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SG_C-PROJECTIVE, INJECTIVE AND FLAT MODULES

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Abstract. In this paper we introduce the concepts of SG_C-projective, injective and flat modules, where C is a semidualizing module and we discuss some connections among SG_C-projective, injective and flat modules.

Key words and Phrases: semidualizing module, SG_C-projective, injective and flat module

Abstrak. Pada paper ini diperkenalkan konsep tentang modul SG_C-projektif, injektif dan flat, dengan C adalah modul semidual, dan dibahas juga beberapa hubungan antara modul SG_C-projektif, injektif dan flat.

Kata kunci: modul semidual, modul SG_C-projektif, injektif dan flat

1. INTRODUCTION

Throughout this paper, R is a commutative ring with identity and R-Mod denotes the category of all R-modules . For an R-module M, we use M^+ to denote the character module $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M.

In relative homological algebra, the notions of Gorenstein-projective (resp. injective and flat) modules play a fundamental role. Bennis et al. in [2] introduced the notion of SG-projective (resp. injective and flat) modules. Over a commutative Noetherian ring, Holm and Jørgensen in [9] introduced the C-Gorenstein projective and C-Gorenstein injective modules using semidualizing modules and their associated projective, injective classes. White in [14] further considered these

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modules when R is a commutative ring and she called C-Gorenstein projective as G_C-projective and C-Gorenstein injective as G_C-injective. In particular, many general results about the Gorenstein projectivity and Gorenstein injectivity in [5, 8] were generalized in [14]. Thus it is natural to ask the following question, What are the counterparts to SG-projective, SG-injective and SG-flat modules with respect to the semidualizing modules?

In this paper, we shall introduce the notions of SG_C-projective, SG_C-injective and SG_C-flat modules with respect to the semidualizing module C, which answer the question above. Also some properties of SG_C-projective, SG_C-injective and SG_C-flat modules are discussed.

This paper is divided into four sections. In Section 2, we recall some known definitions that are needed in the sequel. In Section 3, we introduce and study the SG_C-projective and injective modules. Also, we prove that the class of all SG_C projective modules is projectively resolving and closed under direct sums. We then give some equivalent characterizations of the SG_C-projective and SG_C-injective modules.

In Section 4, we introduce SG_C-flat modules and give their characterizations. Further, we study the relation between SG_C-projective, injective, and flat modules. Moreover, if R is a noetherian ring, we prove every direct limit of finitely generated SG_C-flat R-modules is SG_C-flat.

2. PRELIMINARIES

In this section, we first recall some known definitions and terminologies which we need in the sequel.

Definition 2.1. [14] *An R-module C is semidualizing if it satisfies the following conditions:*

1. *C admits a degreewise finite projective resolution,*
2. *The natural homothety morphism $R \to Hom_R(C, C)$ is an isomorphism, and*
3. *$Ext^i_R(C, C) = 0$ for any $i \geq 1$.*

A free R-module of rank one is semidualizing. If R is Noetherian and admits a dualizing module D, then D is a semidualizing.
From now on, C is a semidualizing R-module.

Definition 2.2. [10] *An R-module is C-projective if it has the form $C \otimes_R P$ for some projective module P. An R-module is called C-injective if it has the form $Hom_R(C, I)$ for some injective module I. Set*

$$\mathcal{P}_C(R) = \{C \otimes_R P \,|\, P\, is\ R - projective\},$$

and

$$\mathcal{I}_C(R) = \{Hom_R(C, I) \,|\, I\, is\ R - injective\}.$$

Definition 2.3. [10] *An R-module is called C-flat if it has the form $C \otimes_R F$ for some flat module F. Set $\mathcal{F}_C(R) = \{C \otimes_R F \,|\, F$ is R-flat$\}$.*

Definition 2.4. [9] *(1). An R-module M is called G_C-projective if there exists a complete PC-resolution of M, which means that*

$$\boldsymbol{PC} : \cdots \to P_1 \to P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

is an exact complex such that $M \cong Coker(P_1 \to P_0)$ and each P_i and P^i is projective and such that the complex $Hom_R(\boldsymbol{PC}, C \otimes_R Q)$ is exact for every projective R-module Q.
(2). M is called G_C-injective if there exists a complete IC-resolution of M, which means that

$$\boldsymbol{IC} : \cdots \to Hom_R(C, I_1) \to Hom_R(C, I_0) \to I^0 \to I^1 \to \cdots$$

is an exact complex such that $M \cong Ker(I^0 \to I^1)$ and each I_i and I^i is injective and such that the complex $Hom_R(Hom_R(C, E), \boldsymbol{IC})$ is exact for every injective R-module E.
(3). M is called G_C-flat if there exists a complete FC-resolution of M, which means that

$$\boldsymbol{FC} : \cdots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots$$

is an exact complex such that $M \cong Coker(F_1 \to F_0)$ and each F_i and F^i is flat and such that the complex $Hom_R(C, E) \otimes_R \boldsymbol{FC}$ is exact for every injective R-module E.

Definition 2.5. [8] *Let $\mathcal{X}$ be a class of R-modules. Then we call $\mathcal{X}$ is projectively (resp. injectively) resolving if it contains all projective (resp. injective) R-modules and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ (resp. $X' \in \mathcal{X}$) the conditions $X \in \mathcal{X}$ and $X' \in \mathcal{X}$ (resp. $X'' \in \mathcal{X}$) are equivalent.*

3. SG_C-PROJECTIVE MODULES

We start with the following definitions.

Definition 3.1. *An R-module M is called a strongly Gorenstein projective module with respect to C (for short SG_C-projective) if there exists a complete resolution of the form*

$$\boldsymbol{SP} : \cdots \to P \to P \to C \otimes_R P \to C \otimes_R P \to \cdots$$

with P a projective R-module such that $M \cong Coker(P \to P)$ and $Hom_R(\boldsymbol{SP}, C \otimes_R Q)$ is exact for any projective R-module Q.

We call the complete resolution of this type as an SPC-resolution.

Definition 3.2. *An R-module M is called a strongly Gorenstein injective module with respect to C (for short SG_C-injective) if there exists a complete resolution of the form*

$$\boldsymbol{SI} : \cdots \to Hom_R(C, I) \to Hom_R(C, I) \to I \to I \to \cdots$$

*with I an injective R-module such that $M \cong Ker(I \to I)$ and $Hom_R(Hom_R(C,E),$ **SI**$)$ is exact for any injective R-module E.*

We call the complete resolution of this type as an SIC-resolution.

We denote by $\mathcal{SGP}_C(R)$ (resp. $\mathcal{SGI}_C(R)$), the class of all strongly Gorenstein projective (resp. injective) R-modules with respect to C. The definition of SG_C projective (resp. injective) module gives the following simple characterization.

Proposition 3.3. (1) *M is SG_C-projective if and only if $Ext_R^{\geq 1}(M, C \otimes_R Q) = 0$ and there exists an SPC-coresolution of the form*

$$X = 0 \to M \to C \otimes_R P \to C \otimes_R P \to \cdots$$

with P a projective R module such that $Hom_R(X, C \otimes_R Q)$ is exact for any projective R-module Q.

(2) *M is SG_C-injective if and only if $Ext_R^{\geq 1}(Hom_R(C,E), M) = 0$ and there exists an SIC-coresolution of the form*

$$Y = \cdots \to Hom_R(C,I) \to Hom_R(C,I) \to M \to 0$$

with I an injective R-module such that $Hom_R(Hom_R(C,E), Y)$ is exact for any injective R-module E.

It is straightforward that the class of SG_C-projective (resp. SG_C-injective) modules is a special class of G_C-projective (resp. G_C-injective) modules. The following proposition shows that the class of projective and C-projective modules are special classes of SG_C-projective modules.

Proposition 3.4. *Let P be a projective R-module. Then the class $\mathcal{P}_C(R)$ of all C-projective R-modules and the class of all projective R-modules are contained in the class $\mathcal{SGP}_C(R)$.*

Proof. Let $C \otimes_R P$ be in $\mathcal{P}_C(R)$. By definition of C, it admits an augmented degreewise finite free resolution of the form

$$X = \cdots \to R^\alpha \to R^\alpha \to C \to 0,$$

and this is a complete SPC-resolution of C with $C \cong Coker(R^\alpha \to R^\alpha)$. Let Q be any projective R-module. Since $Ext_R^{\geq 1}(C,C) = 0$, we have the complex $Hom_R(X, C \otimes Q)$ is exact by [14, Lemma 1.11(b)]. Then, it follows from [14, Lemma 2.5] that the sequence

$$X \otimes_R P = \cdots \to R^\alpha \otimes_R P \to R^\alpha \otimes_R P \to C \otimes_R P \to 0,$$

is a complete resolution of $C \otimes_R P$ with $C \otimes_R P \cong Coker(R^\alpha \otimes_R P \to R^\alpha \otimes_R P)$ and the complex $Hom_R(C \otimes_R P, C \otimes_R Q)$ is exact. Therefore $C \otimes_R P$ is an SG_C-projective R-module.

Using the similar arguments we can prove that the class of projective R-modules is also contained in $\mathcal{SGP}_C(R)$. □

Proposition 3.5. *(1) If $(M_i)_{i\in I}$ is a family of SG_C-projective R-modules, then $\bigoplus M_i$ is SG_C-projective.*
(2) If $(E_i)_{i\in I}$ is a family of SG_C-injective R-modules, then $\prod E_i$ is SG_C-injective.

Proof. (1) Since each M_i is SG_C-projective, we have $Ext_R^{\geq 1}(M_i, C\otimes_R Q) = 0$ and M_i admits an SPC-coresolution

$$Y_i = 0 \to M_i \to C\otimes_R P_i \to C\otimes_R P_i \to \cdots,$$

with $Hom_R(Y_i, C\otimes Q)$ is exact for any projective Q. Now

$$Ext_R^{\geq 1}(\bigoplus_{i\in I} M_i, C\otimes_R Q) \cong \prod_{i\in I} Ext_R^{\geq 1}(M_i, C\otimes_R Q) = 0.$$

Since the class of C-projectives is closed under direct sums, we have the SPC-coresolution

$$\bigoplus\nolimits_{i\in I} Y_i = 0 \to \bigoplus\nolimits_{i\in I} M_i \to \bigoplus\nolimits_{i\in I}(C\otimes_R P_i) \to \bigoplus\nolimits_{i\in I}(C\otimes_R P_i) \to \cdots,$$

$$\text{i.e. } \bigoplus\nolimits_{i\in I} Y_i = 0 \to \bigoplus\nolimits_{i\in I} M_i \to C\otimes_R (\bigoplus\nolimits_{i\in I} P_i) \to C\otimes_R (\bigoplus\nolimits_{i\in I} P_i) \to \cdots$$

with $Hom_R(\bigoplus_{i\in I} Y_i, C\otimes_R Q) \cong \prod_{i\in I} Hom_R(Y_i, C\otimes_R Q)$ exact. Therefore, the class of SG_C-projective modules is closed under direct sums by Proposition 3.3(1).

(2) The proof of this is similar to that of (1). □

The next result gives a simple characterization of the SG_C-projective modules, which is analogous to [2, Proposition 2.9].

Proposition 3.6. *For any R-module M, the following are equivalent:*

(1) *M is SG_C-projective;*
(2) *There exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is a projective module, and $Ext_R^1(M, C\otimes_R Q) = 0$ for any projective R-module Q;*
(3) *There exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is a projective module, and $Ext_R^1(M, Q') = 0$ for any module Q' with finite C-projective dimension;*
(4) *There exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is a projective module; such that, for any projective module Q, the short exact sequence $0 \to Hom_R(M, C\otimes_R Q) \to Hom_R(P, C\otimes_R Q) \to Hom_R(M, C\otimes_R Q) \to 0$ is exact;*
(5) *There exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is a projective module; such that, for any module Q' with finite C-projective dimension, the short exact sequence $0 \to Hom_R(M, Q') \to Hom_R(P, Q') \to Hom_R(M, Q') \to 0$ is exact.*

Proof. Using the standard arguments, this follows immediately from the definition of SG_C-projective module. □

Remark 3.7. *We can also characterize the SG_C-injective modules in a similar way to the description of SG_C-projective modules in Proposition 3.6.*

Proposition 3.8. *The class of all SG_C-projective modules is projectively resolving.*

Proof. Consider an exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules with M' and M'' SG_C-projective. Let X' and X'' be the complete SPC-resolutions of M' and M'' respectively. Since the classes of projective and C-projective R-modules are closed under extensions and using [8, Lemma 1.7] and [13, Lemma 6.20], we can obtain a complex

$$X = \cdots \to P \to P \to C \otimes_R P \to C \otimes_R P \to \cdots$$

with P projective and a degreewise split exact sequence of complexes

$$0 \to X' \overset{f}{\to} X \overset{g}{\to} X'' \to 0$$

such that $M \cong Coker(P \to P)$. By using [8, Lemma 1.7], we have an exact sequence of complexes

$$0 \to Hom_R(X'', C \otimes_R Q) \to Hom_R(X, C \otimes_R Q) \to Hom_R(X', C \otimes_R Q) \to 0.$$

Since the extreme complexes in the above are exact, the associated long exact sequence in homology gives that the middle one also to be exact.

Now assume that M and M'' are SG_C-projective with the complete SPC-resolutions X and X'' respectively. Using the Comparison Lemmas for resolutions in [8, Lemma 1.7] and by [13, Lemma 6.9], we get a morphism of chain complexes $\phi : X \to X''$ inducing g on the degree 0 cokernels. By adding complexes of the form $0 \to P_i' \to P_i'' \to 0$ and

$$0 \to C \otimes_R (P_i)'' \to C \otimes_R (P_i)'' \to 0,$$

one can assume ϕ is surjective. Since both the class of projective and C-projective modules are closed under kernels of epimorphisms, the complex $X' = Ker\phi$ has the form

$$X = \cdots \to P' \to P' \to C \otimes_R P \to C \otimes_R P' \to \cdots$$

with P' projective. By [14, Lemma 1.13], the exact sequence $0 \to X' \to X \to X'' \to 0$ is split and so the similar argument as in the previous paragraph gives that X' is a complete SPC-resolution of M' and hence M' is SG_C-projective.

□

Proposition 3.9. *Let Q be a projective R-module. If M is an SG_C-projective R-module, then $M \otimes_R Q$ is an SG_C-projective R-module.*

Proof. Suppose M is SG_C-projective. Then there exists an SPC-coresolution

$$Y = 0 \to M \to C \otimes_R P \to C \otimes_R P \to \cdots$$

such that $Hom_R(Y, C \otimes_R Q')$ is exact and $Ext_R(M, C \otimes_R Q') = 0$ for any projective R-module Q'. Then we have an exact sequence

$$Y \otimes_R Q = 0 \to M \otimes_R Q \to C \otimes_R (P \otimes_R Q) \to C \otimes_R (P \otimes_R Q) \to \cdots$$

such that $Hom_R(Y \otimes_R Q, C \otimes_R Q') \cong Hom_R(Q, Hom_R(Y, C \otimes_R Q'))$ exact and

$$Ext_R^{\geq 1}(M \otimes_R Q, C \otimes_R Q') \cong Hom_R(Q, Ext_R^{\geq 1}(M, C \otimes_R Q')) = 0.$$

Therefore, $M \otimes_R Q$ is an SG_C-projective R-module by the Proposition 3.3 (1). □

4. SG_C-FLAT MODULES

In this section, we introduce and study the SG_C-flat modules, and also link them with the SG_C-projective and injective modules.

Definition 4.1. *An R-module M is called a strongly Gorenstein flat R-module with respect to C (for short SG_C-flat) if there exists a complete resolution of the form*

$$\boldsymbol{SF} : \cdots \to F \to F \to C \otimes_R F \to C \otimes_R F \to \cdots$$

such that $M \cong Coker(F \to F)$ with F flat and $Hom_R(C, E) \otimes \boldsymbol{SF}$ is exact for any injective R-module E. We call the complete resolution of the above as an SFC-resolution.

We denote by $\mathcal{SGF}_C(R)$, the class of all strongly Gorenstein flat R-modules with respect to C. The SG_C-flat R-modules are particular cases of G_C-flat R-modules. The following Proposition shows that the flat modules are special types of SG_C-flat R-modules.

Proposition 4.2. *Every flat module is an SG_C-flat.*

Proof. This is similar to that of Proposition 3.4. □

Proposition 4.3. *M is SG_C-flat if and only if $Tor^R_{n\geq 1}(Hom_R(C, E), M) = 0$ and there exists an SFC-coresolution of the form*

$$Z = 0 \to M \to C \otimes_R F \to C \otimes_R F \to \cdots$$

with F a flat R-module such that $Hom_R(C, E) \otimes_R Z$ is exact for any injective R-module E.

Proposition 4.4. *Every direct sum of SG_C-flat modules is also SG_C-flat.*

Proof. Since the direct sum of C-flat modules is C-flat and Tor commutes with the direct sums, the proposition follows easily from the Proposition 4.3. □

Next, we have the characterization of SG_C-flat modules similar to that of Proposition 3.6.

Proposition 4.5. *For any module M, the following are equivalent:*

(1) *M is SG_C-flat;*

(2) *There exists a short exact sequence* $0 \to M \to F \to M \to 0$, *where* F *is a flat module, and* $Tor_1^R(Hom_R(C,E),M) = 0$ *for any injective module* E*;*
(3) *There exists a short exact sequence* $0 \to M \to F \to M \to 0$, *where* F *is a flat module, and* $Tor_1^R(E',M) = 0$ *for any module* E' *with finite* C*-injective dimension;*
(4) *There exists a short exact sequence* $0 \to M \to F \to M \to 0$, *where* F *is a flat module; such that the sequence* $0 \to Hom_R(C,E) \otimes M \to Hom_R(C,E) \otimes F \to Hom_R(C,E) \otimes M \to 0$ *is exact for any injective module* E*;*
(5) *There exists a short exact sequence* $0 \to M \to F \to M \to 0$, *where* F *is a flat module; such that the sequence* $0 \to E' \otimes M \to E' \otimes F \to E' \otimes M \to 0$ *is exact for any module* E' *with finite* C*-injective dimension.*

The following proposition is a consequence of Proposition 4.5.

Proposition 4.6. *An* SG_C*-flat module is flat if and only if it has finite flat dimension.*

Proposition 4.7. *A module is finitely generated* SG_C*-projective module if and only if it is finitely presented* SG_C*-flat.*

Proof. ($\Rightarrow$). Let M be a finitely generated SG_C-projective module. Then, by Proposition 3.6, there exists a short exact sequence $0 \to M \to P \to M \to 0$ with P a finitely generated projective R-module and $Ext_R^1(M, C \otimes_R Q) = 0$ for any projective R-module Q. Let E be any injective R-module. Since M is infinitely presented, we have from [7, Theorem 1.1.8], the following isomorphism:

$$\begin{aligned} Tor_1^R(Hom_R(C,E),M) &\cong Hom_R(Ext_R^1(M,C),E) \\ &\cong Hom_R(Ext_R^1(M,C \otimes_R R),E). \end{aligned}$$

Thus, $Tor_1^R(Hom_R(C,E),M) = 0$ since $Ext_R^1(M, C \otimes_R R) = 0$. Therefore, M is an SG_C-flat R-module by Proposition 4.5.
($\Leftarrow$). Now, we assume M to be a finitely presented SG_C-flat module. From Proposition 4.5, we deduce that there exists a short exact sequence $0 \to M \to P \to M \to 0$ with P a finitely generated projective R- module, and $Tor_1^R(Hom_R(C,E),M) = 0$ for every injective R-module E. Let Q be any projective R-module. Then by [7, Theorem 1.1.8], we have the following isomorphism:

$$\begin{aligned} Hom_R(Ext_R^1(M, C \otimes_R Q), E) &\cong Tor_1^R(Hom_R(C \otimes_R Q, E), M) \\ &\cong Tor_1^R(Hom_R(C, Hom_R(Q,E)), M) = 0 \end{aligned}$$

since $Hom_R(Q,E)$ is injective. If we assume E to be faithfully injective, then we have $Ext_R^1(M, C \otimes_R Q) = 0$. Therefore M is an SG_C-projective R-module by Proposition 3.6. □

Theorem 4.8. *Let* M *be an* R*-module and* P *be a flat left* R*-module. Then* M *is* SG_C*-flat if and only if* $M \oplus P$ *is* SG_C*-flat.*

Proof. (⇒). If M is SG_C-flat, then $M \oplus P$ is SG_C-flat by Proposition 4.4.
(⇐). Assume $M \oplus P$ is SG_C-flat. Then there exists an exact sequence $0 \to M \oplus P \to F \to M \oplus P \to 0$ with F flat. Then $(M \oplus P)^+$ is G_C-injective [15, Theorem 3.1], and hence M^+ is G_C-injective by [15, Theorem 2.2]. Consider the pushout of $M \oplus P \to F$ and $M \oplus P \to M$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & M \oplus P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & F & \longrightarrow & F' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M \oplus P & = & M \oplus P & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

and consider the commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & (M \oplus P)^+ & = & (M \oplus P)^+ & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F'^+ & \longrightarrow & F^+ & \longrightarrow & P^+ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M^+ & \longrightarrow & (M \oplus P)^+ & \longrightarrow & P^+ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

Then, F'^+ is G_C-injective by [15, Theorem 2.2], and thus $Ext^1_R(P^+, F'^+) = 0$, the sequence $0 \to F'^+ \to F^+ \to P^+ \to 0$ splits. It follows that F'^+ is injective, and hence F' is flat. Consider the pullback of $F' \to M \oplus P$ and $M \to M \oplus P$:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & M & = & M & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & F'' & \to & F' & \to & P & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & M & \to & M\oplus P & \to & P & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0. & & & & \end{array}$$

Then $0 \to M \to F'' \to M \to 0$ is exact and F'' is flat. Let E be any injective right R-module. Then, $0 = Tor_{i+1}^R(Hom_R(C,E),P) \to Tor_i^R(Hom_R(C,E),M) \to Tor_i^R(Hom_R(C,E),M \oplus P) = 0$ is exact for all $i \geq 1$. Hence $Tor_i^R(Hom_R(C,E),M) = 0$ for all $i \geq 1$, and therefore M is SG_C-flat by Proposition 4.5. □

Theorem 4.9. *Let R be a coherent ring. Then M is an SG_C-flat left R-module if and only if M^+ is an SG_C-injective right R-module.*

Proof. (⇒). Assume M an SG_C-flat left R-module. By Proposition 4.5, there exists an exact sequence $0 \to M \to F \to M \to 0$ in with F flat. Then $0 \to M^+ \to F^+ \to M^+ \to 0$ is exact and F^+ is injective. Let I be an injective right R-module. Then, $Ext_R^i(I,M^+) \cong Tor_i^R(I,M)^+ = 0$ for all $i \geq 1$, and hence M^+ is an SG_C-injective right R-module.
(⇐). If M^+ is an SG_C-injective right R-module, then there exists an exact sequence $0 \to M^+ \to E \to M^+ \to 0$ with E injective. Then there is an injective right R-module E' such that $E \oplus E' = E^{++}$. Let $H = (E' \oplus E)^{\mathbb{N}} \cong (E^{+(\mathbb{N})})^+$. Consider the exact sequence $0 \to M^+ \oplus H \to E \oplus H \oplus H \to M^+ \oplus H \to 0$. Then, $0 \to M \oplus E^{+(\mathbb{N})} \to E^{+(\mathbb{N})} \oplus E^{+(\mathbb{N})} \to M \oplus E^{+(\mathbb{N})} \to 0$ is exact and $E^{+(\mathbb{N})} \oplus E^{+(\mathbb{N})}$ is flat. Let I be any injective right R-module. Then, $Tor_i^R(Hom_R(C,I), M \oplus E^{+(\mathbb{N})}) = Tor_i^R(Hom_R(C,I),M) \oplus Tor_i^R(Hom_R(C,I),E^{+(\mathbb{N})}) = 0$ for all $i \geq 1$ since M is G_C-flat by [15, Theorem 3.1], and thus $M \oplus E^{+(\mathbb{N})}$ is SG_C-flat by Theorem 4.8. □

Let R be a ring and let M, N be left R-modules. Set $T(M) = \{x \in M |\ l_R(x) \neq 0\}$. If $T(M) = 0$, then M is called torsionfree. We denote by τ_N the natural map from $M^* \otimes_R N$ to $Hom_R(M,N)$ via $\phi \otimes x \mapsto \tau_N(\phi \otimes x)(m) = \phi(m)x$ for any $\phi \in M^*$, $x \in N$ and $m \in M$, where $M^* = Hom_R(M,R)$.

Theorem 4.10. *Let M be a finitely presented torsionfree left R-module. Then the following are equivalent:*

(1) *M is SG_C-projective ;*
(2) *M is SG_C-flat;*
(3) *The natural map from $M^* \otimes_R M \to Hom(M,M)$ is an isomorphism;*

(4) *The image of the natural map from* $M^* \otimes_R M \to Hom(M,M)$ *contains* Id_M;
(5) M *is projective ;*
(6) M *is flat.*

Proof. (1) $\Leftrightarrow$ (2). By Proposition 4.7.

(2) $\Rightarrow$ (3). There exists an exact sequence $0 \to M \xrightarrow{f} F \xrightarrow{g} M \to 0$ with F flat. Consider the commutative diagram:

$$\begin{array}{ccccccccc} & & M^* \otimes_R M & \xrightarrow{\tau_F M^* \otimes_R f} & M^* \otimes_R F & \xrightarrow{M^* \otimes_R g} & M^* \otimes_R M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \tau_M & & \\ 0 & \longrightarrow & Hom_R(M,M) & \xrightarrow{Hom_R(M,f)} & Hom_R(M,F) & \xrightarrow{Hom_R(M,g)} & Hom_R(M,M). & & \end{array}$$

Let $\phi \otimes m \in Ker(M^* \otimes_R f)$. Then for any $m' \in M, \tau_F(\phi \otimes f(m))(m') = f(\phi(m')m) = 0$. So $\phi(m')m = 0$ and hence $m = 0$ or $\phi = 0$ since M is torsionfree. It follows that $\phi \otimes m = 0, M^* \otimes_R f$ is monic, and hence τ_M is an isomorphism since τ_F is an isomorphism by [5, Theorem 3.2.14].
(3) $\Rightarrow$ (4) is obvious and (5) $\Rightarrow$ (1) follows from the Proposition 3.4.
(4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follows from [12, Theorem 4.19]. □

Proposition 4.11. *Let* R *be a noetherian ring. Then every direct limit of finitely generated* SG_C*-flat left* R*-modules is* SG_C*-flat.*

Proof. Let $((G_i), (\phi_{ji}))$ be a direct system over I of finitely generated SG_C-flat left R-modules. Let $i, j \in I$ with $i \leq j$. Then there are exact sequences $0 \to G_i \to F_i \to G_i \to 0$ and $0 \to G_j \to F_j \to G_j \to 0$ with F_i, F_j flat. Since $Ext^n_R(G_i, F_j)^+ \cong Tor^R_n(F_j^+, G_j) = 0$ by [5, Theorem 3.2.13] for all $n \geq 1$, then $Ext^1_R(G_i, F_j) = 0$. Consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_i & \longrightarrow & F_i & \longrightarrow & G_i & \longrightarrow & 0 \\ & & \downarrow \phi_{ji} & & \downarrow \psi_{ji} & & \downarrow & & \\ 0 & \longrightarrow & G_j & \longrightarrow & F_j & \longrightarrow & G_j & \longrightarrow & 0. \end{array}$$

Then $((F_i), (\psi_{ji}))$ is a direct system over I. Therefore, $0 \to \varinjlim G_i \to \varinjlim F_i \to \varinjlim G_i \to 0$ is exact by [5, Theorem 1.5.6] and $\varinjlim F_i$ is a flat left R-module. Then, for any injective right R-module E we have,

$$Tor^R_n(Hom_R(C,E), \varinjlim G_i) \cong \varinjlim Tor^R_n(Hom_R(C,E), G_i) = 0,$$

for all $n \geq 1$. Hence $\varinjlim G_i$ is SG_C-flat by Proposition 4.5. □

Theorem 4.12. *Let* R *be an artinian ring and suppose that the injective envelope of every simple left* R*-module is finitely generated. Then* M *is an* SG_C*-injective left* R*-module if and only if* M^+ *is an* SG_C*-flat right* R*-module.*

Proof. (⇒). There exists an exact sequence $0 \to M \to E \to M \to 0$ with E injective. Then, $0 \to M^+ \to E^+ \to M^+ \to 0$ is exact and E^+ is a flat right R-module. Let J be any injective left R-module. Then $J = \bigoplus_\Lambda J_\alpha$, where J_α is an injective envelope of some simple left R-module for any $\alpha \in \Lambda$ by [11, Theorem 6.6.4] for all $i \geq 1$, and hence

$$\begin{aligned} Tor_i^R(M^+, Hom_R(C,J)) &\cong Tor_i^R(M^+, Hom_R(C, \oplus_\Lambda J_\alpha)) \\ &\cong \prod_\Lambda Tor_i^R(M^+, Hom_R(C, J_\alpha)) \\ &\cong \prod_\Lambda Ext_R^i(Hom_R(C, J_\alpha), M)^+ \\ &= 0, \end{aligned}$$

by [5, Theorem 3.2.13] for all $i \geq 1$. Therefore, M^+ is an SG_C-flat right R-module.

(⇐). Assume M^+ is an SG_C-flat right R-module. Then there exists an exact sequence $0 \to M^+ \to F \to M^+ \to 0$ with F flat. Then, $0 \to M^{++\mathbb{N}} \to F^{+\mathbb{N}} \to M^{++\mathbb{N}} \to 0$ is exact and $F^{+\mathbb{N}}$ is an injective left R-module, and so there is an injective left R-module E such that $F^{+\mathbb{N}} \oplus E = (F^{+\mathbb{N}})^{++}$. Set $L = (F^{+\mathbb{N}} \oplus E)^{\mathbb{N}}$. Then $0 \to M^{++\mathbb{N}} \oplus L \to L \to M^{++\mathbb{N}} \oplus L \to 0$ is exact, and thus $0 \to M \oplus F^{+\mathbb{N}} \to F^{+\mathbb{N}} \to M \oplus F^{+\mathbb{N}} \to 0$ is exact. Let J be any injective left R-module. Then $J = \bigoplus_\Lambda J_\alpha$, where J_α is an injective envelope of some simple left R-module for any $\alpha \in \Lambda$ by [11, Theorem 6.6.4]. Thus, $Ext_R^i(Hom_R(C, J_\alpha), M)^+ \cong Tor_i^R(M^+, Hom_R(C, J_\alpha)) = 0$ by [5, Theorem 3.2.13] for all $i \geq 1$ and any $\alpha \in \Lambda$, and hence $Ext_R^i(Hom_R(C,J), M) \cong \prod_\Lambda Ext_R^i(Hom_R(C, J_\alpha), M) = 0$ for all $i \geq 1$. It follows that $M \oplus F^{+\mathbb{N}}$ is an SG_C-injective left R-module, and so M is an SG_C-injective left R-module. □

Lemma 4.13. *Let R be an artinian ring and suppose that the injective envelope of every simple left R-module is finitely generated. Then the class $\mathcal{SGF}_\mathcal{C}(R)$ is closed under arbitrary direct products.*

Proof. Let $M = \prod_{i\in I} M_i$, and $M_i \in \mathcal{SGF}_\mathcal{C}(R)$ for all $i \geq 1$. There exists an exact sequence $0 \to M_i \to F_i \to M_i \to 0$ for all $i \geq 1$. Then, $0 \to \prod_{i\in I} M_i \to \prod_{i\in I} F_i \to \prod_{i\in I} M_i \to 0$ is exact and $\prod_{i\in I} F_i$ is a flat right R-module. Let E be any injective left R-module. Then $E = \bigoplus_\Lambda E_\alpha$, where E_α is an injective envelope of some simple left R-module for any $\alpha \in \Lambda$ by [11, Theorem 6.6.4]. Thus,

$$\begin{aligned} Tor_n^R(\prod_{i\in I} M_i, Hom_R(C,E)) &\cong Tor_n^R(\prod_{i\in I} M_i, Hom_R(C, \bigoplus E_\alpha) \\ &\cong \prod_\Lambda \prod_{i\in I} Tor_n^R(M_i, Hom_R(C, E_\alpha)) \\ &= 0 \end{aligned}$$

by [5, Theorem 3.2.26] for all $n \geq 1$. Therefore, M is an SG_C-flat right R-module. □

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