anti-invariant submanifold of* $\bar{M}$ *and* $\xi \in \Gamma(TN_1)$*. Then we have:*

connection on $\bar{M}$	N_2
∇	*Einstein*
$\tilde{\nabla}$	*nearly quasi Einstein*
$\overset{*}{\nabla}$	*Einstein*

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