J. Indones. Math. Soc.
Vol. 25, No. 01 (2019), pp. 75–84.

METRIC DIMENSION OF GRAPH JOIN TWO PATHS P_2 AND P_t

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Abstract. The following metric dimension of join two paths $P_2 + P_t$ is determined as follows. For every $k = 1, 2, 3, \ldots$ and $t = 2 + 5k$ or $t = 3 + 5k$, the dimension of $P_2 + P_t$ is $2 + 2k$; whereas for $t = 4 + 5k, t = 5(k + 1)$ or $t = 1 + 5(k + 1)$, the dimension is $3 + 2k$. In case $t \geq 7$, the dimension is determined by a chosen (maximal) ordered basis for P_2+P_t, in which the integers $1, 2$ are the two consecutive vertices of P_2 and the next integers $3, 4, \ldots, t + 2$ are the t consecutive vertices of P_t. If $t \geq 10$, the ordered binary string contains repeated substrings of length 5. For $t < 7$, the dimension is easily found using a computer search, or even just using hand computations.

Key words and Phrases: graph join of two paths, (metric) dimension, maximal basis

Abstrak. Dimensi metrik dari graf gabungan dua lintasan $P_2 + P_t$ ditentukan sebagai berikut. Untuk setiap $k = 1, 2, 3, \ldots$ dan $t = 2 + 5k$ atau $t = 3 + 5k$, dimensi dari $P_2 + P_t$ adalah $2 + 2k$; sedangkan untuk $t = 4 + 5k, t = 5(k + 1)$ atau $t = 1+5(k+1)$, dimensinya adalah $3+2k$. Pada kasus $t \geq 7$, ukuran dimensi ditentukan dengan memilih sebuah basis (maksimal) terurut dari $P_2 + P_t$, di mana bilangan 1, 2 adalah titik-titik berurutan dari P_2 dan bilangan selanjutnya: $3, 4, \ldots, t + 2$; adalah titik-titik berurutan dari P_t. Jika $t \geq 10$, untaian biner terurut tersebut memuat subuntaian periodik dengan perioda 5. Untuk $t < 7$, dimensinya mudah didapat dengan menggunakan komputer, atau bahkan hanya dengan menggunakan komputasi manual.

Kata kunci: graf gabungan dua lintasan, dimensi (metrik), basis maksimal

1. INTRODUCTION

In the last two or three decades, there have been many and fast developments of graph theory, including its related terminologies. The theoretical developments

2010 Mathematics Subject Classification: 05C12; 05C76.
Received: 26-02-2018, accepted: 26-09-2018.

include many new concepts and notions such as various kinds of graphs and graph labelings as surveyed by Gallian [1], and in particular the notion of metric dimension.

Metric dimension in graph theory was first introduced in the mid-1970s by Harary and Melter [3] and independently by Slater [6]. With respect to terminology, the authors tend to follow Kuziak, Rodrígues-Veláquez and Yero [4, 5] that use the term 'metric generator' and 'metric basis' instead of 'resolving set' and 'minimum resolving set' used by many authors, for example by Chartrand, Eroh, Johnson and Oellermann [1]. In this paper, the authors simplify the terms by using two shorter terms: 'generator' and 'basis'.

In Section 2, the preliminary definitions and notations are introduced, followed by Section 3 containing a complete derivation of the metric dimension of the join two paths P_2 and P_t with $t \geq 7$, based on an ordered binary string representation of a vertex set W. It is shown that for $t \geq 10$, the string contains repeated substrings of length 5. Section 4 provides the dimension of smaller join graphs $P_2 + P_t$ with $t < 7$. The final section contains a brief conclusion of the results.

2. NOTATIONS AND DEFINITIONS

Let $G = (V, E)$ be a connected graph of n vertices labeled by positive integers. Let $W = \{w_1, w_2, ..., w_m\} \subseteq V$ be an ordered set with $w_1 < w_2 < \cdots < w_m$ and let $d(x, y)$ be the usual distance between two vertices $x, y \in V$. Then the representation of $u \in V$ with respect to W is defined to be the m-tuple $r(u|W) = [d(u, w_1), d(u, w_2), ..., d(u, w_m)]$. The ordered set W is called a *generator* for G if for every $u, v \in V, r(u|W) = r(v|W)$ implies $u = v$. The order of W will be denoted by $|W|$. The dimension $dim(G)$ (commonly called *metric dimension*) of the graph G is the minimum cardinality of a generator for G, and this generator is called a basis for G. In the sequel, all the sets of integers are assumed to be ordered sets, unless stated otherwise.

Let $t \geq 1, n = t + 2$ and let $P_2 = (V_2, E_2)$ and $P_t = (V_t, E_t)$ be two paths of length 1 and $t - 1$, respectively, where $V_2 = \{1, 2\}, V_t = \{3, 4, ..., n\}, E_2 = \{\{1, 2\}\}$ and $E_t = \{\{3, 4\}, \{4, 5\}, ..., \{n - 1, n\}\}$, or $E_t = \emptyset$ in case $t = 1$. The join graph $P_2 + P_t = (V_{2,t}, E_{2,t})$ is the graph with vertex set $V_{2,t} = V_2 \cup V_t$ and edge set $E_{2,t} = E_2 \cup E_t \cup \{\{u, v\}|u \in V_2, v \in V_t\}$. The definition of dimension implies that, if $t \leq t'$, then $dim(P_2 + P_t) \leq dim(P_2 + P_{t'})$. Clearly, $|V_{2,t}| = t + 2 = n$ and $|E_{2,t}| = 3t$. It is easy to conclude that if $u, v \in V_2$ or $u, v \in V_t$ then

$$d(u, v) = \begin{cases} 1, & \text{if } |u - v| \leq 1, \\ 2, & \text{if } |u - v| > 1, \end{cases}$$

whereas, if $u \in V_2$ and $v \in V_t$ then $d(u, v) = 1$.

Any ordered set $W = \{w_1, w_2, ..., w_m\} \subseteq V_{2,t}$ is of the form $W = W[1] \cup W[2]$, where $W[1] \subseteq V_2 = \{1, 2\}$ and $W[2] \subseteq V_t$. More precisely, this set will be treated as a pair of ordered sets $W[1] \subseteq V_2$, $W[2] \subseteq V_t$, written $W = W[1]|W[2]$, and $w_i < w_j$ if and only if $i < j$.

Definition 2.1. *Let $P_2 + P_t = (V_{2,t}, E_{2,t})$ and $W = W[1]|W[2] \subseteq V_{2,t}$. The binary representation of W is the* split binary string $\mathbf{b}_W = b_1 b_2 | b_3 ... b_n$, *where.*

$$b_i = \begin{cases} 1, & \text{if } i \in W, \\ 0, & \text{otherwise.} \end{cases}$$

In this definition, the first and the second part of the split string $\mathbf{b}_W$ are denoted by $\mathbf{b}_{W[1]} = b_1 b_2$ and $\mathbf{b}_{W[2]} = b_3 b_4 ... b_n$, respectively, that is $\mathbf{b}_W = \mathbf{b}_{W[1]} | \mathbf{b}_{W[2]}$.

The following definition using the fact that $0 < 1$.

Definition 2.2. *Let $U = \{u_1, u_2, ..., u_m\}, W = \{w_1, w_2, ..., w_m\} \subseteq V_{2,t}$ be represented by $\mathbf{b}_U = b_1 b_2 | b_3 ... b_n$ and $\mathbf{b}_W = c_1 c_2 | c_3 ... c_n$, respectively. If $b_1 > c_1$ or there exists a positive integer j with $1 < j \leq t + 2$ such that $b_j > c_j$ and for every $i < j, b_i = c_i$, then U is said lower than W, written "$U < W$". Equivalently, W is said higher than U and this is denoted by "$W > U$".*

The above definition leads to the fact that there is one and only one subset of $V_{2,t}$ that higher than any other set of the same size.

Definition 2.3. *A basis $B \subseteq V_{2,t}$ is called the* maximal basis *for $P_2 + P_t$ if for any basis $B' \neq B, B' < B$. This unique maximal basis will be denoted by Max_t.*

3. DIMENSION OF $P_2 + P_t$ FOR $t \geq 7$

The following lemma provides a necessary condition of a (maximal) basis.

Lemma 3.1. *If $B \subseteq V_{2,t}$ is a basis represented by $\mathbf{b}_B = b_1 b_2 | b_3 b_4 ... b_n$ then $1 \in B$ or $2 \in B$. If B is maximal, then $2 \in B$.*

Proof. Any representation $\mathbf{b}_B = 00|b_3 ... b_n$ of $B \subseteq V_{2,t}$ would imply $r(1|B) = (1, 1, ..., 1) = r(2|B)$. Clearly, if B is maximal then $2 \in B$. $\square$

Example 3.2. It is easy to show that $Max_6 = \{2, 5, 7, 8\}$. In fact, the size of Max_6 cannot be smaller. By the preceding lemma, 2 cannot be deleted from W. If $|W[2]| = 2$, then $W[2]$ would be represented by any of the following 15 binary strings 110000, 101000, ..., 010100, ..., 000101, or 000011. The binary strings 010100 and 001010 cannot represent a generator because they lead to equations $d(1|W) = d(5|W)$ and $d(1|W) = d(6|W)$, respectively. By applying the next theorem, which is is still valid for $t = 6$ (although its statement restricts only for $t \geq 7$), any of the remaining 13 binary strings does not represent a generator for $P_2 + P_6$.

In this Example 3.2, and also in Example 4.2, it is shown that $dim(P_2+P_6) = 4$. Therefore for $t \geq 7, dim(P_2 + P_t) \geq 4$. The following theorem provides all the possibilities when a binary string $\mathbf{b}_W = 01|b_3...b_n$ with at least four bit-1s, accordingly with $|W| \geq 4$; cannot represent a generator $W \subseteq V_{2,t}$ with $t \geq 7$.

Theorem 3.3. *Let $t \geq 7, W \subseteq V_{2,t}$ be represented by $\mathbf{b}_W = 01|b_3...b_n$ and $i_1, i_2 \in V_t - W[2]$ with $i_1 < i_2$. Then $r(i_1|W) = r(i_2|W)$ if and only if $\mathbf{b}_W$ is of at least one of the following forms in which the underline* 0 (that is $\underline{0}$) *refers to the position of integer i_1 and i_2 in W:*

$\mathbf{b}_W = 01|\mathbf{r}0\underline{00}0\mathbf{s}; \mathbf{b}_W = 01|\underline{00}01\mathbf{s}; \mathbf{b}_W = 01|\mathbf{r}10\underline{00}; \mathbf{b}_W = 01|\mathbf{r}0\underline{0}1\underline{0}0\mathbf{s}; \mathbf{b}_W = 01|\underline{0}1\underline{0}0\mathbf{s}; \mathbf{b}_W = 01|\mathbf{r}0\underline{0}1\underline{0}; \mathbf{b}_W = 01|\underline{0}0\mathbf{r}0\underline{0}; \mathbf{b}_W = 01|\underline{0}0\mathbf{r}10\underline{0}01\mathbf{s}\mathbf{b}_W = 01|\mathbf{r}10\underline{0}01\mathbf{s}0\underline{0}\mathbf{b}_W = 01|\mathbf{r}10\underline{0}010\underline{0}01\mathbf{s}; 3(tsk[a])$.

where $\mathbf{r}$ *and* $\mathbf{s}$ *are binary strings, possibly empty, and the form* (a) *is the only form containing four consecutive bit-*0*s.*

The restriction $i_1, i_2 \notin W[2]$ is needed since if, say $i_2 \in W[2]$, then obviously $0 < d(i_1, i_2) \neq d(i_2, i_2) = 0$ and there is no possibility $r(i_1|W) = r(i_2|W)$.

PROOF. Let $m = |W|$. Firstly we prove that any of the ten forms stated in the theorem is a sufficient condition for the existence of $i_1, i_2 \in W - W[2]$ with $d(i_1|W) = d(i_2|W)$. There are three cases:

i$_2 - i_1 = 1; i_2 - i_1 = 2; i_2 - i_1 > 2.3(tsk[r])$

(i) The case $i_2 - i_1 = 1$ can occur only when one of the first three forms of binary string (a), (b) and (c) occurs. If the form (a) occurs then $d(i_1, 2) = 1$ (or $d(i_2, 2) = 1$) is the only case that the integer i_1 (or i_2) at distant 1 with another integer. So, we have an equality of two m-sequences, $r(i_1|W) = (1, 2, 2, 2, ...) = r(i_2+1|W)$ with $3 < i_1 < i_1+1 = i_2 < n$. The same reasoning applies to (b) and (c) because if $i_1 = 3$ (that can be obtained by shifting $\underline{000}$ in (a) to the most left of $\mathbf{b}_{W[2]}$), then (a) becomes (b) and similarly, if $i_2 = n$, then (a) becomes (c).

(ii) The second case $i_2 - i_1 = 2$ holds only when one of the three forms (d), (e) or (f) occurs. If (d) occurs with $3 < i_1 < i_1 + 2 = i_2 < n$ then $d(i_1, 2) = 1$ and $d(i_1, i_1 + 1) = 1$ (or $d(2, i_2) = 1$ and $d(i_2 - 1, i_2) = 1$) are the only two cases that the integer i_1 (or i_2) at distant 1 with another integer. There may or may not exist an integer $u < i_1$ with $d(u, i_1) = 2$ or an integer $v > i_2$ with $d(v, i_2) = 2$. In this case, depending on the existence of such integer u and v, this form (d) will result an equality of m-sequences $r(i_1|W) = r(i_2|W)$ of the form (1, 1, 2, ..., 2), (1, 2, ..., 1, ..., 2) or (1, 2, ..., 2,1), where $3 < i_1 < i_1 + 1 < i_1 + 2 = i_2 < n$. As in the first case, if $i_1 = 3$, then (d) becomes (e) with $r(3|W) = r(5|W) = (1, 1, 2, ..., 2)$ and if $i_2 = n$, (d) becomes (f) with $r(n - 2|W) = r(n|W) = (1, 2, ..., 2, 1)$.

(iii) The third case $i_2 - i_1 > 2$ occurs only in one of the forms (g) - (j) with $r(i_1|W) = r(i_2|W) = (1, 2, 2, ..., 2)$ where in (g), we have $i_1 = 3$ and $i_2 = n$ and in (h), $i_1 = 3$ and $i_1 + 4 \leq i_2 \leq n - 2$, in (i), $5 \leq i_1 \leq n - 4 < i_2 = n$ and in (j), $5 \leq i_1 < i_1 + 4 \leq i_2 \leq n - 2$. The reasons are similar with those of the previous cases.

To prove the necessary condition, it must be shown that if there exist two integers $i_1, i_2 \in W - W[2]$ with $i_1 < i_2$ and $d(i_1|W) = d(i_2|W)$, then the set W must be one of the forms (a) to (j), no other form is possible. For example, in case $i_2 - i_1 > 2$, then there are exactly only four possible subcases $d(i_1|W) = d(i_2|W)$:

1. $i_1 = 3, i_2 = n$. Since $i_2 - i_1 > 2$, we must have $i_1 = 3 < n-3 < n = i_2$, as indicated by the form (g). In this subcase, the inequality $3 < n-3$ must be satisfied. Otherwise $3 = n-3$ or $n = 6$, contradicts the assumption $n \geq 7$.
2. $i_1 = 3, i_2 < n$. The condition $i_2 - i_1 > 2$ implies $i_1 = 3 < 5 < i_2 < n$, as shown by the form (h).
3. $i_1 > 3, i_2 = n$. Since $i_2 - i_1 > 2$, we also have $3 < i_1 < n-2 < n = i_2$, as indicated by the form (i). It is not possible to have $i_1 = n-2$ which implies $i_1 = i_2 - 2$ and so $i_2 - i_1 = 2$.
4. $i_1 > 3, i_2 < n$. From $i_2 - i_1 > 2$, it can be deduced that $3 < i_1 < i_1 + 2 < i_2 < n$, as suggested by the form (j).

The remaining two cases $i_2 - i_1 = 1$ and $i_2 - i_1 = 2$, can be proved analogously. □

We will call the explicit-displayed binary substrings of the form $\mathbf{b}_{W[2]}$s mentioned in Theorem 3.3 as *improper strings*. For example, the substring $\mathbf{b}_{W[2]} = \mathbf{r}0\underline{00}0\mathbf{s}$ of the form (c) is an improper string. In general, a binary string $\mathbf{r}$ is called improper if for any vertex-set W represented by $\mathbf{b}_{W[1]}|\mathbf{r}$ (that is $\mathbf{b}_{W[2]} = \mathbf{r}$), then there always exist two distinct integers $i_1, i_2 \in W - W[2]$ such that $r(i_1|W) = r(i_2|W)$. Notice that in Theorem 3.3, the main reason why string of the form (a) improper is because it contains four consecutive bit-0s.

Example 3.4. By inspection, $B = \{2, 5, 7, 9\}$ is a basis for $P_2 + P_t$. This basis cannot be higher since if 5 is replaced by 6, then the resulting set $U = \{2, 6, 7, 9\}$ will be represented by improper string $\mathbf{b}_U = 01|\underline{00}01101$ with $r(3|U) = 1222 = r(4|U)$ and if 7 is replaced by 8, then the resulting set $V = \{2, 5, 8, 9\}$ will be represented by the improper string $\mathbf{b}_V = 01|0\underline{0}1\underline{0}011$ with $r(4|V) = 1122 = r(6|V)$. Therefore, $B = Max_7$.

Let $\mathbf{b} = b_1 b_2 ... b_k$ and $\mathbf{c} = c_1 c_2 ... c_l$ be two binary strings. A concatenation between $\mathbf{b}$ and $\mathbf{c}$ is the binary string $\mathbf{bc} = b_1 b_2 ... b_k c_1 c_2 ... c_l$. In particular, k times concatenations of $\mathbf{b}$ with itself, that is $\mathbf{bb}...\mathbf{b}$ ($\mathbf{b}$ repeated k times), will be written as $\mathbf{b}^k$. Let $\mathbf{b}^0$ be the empty string and $\mathbf{bcbc}$, $\mathbf{bcbcb}$, $\mathbf{bcbcbc}$,..., etc. be written as $(\mathbf{bc})^2$, $(\mathbf{bc})^2\mathbf{b}$ or $\mathbf{b}(\mathbf{cb})^2$, $(\mathbf{bc})^3$, ... and so on. Two particular binary strings $\boldsymbol{\alpha} = 001$ and $\boldsymbol{\beta} = 01$ will play important roles.

Lemma 3.5. *Let $t \geq 7$ and $W = \{w_1, ..., w_m\} \subseteq V_{2,t}$ be a basis represented by $\mathbf{b}_W = 01|b_3 b_4 ... b_n$. If W is maximal, then $w_2 = 5$, $w_3 = 7$, and therefore $\mathbf{b}_W = 01|\boldsymbol{\alpha\beta} b_8..b_n$.. In particular, $Max_7 = \{2, 5, 7, 9\}, Max_8 = \{2, 5, 7, 10\}, Max_9 = \{2, 5, 7, 10, 11\}$ and $Max_{10} = \{2, 5, 7, 10, 12\}$.*

Proof. Each of the four sets is obviously a basis of their respective graph join. If $w_2 > 5$ then $\mathbf{b}_W = 01|\underline{00}0...$ contains improper substring of the form (a) stated in Theorem 3.3. Suppose $w_3 > 7$. Then, $\mathbf{b}_W = 01|0\underline{0}1\underline{0}0 b_8 ... b_{12}$ and $r(4|W) = r(6|W)$, as indicated by the form (f). Therefore, 5 and 7 are the only option for

w_2 and w_3. The four sets {2, 5, 7, 9}, {2, 5, 7, 10}, {2, 5, 7, 10, 11}, {2, 5, 7, 10, 12} are represented by $01|0010101, 01|00101001, 01|001010011, 01|(00101)^2$, respectively, and any of these binary string does not contain improper string of the form (a) - (j). This proves that the sets are bases for $P_2+P_7, P_2+P_8, P_2+P_9$ and P_2+P_{10}, respectively. It is easy to prove that the four sets are maximal bases by showing every integer in each set cannot be made larger. For example, {2, 5, 7, 10, 12} cannot be replaced by higher set $W=\{2,5,7,11,12\}$ since there would be equality $r(3|W)=[2,2,...,2]=r(9|W)$. □

Notice that $Max_7=\{2,5,7,9\}$ is also a basis of P_2+P_8 but it is not maximal.

The most important role played by the binary string $\boldsymbol{\alpha\beta}$ is described by the following two theorems.

Theorem 3.6. *For every $k=1,2,3,...,$ we have*

$$Max_{5(k+1)}=\{2\}\cup\bigcup_{i=0}^{k}\{5(i+1),2+5(i+1)\}. \tag{1}$$

This maximal basis is represented by the split binary string

$$\mathbf{b}_{Max_{5(k+1)}}=01|(\boldsymbol{\alpha\beta})^{k+1}. \tag{2}$$

Consequently,

$$dim(P_2+P_{5(k+1)})=2(k+1). \tag{3}$$

PROOF. We will show that $W=\{2,5,7,10,12,15,17,...,4+5(k+1),2+5(k+1)\}=\{2\}\cup\{5,7\}\cup\{10,12\}\cup\{15,17\}\cup...\cup\{5(k+1),2+5(k+1)\}$ is the maximal bases for $P_2+P_{5(k+1)}$. Clearly, W is represented by the binary string (2) and deleting any integer in W would result one of the forms (a) - (j) stated in Theorem 3.3. Furthermore, the string (2), which does not contain three consecutive bit-0s, cannot be of the form (a), (b), (c), (h) or (i). Observe that W is of the form (1), consists of $2k+3$ integers initialized by 2 and 5. Moreover, the last 3-bit of its binary representation (2) is 101 (because the last two integers $5(k+1)$ and $2+5(k+1)$ of W differ by 2). Consequently, any of the forms (e), (f) and (g) cannot be the form of the string (2), because neither initialized by 2 and 5 nor the last 3-bit of its binary representation is 101. Each of forms (d) and (j) cannot be the form of the string (2) because it contains a substring $\boldsymbol{\alpha}00$, contradicts the fact that $\boldsymbol{\alpha}$ in (2) is always followed by $\boldsymbol{\beta}=01$. This proves that W is a basis. The proof that this basis is maximal can be done with the same way as the proof that $Max_7=\{2,5,7,9\}$ given in Example 3.4. □

The advantage of the binary representation can be seen from the fact that although $Max_{10}=\{2,5,7,10,12\}$ is the same as Max_{11}, they have different representations. In fact, Max_{10} is represented by $01|\boldsymbol{\alpha\beta\alpha\beta}$ whereas Max_{11} is represented by $01|\boldsymbol{\alpha\beta\alpha\beta}0$ as stated by the next theorem. Notice that $\{2,5,7,10,12,13\}$

is a basis for $P_2 + P_{12}$, but $Max_{12} = \{2, 5, 7, 10, 12, 14\}$. This situation occurs for every $t = 2 + 5k$, where $k = 1, 2, 3, ...$

Theorem 3.7. *For every $k = 1, 2, 3, ...$ and $j = 1, 2, 3, 4$, we have*

$$Max_{j+5(k+1)} = \{2\} \cup \bigcup_{i=0}^{k} \{5(i+1), 2+5(i+1)\} \cup U_j, \tag{4}$$

where $U_1 = \emptyset, U_2 = \{4 + 5(k+1)\}, U_3 = \{5(k+2)\}, U_4 = \{5(k+2), 1 + 5(k+2)\}$, respectively. Each of these maximal bases is represented by the split binary string

$$\mathbf{b}_{Max_{j+5(k+1)}} = 01|(\boldsymbol{\alpha\beta})^{k+1}\mathbf{u}_j. \tag{5}$$

where $\mathbf{u}_1 = 0, \mathbf{u}_2 = \boldsymbol{\beta}, \mathbf{u}_3 = \boldsymbol{\alpha}, \mathbf{u}_4 = \boldsymbol{\alpha}1$, respectively. Consequently,

$$dim(P_2 + P_t) = \begin{cases} 2k+2, & if\ t = 2 + 5k \text{ or } 3 + 5k; \\ 2k+3, & if\ t = 4 + 5k, 5(k+1) \text{ or } 1 + 5(k+1). \end{cases} \tag{6}$$

Proof. Using the notations and results from Theorem 3.6, equation (4) can be rewritten as $Max_{j+5(k+1)} = Max_{5(k+1)} \cup U_j$. If $U_1 = \emptyset$, then $Max_{1+5(k+1)} = Max_{5(k+1)}$. However, the binary representation of $Max_{1+5(k+1)}$ has an additional bit-0 at the end of the expression (2). The proof that this set is a basis for $P_2 + P_{1+5(k+1)}$ almost exactly the same as the proof of Theorem 3.6, except in using the fact that the last 4-bit of this set is 1010. In proving that this basis is maximal, it is enough to prove that if the last bit $2 + 5(k+1)$ be made larger to $3 + 5(k+1)$, then the resulting binary representation would be the improper string (d) stated in Theorem 3.3. Analogously, the case $j = 2$, 3 or 4 can be proved using the facts that the last 4-bit binary representation of the set $Max_{5(k+1)} \cup U_j$ are 0101, 1001, 0011, respectively, and any of the last 4 integers in this set cannot be made larger. □

Example 3.8. Applying (1) and (2) of Theorem 3.6 with $k = 2$ gives $t = 15$, $Max_{15} = \{2, 5, 7, 10, 12, 15, 17\} = Max_{14}$ and $\mathbf{b}_{Max_{15}} = 01|001010010100101 = 01|(\boldsymbol{\alpha\beta})^3$ whereas applying (4) and (5) of Theorem 3.7 with $k = 2$ and $j = 4$ results $t = 19$ with $Max_{19} = \{2, 5, 7, 10, 12, 15, 17, 20, 21\}$ and $\mathbf{b}_{Max_{19}} = 01|(\boldsymbol{\alpha\beta})^3 0011 = 01|0010100101001010011$. Therefore, $dim(P_2 + P_{15}) = 2(3) + 1 = 7$ and $dim(P_2 + P_{19}) = dim(P_2 + P_{20}) = 9$. In general for every $t \geq 7, dim(P_2 + P_{t+5}) = dim(P_2 + P_t) + 2$ and for every $k = 2, 3, 4, ..., dim(P_2 + P_{5k-1}) = dim(P_2 + P_{5k}) = dim(P_2 + P_{5k+1})$ is an odd number whereas $dim(P_2 + P_{5k+2}) = dim(P_2 + P_{5k+3})$ is an even number. The binary representations of Max_t with $15 \leq t \leq 29$ are

01—$(\boldsymbol{\alpha\beta})^3, 01|(\boldsymbol{\alpha\beta})^3 0, 01|(\boldsymbol{\alpha\beta})^3 01, 01|(\boldsymbol{\alpha\beta})^3 001, 01|(\boldsymbol{\alpha\beta})^3 0011, 01|(\boldsymbol{\alpha\beta})^4, 01|(\boldsymbol{\alpha\beta})^4 0,$
$01|(\boldsymbol{\alpha\beta})^4 01, 01|(\boldsymbol{\alpha\beta})^4 001, 01|(\boldsymbol{\alpha\beta})^4 0011, 01|(\boldsymbol{\alpha\beta})^5, 01|(\boldsymbol{\alpha\beta})^5 0, 01|(\boldsymbol{\alpha\beta})^5 01, 01|(\boldsymbol{\alpha\beta})^5 001,$
$01|(\boldsymbol{\alpha\beta})^5 0011; 5(tsk[a])$
and this list can be extended indefinitely for larger values of t.

4. DIMENSION OF $P_2 + P_t$ FOR $t < 7$.

The reverse of a binary string $\mathbf{s} = s_1 s_2 ... s_m$ is defined as the string $s = s_m s_{m-1} ... s_1$. For any set $W = W[1]|W[2] \subseteq V_{2,t}$ represented by $\mathbf{b}_W = b_1 b_2 b_3 ... b_n$, we define $W[1]^R$, the reverse of $W[1] \subseteq V_2$, as the set with binary representation $\mathbf{b}_{W[1]^R} = b_2 b_1$ and similarly the reverse of $W[2] \subseteq V_t$ is defined as the set $W[2]^R$ represented by the binary string $\mathbf{b}_{W[2]^R} = b_n b_{n-1},,, b_2 b_1$.
Consequently, if $W[1] = \{x_1, x_2\} \subseteq V_2$ and $W[2] = \{y_1, y_2, ..., y_k\} \subseteq V_t$, then $W[1]^R = \{2x_2 \bmod 3, 2x_1 \bmod 3\}, W[2]^R = \{3+n-y_k, 3+n-y_{k-1}, ..., 3+n-y_1\}$ and $W^R = W[1]^R \cup W[2]^R$. That is if $W \subseteq V_2$ (or $W \subseteq V_t$) then W^R is the 'mirror' of W in opposite direction with respect to the set V_2 (or V_t). We define $W^R = W[1]^R | W[2]^R$ and $\mathbf{b}_W^R = \mathbf{b}_{W[1]^R} | \mathbf{b}_{W[2]^R}$. So, $\mathbf{b}_W^R = \mathbf{b}_{W^R}$.

A set B is called the *proper minimal basis* for $P_2 + P_t$ if for any basis $B', B \leq B'$ and will be denoted by min_t. We also define $Min_t = Max_t^R$, which is also a basis as justified by the following proposition.

Proposition 4.1. *Let $B \subseteq V_{2,t}$ be a basis of $P_2 + P_t$. Then B^R is also a basis. In particular, Min_t is a basis.*

Proof. Let the *ordered* set $B = \{x_1, x_2, x_3, \ldots, x_m\} = (B[1]|B[2]) \subseteq V_{2,t}$ be the basis. Without loss of generality, we may assume that $B[1] = \{x_1\}$. By using the fact that $(i-x) \neq (j-x)$ if and only if $(i-(L-x)) \neq (j-(L-x))$, then for every $i, j \in V_{2,t}$, inequality $d(i|\{x_1, x_2, x_3, \ldots, x_m\}) \neq d(j|\{x_1, x_2, x_3, \ldots, x_m\})$ is true if and only if $d(i|\{3-x_1, n+3-x_m, n+3-x_{m-1}, \ldots, n+3-x_2\}) \neq d(j|\{3-x_1, n+3-x_m, n+3-x_{m-1}, \ldots, n+3-x_2\})$ is also true. That is, $d(i|(B[1]|B[2])) \neq d(j|(B[1]|B[2]))$ if and only if $d(i|(B^R[1]|B^R[2])) \neq d(j|(B^R[1]|B^R[2]))$. Equivalently, B is a basis if and only if B^R is a basis. In particular, since Max_t is a basis, then $Min_t = Max_t^R$ is also a basis. $\square$

Min_t is called the *improper minimal basis*. Clearly, $min_t \leq Min_t$.

Example 4.2. By a computer search, there are 34 bases $W_1 < W_2 < ... < W_{34}$ for $P_2 + P_6$. The following is the list of eight of these bases including their binary representations.

$$\begin{array}{ll} W_1 = \{1,2,4,6\} \approx 11|010100, & W_2 = \{1,2,5,7\} \approx 11|001010, \\ W_3 = \{1,3,4,6\} \approx 10|110100, & W_4 = \{1,3,4,8\} \approx 10|110001, \\ W_{31} = \{2,4,6,8\} \approx 01|010101, & W_{32} = \{2,5,6,7\} \approx 01|001110, \\ W_{33} = \{2,5,6,8\} \approx 01|001101, & W_{34} = \{2,5,7,8\} \approx 01|001011. \end{array}$$

Since $\mathbf{b}_{W_{34}} = 01|001011$ represents the maximal basis $Max_6 = W_{34}$, its reverse ${W_{34}}^R = Min_6 = W_3$ is the improper minimal basis represented by $\mathbf{b}_{W_{34}}^R = 10|110100 = \mathbf{b}_{W_3}$. Here, $Min_6 > W_1 = min_t$, the proper minimal basis for $P_2 + P_6$, and $\mathbf{b}_{W_1} = {\mathbf{b}_{W_2}}^R$. The string $\mathbf{b}_{W_4} = 10|110001$ represents the basis W_4, so does ${\mathbf{b}_{W_4}}^R = 01|100011 = \mathbf{b}_{{W_4}^R}$, which actually represents the basis $W_{26} = \{2,3,7,8\} = {W_4}^R$.

Theorem 4.3. *If $t \leq 6$ then $dim(P_2 + P_t) \leq 4$. For $t \geq 7$, any binary string of the form $11|b_3...b_n$ cannot be a representation of a generator and any binary string of the form $10|b_3...b_n$ cannot be a representation of a maximal basis.*

PROOF. By the preceding example, the first statement is obvious. The second statement obviously derived from Theorem 3.3 □.

Some bases for $P_2 + P_t$ with $t < 7$ are provided in the following example and written using the notations defined at the beginning of this section.

Example 4.4. $min_1 = \{1,2\} < Min_1 = \{1,3\} < Max_1 = \{2,3\} = \{2\}|\{3\} = \{1\}^R|\{3\}^R = \{1,3\}^R = Min_1^R$ represented by $11|0, \mathbf{b}_{Min_1} = 10|1 = (01)^R|(1)^R = (01|1)^R$, which is the reverse of $01|1 = \mathbf{b}_{Max_1^R}$; whereas $min_2 = \{1,2,3\} < Min_2 = \{1,3,4\} < Max_2 = \{2,3,4\} = \{1,3,4\}^R$ represented by $11|10, 10|11, 01|11 = (10|11)^R$ respectively. Likewise, $min_3 = \{1,2,3\} < Min_3 = \{1,3,4\} < Max_3 = \{2,4,5\} = \{2\}|\{4,5\} = \{1\}^R|\{3,4\}^R = \{1,3,4\}^R = {Min_3}^R$ represented by $11|100$, $\mathbf{b}_{Min_3} = 10|110, \mathbf{b}_{Max_3} = 01|011 = (10)^R|(110)^R = (10110)^R = \mathbf{b}^R_{Min_3}$. For $t = 4, min_4 = Min_4 = \{1,3,4\} < Max_4 = \{2,5,6\}$ are represented by $10|1100, 01|0011$ respectively. Observe that any set of the form $W = \{1,2,x\}$ cannot be a basis for $P_2 + P_4$ since in this case there always exist $i, j \in \{3,4,5,6\}$ with $r(i|W) = r(j|W)$. For example if $x = 3$ then $r(5|W) = [1,1,2] = r(6|W)$ and if $x = 4$ then $r(3|W) = [1,1,1] = r(5|W)$. For $t = 5, min_5 = \{1,3,7\} < Min_5 = \{1,4,5\} < Max_5 = \{2,5,6\}$ are represented by $10|10001, 10|01100, 01|00110$. The minimal and maximal bases for $P_2 + P_6$ are already given in Example 4.2.

5. Conclusion

In the case of $t \geq 7$ large enough, the binary representations of Max_t have (repeated) substrings of length 5 that ease the search for their dimension. In particular when $t = 5(k+1)$, the binary representation of Max_t is of the form $01|(\boldsymbol{\alpha\beta})^{k+1}$, where $\boldsymbol{\alpha} = 001, \boldsymbol{\beta} = 01$ and $k = 1, 2, 3, \ldots$. Furthermore, for three consecutive t-values $5k-1, 5k, 5k+1$, the dimensions of $P_2 + P_t$ are the same odd number $2k+1$ and for two t-values $5k+2$ and $5k+3$, the dimensions of $P_2 + P_t$ are the same even number $2(k+1)$.

When $t = 1 + 5k$, the binary representation of Max_t is $01|(\boldsymbol{\alpha\beta})^k 0$ and this is only valid when $k > 1$. The case $k = 1$ results in $t = 6$, with Max_6 represented by $01|\boldsymbol{\alpha\beta}1$. In general, the regular pattern of the $dim(P_2 + P_t)$ begins at $t = 2 + 5 \cdot 1 = 7$ with $Max_7 = 01|\boldsymbol{\alpha\beta}01$. From $t = 7$, the dimension of $P_2 + P_t$ increases by 2 as t increases by 5. This regular behavior for $t \geq 7$ is due to the fact that the binary representation of maximal bases for $P_2 + P_t$ contains a substring $\boldsymbol{\alpha\beta}$ of length 5 which is repeated at least k times when $t \geq 2 + 5k$.

In the case of $t < 7$, there is no regular pattern to describe the basis and dimension of $P_2 + P_t$. Moreover, the graph $P_2 + P_4$ does not have minimal, proper basis whereas for $t = 1, 2, 3, 5$ or 6; the graph $P_2 + P_t$ does.

REFERENCES

[1] Chartrand, G., Eroh, L., Johnson, M.A., and Oellermann, O.R., "Resolvability in graphs and the metric dimension of a graph", *Discrete Appl. Math.*, **105** (2000), 99-113.
[2] Gallian, J.A., "A dynamic survey of graph labeling", *Electron J Comb.*, **19** (2016), 1-408.
[3] Harary, F. and Melter, R.A., "On the metric dimension of a graph", *Ars Combin.*, **2** (1976), 191-195.
[4] Kuziak, D., Rodrígues-Veláquez, J.A., and Yero, I.G., "Computing the metric dimension of a graph from primary subgraphs", *Discuss. Math.*, **37** (2017), 273-293.
[5] Kuziak, D., Yero, I.G., and Rodrígues-Veláquez, J.A., "On the strong metric dimension of corona product graphs and join graphs", *Discrete Appl. Math.*, **161** (2013), 1022-1027.
[6] Slater, P.J., "Leaves of trees", *Congr. Numer.*, **14** (1975), 549-559.