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FIXED POINT THEOREMS FOR NON-SELF MAPPINGS WITH NONLINEAR CONTRACTIVE CONDITION IN STRICTLY CONVEX FCM-SPACES

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Abstract. In this paper we define convex, strict convex and normal structures for sets in fuzzy cone metric spaces. Also, existence and uniqueness of a fixed point for non-self mappings with nonlinear contractive condition will be proved, using the notion of strictly convex structure. Moreover, we give some examples illustrate our results.

Key words and Phrases: fixed Point; convex structure; normal structure; fuzzy normed space.

Kata kunci: titik tetap; struktur convex; struktur normal; ruang bernorma fuzzy.

1. INTRODUCTION

The Banach Contraction Mapping Principle [2] is one of the most important theorems in functional analysis. There are many generalizations of this theorem for classical metric spaces. One of the most important of them is the introduction of a nonlinear contractive principle by Boyd and Wong [3]. Huang and Zhang [11] introduced the notion of cone metric spaces by replacing real numbers with an ordered Banach space and proved some fixed point theorems for contractive mappings between these spaces. They described the convergence in cone metric spaces, introduced the notion of completeness and proved some fixed point theorems of contractive mappings on these spaces.

On the other hand, after the theory of fuzzy sets which introduced by Zadeh [21], there has been a great effort to obtain fuzzy analogues of classical theories. In particular, Kramosil and Michalek in [15] introduced the fuzzy metric space. Later on, George and Veeramani in [9] gave a stronger form of metric fuzziness.

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The notion of fuzzy cone metric spaces, as a generalization of the corresponding notions of fuzzy metric spaces by George and Veeramani was introduced by Ōner, Kandemir and Tanay [14]. They studied topology, convergence of sequences, continuity of mappings, defined the completeness of these spaces, etc. Also, they gave the fuzzy cone Banach contraction theorem.

In 1970 Takahashi [20] was defined convex and normal structures for sets in metric spaces and generalized some important fixed point theorems previously proved for Banach spaces. In 1987 Hadžić [10] introduced the notion of convex structure for sets in Menger probabilistic metric spaces and proved fixed point theorem for mappings in probabilistic metric spaces with a convex structure. Ješić [12] defined convex, strictly convex and normal structure in intuitionistic fuzzy metric spaces. Recently, Ješić et al. [13] have introduced convex, strictly convex and normal structure in Menger PM-spaces.

Furthermore, in convex spaces occur cases where the involved function is not necessarily a self-mapping of a closed subset. Assad and Kirk [1] first considered non-self mappings in a metric spaces (X, d). They proved that for some non-self (single- valued) mapping $f : C \to X$, which satisfied Banach Contraction Mapping Principle $d(fx, fy) \leq \lambda d(x, y)$ for all $x, y \in C$ and $\lambda \in (0, 1)$, where X is complete metrically convex space in the sense of Menger (i.e. for every $x, y \in X$ $(x \neq y)$, there exists $z \in X$ such that $d(x, y) = d(x, z) + d(z, y)$), then the condition $f(\partial C) \subseteq C$ is sufficient to guarantee the existence of fixed point for mapping f, where ∂C is boundary of set C. In recent years many generalizations of mentioned theorem were proved (see e.g. [4], [5], [6], [7], [8] and [16]).

In this paper, using the notion of strictly convex structure for fuzzy cone metric space the existence and uniqueness of a fixed point for non-self mappings with non- linear contractive condition for function $\phi : P \to P$, will be proved. In the proof of the main result topological methods for characterization spaces with nondeterministic distances will be used.

2. PRELIMINARIES

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, $int(P)$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Longrightarrow \|x\| \leq K \|y\| .$$

The least positive number satisfying above is called the normal constant of P [11]. Rezapour and Hamlbarani [17] showed that there are no cones with normal constant $K < 1$ and there exist cones of normal constant 1, and cones of normal constant $M > K$ for each $K > 1$.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \le x_2 \le \cdots \le x_n \cdots \le y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ $(n \to \infty)$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose E is a Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\le$ is partial ordering with respect to P.

Proposition 2.1. *([17, 19]). Let P be a cone of E. Then*

(a) $P + int(P) \subset int(P)$;
(b) For every $\alpha \in \mathbb{R}^+$, we have $\alpha\, int(P) \subseteq int(P)$;
(c) For each $\theta \le c_1$ and $\theta \le c_2$, there is an element $\theta \le c$ such that $c \le c_1$, $c \le c_2$.

Definition 2.2. *([11]). A cone metric space is an ordered (X, d), where X is any set and $d : X \times X \to E$ is a mapping satisfying:*

(CM1) $d(x, y) \ge \theta$ for all $x, y \in X$,
(CM2) $d(x, y) = \theta$ if and only if $x = y$,
(CM3) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(CM4) $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$.

In [19], for $c \in E$ with $c \gg \theta$ and $x \in X$, define $B(x, c) = \{y \in X : d(x, y) \ll c\}$ and $\beta = \{B(x, c) : x \in X, c \in E \text{ with } c \gg \theta\}$, then show that

$$\tau_c = \{U \subset X : \forall x \in U, \exists B(x, c) \in \beta, x \in B(x, c) \subset U\}$$

is a topology on X.

Definition 2.3. *([11]). Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then*

(i) The sequence $\{x_n\}$ is said to converge to x if for any $c \in E$ with $c \gg \theta$ there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n \ge n_0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(ii) The sequence $\{x_n\}$ is said to be a Cauchy sequence if for any $c \in E$ with $c \gg \theta$ there exists a natural number n_0 such that $d(x_n, x_m) \ll c$ for all $n, m \ge n_0$.
(iii) (X, d) is said to be a complete cone metric space if every Cauchy sequence is convergent.

Definition 2.4. *([18]) A binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a continuous t-norm if $([0, 1], T)$ is a topological monoid with unit 1 such that $T(a, b) \le T(c, d)$ whenever $a \le c, b \le d$ for all $a, b, c, d \in [0, 1]$.*

Some typical examples of t-norm are the following:

$$T(a,b) = ab, \quad \text{(product)}$$
$$T(a,b) = \min\{a,b\}, \quad \text{(minimum)}$$
$$T(a,b) = \max\{a+b-1,0\}, \quad \text{(Lukasiewicz)}$$
$$T(a,b) = \frac{ab}{a+b-ab}, \quad \text{(Hamacher)}$$

Remark 2.5. *([9]). For any $r_1 > r_2$, we can find a r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 we can find a r_5 such that $r_5 * r_j \geq r_4$, where $r_j \in (0,1)$ for $j = 1, 2, \cdots, 5$.*

Definition 2.6. *([9]) A triple (X, M, T) is called a fuzzy metric space (briefly, a FM-space) if X is an arbitrary (non-empty) set, T is a continuous t-norm and M is a fuzzy set on $X \times X \times [0,\infty)$ such that the following axioms hold:*

(FM-1) $M(x,y,t) > 0$ *for all* $x, y \in X$ *and* $t > 0$,
(FM-2) $M(x,y,t) = 1$ *for every* $t > 0$ *if and only if* $x = y$,
(FM-3) $M(x,y,t) = M(y,x,t)$ *for all* $x, y \in X$ *and* $t > 0$,
(FM-4) $M(x,z,t+s) \geq T\{M(x,y,t), M(y,z,s)\}$ *for all* $x, y, z \in X$ *and for all* $t, s \in [0,\infty)$.
(FM-5) $M(x,y,.) : (0,\infty) \to [0,1]$ *is continuous.*

Remark 2.7. *([9]) Let (X, M, T) be a fuzzy metric space, $\tau = \{A \subset X : x \in A \longleftrightarrow \exists\, t > 0$ and $r \in (0,1)$ such that $B(x,r,t) \subset A\}$ is a topology on X.*

3. Fuzzy Cone Metric Spaces

In this section, we introduce the fuzzy cone metric space and the topology induced by this space. Then we give some properties.

Definition 3.1. *([14]) A triple (X, M, T) is said to be a fuzzy cone metric space (briefly, a FCM-space) if P is a cone in E, X is an arbitrary set, T is a continuous t-norm and M is a fuzzy set on $X^2 \times int(P)$ satisfying the following axioms:*

(FCM-1) $M(x,y,t) > 0$ *for all* $x, y \in X$ *and* $t \in int(P)$;
(FCM-2) $M(x,y,t) = 1$ *for every* $t \in int(P)$ *if and only if* $x = y$,
(FCM-3) $M(x,y,t) = M(y,x,t)$ *for all* $x, y \in X$ *and* $t \in int(P)$,
(FCM-4) $M(x,z,t+s) \geq T\{M(x,y,t), M(y,z,s)\}$ *for all* $x, y, z \in X$ *and for all* $t, s \in int(P)$,
(FCM-5) $M(x,y,.) : int(P) \to [0,1]$ *is continuous.*

Remark 3.2. *If we take $E = \mathbb{R}$, $P = (0,\infty)$ and $T(a,b) = ab$, then every fuzzy metric spaces became a fuzzy cone metric spaces.*

Example 3.3. *([14]) Let $E = \mathbb{R}^2$. Then $P = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\}$ is a normal cone with normal constant $K = 1$. Let $X = \mathbb{R}$, $T(a,b) = ab$ and $M : X^2 \times int(P) \to [0,1]$ defined by*

$$M(x,y,t) = \frac{1}{e^{\frac{|x-y|}{\|t\|}}}$$

for all $x, y \in X$ *and* $t \gg \theta$. *Then* (X, M, T) *is a fuzzy cone metric spaces.*

Example 3.4. *(*[14]*) Let* P *be any cone,* $X = \mathbb{N}$, $T(a,b) = ab$, $M : X^2 \times int(P) \to [0,1]$ *defined by*

$$M(x,y,t) = \begin{cases} x/y, & \text{if } x \leq y; \\ y/x, & \text{if } y \leq x. \end{cases}$$

for all $x, y \in X$ *and* $t \gg \theta$. *Then* (X, M, T) *is a fuzzy cone metric spaces.*

Lemma 3.5. *(*[14]*)* $M(x,y,.) : int(P) \to [0,1]$ *is nondecreasing for all* $x, y \in X$.

Definition 3.6. *(*[14]*) Let* (X, M, T) *be a fuzzy cone metric space. For* $t \gg \theta$, *the open ball* $B(x,r,t)$ *with center* x *and radius* $r \in (0,1)$ *is defined by*

$$B_x(r,t) = \{y \in X : M(x,y,t) > 1 - r\}.$$

Proposition 3.7. *(*[14]*) Let* (X, M, T) *be a fuzzy cone metric space. Define*

$$\tau_{fc} = \{A \subset X : x \in A \Longleftrightarrow \exists\, r \in (0,1), \text{ and } t \gg \theta \text{ such that } B_x(r,t) \subset A\},$$

then τ_{fc} *is a topology on* X.

Proposition 3.8. *(*[14]*) Let* (X, M, T) *be a fuzzy cone metric space. Then* (X, τ_{fc}) *is a Hausdorff and first countable.*

Definition 3.9. *(*[14]*) Let* (X, M, T) *be a fuzzy cone metric space,* $x \in X$ *and* $\{x_n\}$ *be a sequence in* X. *Then*

(i) $\{x_n\}$ *is said to be converge to* x *if for any* $t \gg \theta$ *and any* $r \in (0,1)$ *there exists a natural number* n_0 *such that* $M(x_n, x, t) > 1 - r$ *for all* $n \geq n_0$. *We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$.

(ii) $\{x_n\}$ *is said to be a Cauchy sequence if for any* $t \gg \theta$ *and any* $r \in (0,1)$ *there exists a natural number* n_0 *such that* $M(x_n, x_m, t) > 1 - r$ *for all* $n, m \geq n_0$.

(iii) *A fuzzy cone metric space is said to be complete if every Cauchy sequence is convergent.*

Definition 3.10. *Let* (X, M, T) *be a fuzzy cone metric space and* $A \subset X$. *The closure of the set* A *is the smallest closed set containing* A, *denoted by* $\overline{A}$.

Definition 3.11. *A subset* K *of a fuzzy cone metric space is called compact if following statement holds:*

$$K \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \Longrightarrow K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \text{ for some } \alpha_1, \cdots, \alpha_n \in \Lambda$$

for every collection $\{U_\alpha : \alpha \in \Lambda\}$ *of open sets* $U_\alpha \subseteq X$.

Lemma 3.12. *Let* (X, M, T) *be a fuzzy cone metric space with a continuous t-norm* T *and* $K \subseteq X$. *Then,* K *is compact if and only if for every collection of closed sets* $\{F_\alpha\}$ *such that* $F_\alpha \subseteq K$ *it holds that*

$$\bigcap_{\alpha\in\Lambda} F_\alpha \Longrightarrow \bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset \text{ for some } \alpha_1, \cdots, \alpha_n \in \Lambda\ .$$

Lemma 3.13. *Let (X, M, T) be a fuzzy cone metric space with a continuous t-norm T and $K \subseteq X$. Then $x \in \overline{K}$ if and only if there exists a sequence $\{x_n\}$ in K such that $x_n \to x$.*

Definition 3.14. *Let (X, M, T) be a fuzzy cone metric space with a continuous t-norm T and $A \subseteq X$. The fuzzy diameter of A is given by*

$$\delta_A(t) = \inf_{x,y\in A} \sup_{s<t} M(x, y, s).$$

The diameter of the set A is defined as

$$\delta_A = \sup_{t>0} \inf_{x,y\in A} \sup_{s<t} M(x, y, s).$$

If there exists a number $\lambda \in (0, 1)$ such that $\delta_A = 1 - \lambda$, then the set A is called fuzzy semi-bounded. If $\delta_A = 1$, then A is called fuzzy bounded.

Lemma 3.15. *Let (X, M, T) be a fuzzy cone metric space with a continuous t-norm T and $A \subseteq X$. A set A is fuzzy bounded if and only if for each $\lambda \in (0, 1)$ there exists $t \gg \theta$ such that $M(x, y, t) > 1 - \lambda$ for all $x, y \in A$.*

Proof. The proof follows immediately from the definition of $\sup A$ and $\inf A$ of non-empty sets. □

It is not difficult to see that every metrically bounded set is also fuzzy bounded if it is considered in the induced FCM-space.

Theorem 3.16. *Every compact subset A of a fuzzy cone metric space (X, M, T) with continuous t-norm T is fuzzy semi-bounded.*

Proof. Let A be a compact subset of X. Let fix $\epsilon \gg \theta$ and $\lambda \in (0, 1)$. Now, we will consider an (ϵ, λ)-cover $\{B_x(\epsilon, \lambda) : x \in A\}$. Since A is compact, there exist $x_1, x_2, \cdots, x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} B_{x_i}(\epsilon, \lambda)$. Let $x, y \in A$. Then there exists $i \in \{1, \cdots n\}$ such that $x \in B_{x_i}(\epsilon, \lambda)$ and exists $j \in \{1, \cdots n\}$ such that $y \in B_{x_j}(\epsilon, \lambda)$. Thus we have $M(x, x_i, \epsilon) > 1 - \lambda$ and $M(y, x_j, \epsilon) > 1 - \lambda$. Now, let $m = \min\{M(x_i, x_j, \epsilon) : 1 \leq i, j \leq n\}$. It is obvious that $m > 0$ and we have

$$M(x, y, \epsilon) \geq T(M(x, x_i, \epsilon), M(x_i, x_j, \epsilon), M(x_j, y, \epsilon)) \geq T(1 - \lambda, m, 1 - \lambda) > 1 - \delta,$$

for some $0 < \delta < 1$. If we take $\epsilon_1 = 3\epsilon$, we have $M(x, y, \epsilon) > 1 - \delta$ for all $x, y \in A$. Hence we obtain that A is fuzzy semi-bounded set. □

Proposition 3.17. *Let (X, M, T) be a fuzzy cone metric space with a continuous t-norm T and τ_{fc} be the topology induced by fuzzy cone metric space. Then for any nonempty subset $S \subseteq X$ we have*

(i) S is closed if and only if for any sequence $\{x_n\}$ in X which converges to x, we have $x \in S$;

(ii) if we define $\overline{S}$ to be the intersection of all closed subset of X which contain S, then for any $x \in \overline{S}$ and for any $0 < \lambda < 1$ and $\epsilon \gg \theta$, we have $N_x(\epsilon, \lambda) \cap S \neq \emptyset$.

Proof. (i) Assume that S is closed and let $\{x_n\}$ be a sequence in S such that $\lim_{n\to\infty} x_n = x$. Let us prove that $x \in S$. Assume not, i.e. $x \notin S$. Since S is closed, then there exists $0 < \lambda < 1$ and $\epsilon \gg \theta$ such that $B_x(\lambda, \epsilon) \cap X = \emptyset$. Since $\{x_n\}$ converges to x, then there exists $N \geq 1$ such that for any $n \geq N$ we have $x_n \in B_x(\lambda, \epsilon)$. Hence $x_n \in B_x(\lambda, \epsilon) \cap S$, which leads to a contradiction. Conversely assume that for any sequence $\{x_n\}$ in S which converges to x, we have $x \in S$. Let us prove that S is closed. Let $x \notin S$. We need to prove that there exists $0 < \lambda < 1$ and $\epsilon \gg \theta$ such that $B_x(\lambda, \epsilon) \cap S = \emptyset$. Assume not, i.e. for any $0 < \lambda < 1$ and $\epsilon \gg \theta$, we have $B_x(\lambda, \epsilon) \cap S \neq \emptyset$. So for any $n \geq 1$, choose $x_n \in B_x(\frac{1}{n}, \epsilon)$. Clearly we have $\{x_n\}$ converges to x. Our assumption on S implies $x \in S$, a contradiction.
(ii) Clearly $\overline{S}$ is the smallest closed subset which contains S. Set

$$S^* = \{x \in X : \text{for any } \epsilon \gg \theta, \text{ there exists } a \in S \text{ such that } M(x, a, \epsilon) > 1 - \lambda\}.$$

We have $S \subset S^*$. Next we prove that S^* is closed. For this we use property (i). Let $\{x_n\}$ be a sequence in S^* such that $\{x_n\}$ converges to x. Let $0 < \lambda < 1$ and $\epsilon \gg \theta$. Since $\{x_n\}$ converges to x, there exists $N \geq 1$ such that for any $n \geq N$ we have

$$M\left(x, x_N, \left(\frac{t}{2}\right)\right) > 1 - \lambda.$$

Let $\lambda_0 = M\left(x, x_N, \left(\frac{t}{2}\right)\right) > 1 - \lambda$. Since $\lambda_0 > 1 - \lambda$, we can find an μ, $0 < \mu < 1$, such that $\lambda_0 > 1 - \mu > 1 - \lambda_0$. Now for a given λ_0 and μ such that $\lambda_0 > 1 - \mu$ we can find λ_1, $0 < \lambda_1 < 1$, such that

$$\lambda_0 * (1 - \lambda_1) \geq 1 - \mu.$$

Now since $x_n \in S^*$, there exists $a \in X$ such that

$$M\left(x_n, a, \left(\frac{\epsilon}{2}\right)\right) > 1 - \lambda_1$$

Hence

$$M(x, a, \epsilon) \geq T\left(M\left(x, x_N, \left(\frac{t}{2}\right)\right), M\left(x_n, a, \left(\frac{\epsilon}{2}\right)\right)\right) > T(\lambda_0, 1-\lambda_1) \geq 1-\mu > 1-\lambda,$$

which implies $x \in S^*$. Therefore S^* is closed and contains S. The definition of $\overline{S} \subset S^*$, which implies the conclusion of (ii). □

Note that, every compact subset of a Hausdorff topological space is closed.

Proposition 3.18. *Let $(X, M, *)$ be a fuzzy cone metric space with a continuous t-norm T and τ_{fc} be the topology induced by fuzzy cone metric type. Let S be a nonempty subset of X. The following properties are equivalent:*

(i) S is compact.
(ii) For any sequence $\{x_n\}$ in S, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges, and if $\{x_{n_k}\}$ converges to x then $x \in S$.

Proof. (i) Assume that S is a nonempty compact subset of X. It is easy to see that any decreasing sequence of nonempty closed subsets of S have a nonempty intersection. Let $\{x_n\}$ be a sequence in S. Set $C_m = \{x_m : m \geq n\}$. Then we have

$\bigcap_{n\geq 1}\overline{C_n}\neq\emptyset$. Then for $0<\lambda<1$, $t\gg\theta$ and for any $n\geq 1$, there exists $m_n\geq n$ such that $M(x,x_{m_n},\epsilon)>1-\lambda$. This clearly implies the existence of a subsequence of $\{x_n\}$ which converges to x. Since S is closed, then we must have $x\in S$.
Conversely let S be a nonempty subset of X such that the conclusion of (ii) is true. Let us prove that S is compact. First note that for any $0<\lambda<1$, $t\gg\theta$, there exists $x_1,x_2,\cdots,x_n\in A$ such that

$$S\subseteq\bigcup_{i=1}^{n}B_{x_i}(\lambda,\epsilon).$$

Assume not, then there exists $0<\lambda_0<1$, such that for any finite number of points $x_1,x_2,\cdots,x_n\in X$, we have

$$S\nsubseteq\bigcup_{i=1}^{n}B_{x_i}(\lambda,\epsilon).$$

Fix $x_1\in X$. Since $S\nsubseteq B_{x_1}(\lambda,\epsilon)$, there exists $x_2\in S\setminus B_{x_1}(\lambda,\epsilon)$. By induction we build a sequence $\{x_n\}$ such that

$$x_{n+1}\in S\setminus(B_{x_1}(\lambda,\epsilon)\cup\cdots\cup B_{x_n}(\lambda,\epsilon))$$

for all $n\geq 1$. Clearly we have $M(x_n,x_m,\epsilon)<1-\lambda_0$ for all $n,m\geq 1$, with $n\neq m$. This condition implies that no subsequence of $\{x_n\}$ will be Cauchy or convergent. This contradicts our assumption on X. Next let $\{O_\alpha\}_{\alpha\in J}$ be an open cover of S. Let us prove that only finitely many O_α cover S. Fix $\epsilon\gg\theta$, first note that there exists $0<\lambda_0<1$ such that for any $x\in S$, there exists $\alpha\in J$ such that $N_x(\lambda_0,\epsilon)\subset O_\alpha$. Assume not, then for any $0<\lambda<1$, there exists $x_\lambda\in X$ such that for any $\alpha\in J$, we have $B_x(\lambda,\epsilon)\nsubseteq O_\alpha$. In particular, for any $n\geq 1$, there exists $x_n\in X$ such that for any $\alpha\in J$, we have $N_x(\epsilon,1/n)\nsubseteq O_\alpha$. By our assumption on S, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to some point $x\in X$. Since the family $\{O_\alpha\}_{\alpha\in J}$ covers X, there exists $\alpha_0\in J$ such that $x\in O_{\alpha_0}$. Since O_{α_0} is open, there exists $0<\lambda_0<1$, and $\epsilon_0\gg\theta$ such that $B_x(\lambda,\epsilon)\subset O_{\alpha_0}$. Fix $\epsilon\gg\theta$ and let $\epsilon_1=\epsilon$, for any $n_1\geq 1$ and $a\in N_{x_{n_1}}(\frac{1}{n_1},\epsilon)=N_{x_{n_1}}(\frac{1}{n_1},\epsilon_1)$ we have

$$M(x,a,\epsilon_0)\geq T(M(x,x_{n_k},\epsilon_0-\epsilon_1),M(x_{n_k},a,\epsilon_1))>M(x,x_{n_k},\epsilon_0-\epsilon_1)>1-\frac{1}{n_k}$$

for n_k large enough, we will get $M(x,a,\epsilon)>1-\lambda_0$ for any $a\in B_{x_{n_k}}(\frac{1}{n_k},\epsilon)$. In the other words, we have $B_{x_{n_k}}(\frac{1}{n_k},t)\subset B_x(r_0,t_0)$, which implies

$$B_{x_{n_k}}(\frac{1}{n_k},\epsilon)\subset O_{\alpha_0}.$$

This is in clear contradiction with the way the sequence $\{x_n\}$ was constructed. Therefore there exists $0<\lambda_0<1$ such that for any $x\in S$, there exists $\alpha\in J$ such that $B_x(\lambda_0,\epsilon)\subset O_\alpha$. For such λ_0, there exist $x_1,x_2,\cdots,x_n\in X$ such that

$$S\subset B_{x_1}(\lambda_0,\epsilon)\cup\cdots\cup B_{x_n}(\lambda_0,\epsilon)$$

But for any $i=1,2,\cdots,n$ there exists $\alpha\in J$ such that $N_{x_i}(\epsilon,\lambda_0)\subset O_\alpha$, i.e. $S\subset O_{\alpha_1}\cup\cdots\cup O_{\alpha_n}$. This completes the proof that S is compact. $\square$

Lemma 3.19. *In a fuzzy cone metric space (X, M, T) with a continuous t-norm T, every compact set is closed and bounded.*

4. Convex Structure, Normal structure and Strictly Convex Structure on FM-Spaces

Takahashi [20] introduced the notion of metric spaces with a convex structure. This class of metric spaces includes normed linear spaces and metric spaces of the hyperbolic type.

Definition 4.1. *Let (X, d) be a cone metric space. We say that a cone metric space possesses a Takahashi convex structure if there exists a function $W : X^2 \times [0, 1] \to X$ which satisfies*

$$d(z, W(x, y, \mu)) \leq \mu d(z, x) + (1 - \mu)d(z, y),$$

for all $x, y, z \in X$ and arbitrary $\mu \in [0, 1]$. A cone metric space (X, d) with Takahashi's structure is called convex cone metric space.

In this section, we introduce a generalization of Takahashi's definition to the case of a fuzzy cone metric space.

Definition 4.2. *Let (X, M, T) be a fuzzy cone metric space with continuous t-norm T. A mapping $S : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if for every $(x, y) \in X \times X$ holds $S(x, y, 0) = y$, $S(x, y, 1) = 1$ and for all $x, y, z \in X$, $\mu \in [0, 1]$ and $t \gg \theta$*

$$M(S(x, y, \mu), z, 2t) \geq T\left(M\left(x, z, \frac{t}{\mu}\right), M\left(x, z, \frac{t}{1 - \mu}\right)\right). \tag{1}$$

Example 4.3. *Let $X = \mathbb{R}$, $E = \mathbb{R}$ and $P = [0, \infty)$ be a cone. Let $S : X \times X \times [0, 1] \to X$ defined by*

$$S(x, y, \mu) = \mu x + (1 - \mu)y$$

for all $x, y \in \mathbb{R}$ and $\mu \in (0, 1)$ is a convex structure on fuzzy cone metric space $(\mathbb{R}, M, T)$ induced by a metric $d(x, y) = |x - y|$ on X, where $T(a, b) = \min\{a, b\}$ is continuous t-norm for $a, b \in [0, 1]$ and

$$M(x, y, t) = \begin{cases} 0, & \text{if } t \leq d(x, y); \\ 1, & \text{if } d(x, y) < t. \end{cases}$$

for all $x, y \in X$ and $t \gg \theta$. Let us prove this assertion. Firstly, we have that $S(x, y, 0) = y$ and $S(x, y, 1) = x$ for all $x, y \in X$. Now, let us prove that inequality (1) is satisfied. If we assume that

$$\min\left\{M\left(x, z, \left(\frac{t}{\mu}\right)\right), M\left(y, z, \left(\frac{t}{1 - \mu}\right)\right)\right\} = 0$$

then inequality (1) is a trivially satisfied because we get $M(S(x, y, \mu), z, 2t) \geq 0$. Now we will assume that $M\left(x, z, \left(\frac{t}{\mu}\right)\right) = 1$ and $M\left(y, z, \left(\frac{t}{1-\mu}\right)\right) = 1$. Then we *have that* $\frac{t}{\mu} > d(x,z)$ *and* $\frac{t}{1-\mu} > d(y,z)$*, i.e.* $t > \mu d(x,z)$ *and* $t > (1-\mu)d(y,z)$*. Hence, we obtain*

$$\begin{aligned} 2t &> \mu d(x,z) + (1-\mu)d(y,z) = \mu|x-z| + (1-\mu)|y-z| \\ &\geq |\mu x - \mu z + (1-\mu)y - (1-\mu)z| \\ &= |\mu x + (1-\mu)y - z| = d(\mu x + (1-\mu)y, z) \\ &= d(S(x,y,\mu), z), \end{aligned}$$

that is, $2t - d(S(x,y,\mu),z) > 0$ *and so* $M(S(x,y,\mu),z,2t) = 1$*. Therefore, inequality (1) holds for all* $x,y,z \in X$ *and* $t \gg \theta$*.*

It is easy to see that every cone metric space (X,d) with a convex structure S can be consider as a fuzzy cone metric space (X,M,T) (the associated fuzzy metric space) with the same function S. A fuzzy cone metric space (X,M,T) with a convex structure S is called a convex fuzzy cone metric space.

Here we give some terminology will be used in the sequel.

Definition 4.4. *A point* $x \in A$ *is called diametral if* $\inf_{y\in A}\sup_{s<t} M(x,y,s) = \delta_A(t)$ *holds for all* $t > 0$*.*

Definition 4.5. *Let* (X,M,T) *be a fuzzy cone metric space and* $A \subseteq X$*. The fuzzy diameter of* A *is given by*

$$\delta_A(t) = \sup_{\epsilon<t}\inf_{x,y\in A} M(x,y,\epsilon).$$

The diameter of the set A *is defined by*

$$\delta_A = \sup_{t>0}\delta_A(t).$$

If there exists $\lambda \in (0,1)$ *such that* $\delta_A = 1-\lambda$ *the set* A *will be called* fuzzy semi-bounded*. If* $\delta_A = 1$ *the set* A *will be called* fuzzy bounded*.*

Lemma 4.6. *Let* (X,M,T) *be a fuzzy cone metric space. A set* $A \subseteq X$ *is fuzzy bounded if and only if for each* $\lambda \in (0,1)$ *there exists* $t \gg \theta$ *such that* $M(x,y,t) > 1-\lambda$ *for all* $x,y \in A$*.*

Definition 4.7. *Let* (X,M,T) *be a fuzzy cone metric space with continuous* t*-norm* T *and a convex structure* $S(x,y,\mu)$*. A subset* $A \subseteq X$ *is said to be a convex if for every* $x,y \in A$ *and* $\mu \in [0,1]$ *it follows that* $S(x,y,\mu) \in A$*.*

Lemma 4.8. *Let* (X,M,T) *be a fuzzy cone metric space with continuous* t*-norm* T *and let* $\{K_\alpha\}_{\alpha\in\Lambda}$ *be a family of convex subsets of* X*. Then the intersection* $K = \bigcap_{\alpha\in\Lambda} K_\alpha$ *is a convex set.*

Proof. If $x,y \in K$, then $x,y \in K_\alpha$ for every $\alpha \in \Lambda$. It follows that $S(x,y,\mu) \in K_\alpha$ for every $\alpha \in \Lambda$, i.e., $S(x,y,\mu) \in K$, which means that K is convex. □

Definition 4.9. *A convex fuzzy cone metric space* (X,M,T) *with a convex structure* $S : X \times X \times [0,1] \to X$ *and continuous* t*-norm* T *will be called strictly convex* *if, for arbitrary $x, y \in X$ and $\mu \in (0,1)$ the element $z = S(x, y, \mu)$ is the unique element which satisfies*

$$M\left(x, y, \frac{t}{\mu}\right) = M(x, y, t), \quad M\left(x, y, \frac{t}{1-\mu}\right) = M(x, y, t), \tag{2}$$

for all $t \gg \theta$.

Lemma 4.10. *Let (X, M, T) be a fuzzy cone metric space with a convex structure $S(x, y, \lambda)$ and continuous t-norm T. Suppose that for every $\mu \in (0,1)$, $t > 0$ and $x, y, z \in X$ hold*

$$M(S(x, y, \mu), z, t) > \min\{M(z, x, t), M(z, y, t)\}. \tag{3}$$

If there exists $z \in X$ such that

$$M(S(x, y, \mu), z, t) = \min\{M(z, x, t), M(z, y, t)\} \tag{4}$$

is satisfied, for all $t \gg \theta$, then $S(x, y, \mu) \in \{x, y\}$.

Proof. Let us assume that (4) holds for some $z \in X$ and for all $t \gg \theta$. Since (3) holds, it follows that $\mu = 0$ or $\mu = 1$ and, consequently we have that $S(x, y, 0) = y$ or $S(x, y, 1) = x$,which proves the lemma. □

Lemma 4.11. *Let (X, M, T) be a fuzzy metric space with a convex structure $S(x, y, \mu)$ and continuous t-norm T. Then for arbitrary $x, y \in X$, $x \neq y$ there exists $\mu \in (0,1)$ such that $S(x, y, \mu) \notin \{x, y\}$.*

Proof. Suppose that for every $\mu \in (0,1)$, it holds that $S(x, y, \mu) \in \{x, y\}$. From (2) it follows that $M(x, y, t) = 1$ for all $t \gg \theta$ which means that $x = y$ and so the proof is achieved. □

Definition 4.12. *A fuzzy cone metric space (X, M, T) with continuous t-norm T possesses a normal structure if, for every closed , fuzzy semi-bounded and convex set $Y \subset X$, which consists of at least two different points, there exists a point $x \in Y$ which is non-diametral, i.e., there exists $t_0 > 0$ such that*

$$\delta_Y(t_0) < \inf_{y \in Y} \sup_{s < t_0} M(x, y, s)$$

holds.

It is obvious that compact and convex sets in convex metric space possess a normal structure (see [20]).

Definition 4.13. *Let (X, M, T) be a convex fuzzy cone metric space with continuous t-norm T and $Y \subseteq X$. The closed convex shell of a set Y denoted by $cov(Y)$, is the intersection of all closed, convex sets that contain Y.*

It is easy to see that the set $cov(Y)$ exists, since the collection of closed, convex sets that contain Y is non-empty, because the fact that X belongs to this collection. From Lemma 4.8, it follows that this intersection is convex set. Also, this intersection is closed as an intersection of closed sets.

Example 4.14. *Let $E = \mathbb{R}$. Then $P = \{x \in \mathbb{R} : x \geq 0\}$ is a normal cone with normal constant $K = 1$. Let $X = \mathbb{R}$, $T(a, b) = \min\{a, b\}$ and $M : X^2 \times int(P) \to [0, 1]$ defined by*

$$M(x, y, t) = H(t - d(x, y))$$

for all $x, y \in X$ and $t \gg \theta$, where $d : X^2 \to E$ is a cone metric space defined by $d(x, y) = |x - y|$ and

$$H(t) = \begin{cases} 0, & \text{if } d(x, y) \leq t; \\ 1, & \text{if } d(x, y) > t. \end{cases}$$

Then (X, M, T) is a fuzzy cone metric spaces.
The mapping $S : \mathbb{R} \times \mathbb{R} \times [0, 1] \to [0, 1]$ defined by

$$S(x, y, \mu) = \mu x + (1 - \mu)y$$

for all $x, y \in \mathbb{R}$ and $\mu \in (0, 1)$ is a convex structure on fuzzy cone metric space. For arbitrary $x, y \in X$ and $\mu \in (0, 1)$ the element $z = S(x, y, \mu) = \mu x + (1 - \mu)y$ is the unique element which satisfies

$$\begin{aligned} M(z, x, t) &= H(t - d(x, z)) = H(t - d(x - \mu x, z - \mu x)) \\ &= H\left(\frac{t}{1 - \mu} - \frac{d((1 - \mu)x, z - \mu x)}{1 - \mu}\right) \\ &= H\left(\frac{t}{1 - \mu} - d\left(x, \frac{z}{1 - \mu} - \frac{\mu x}{1 - \mu}\right)\right) \\ &= H\left(\frac{t}{1 - \mu} - d(x, y)\right) \\ &= M\left(x, y, \frac{t}{1 - \mu}\right). \end{aligned}$$

In a similar way it can proved that the second equality in (2) is satisfied. Hence we obtained that the fuzzy cone metric space is strictly convex with a given convex structure $S(x, y, \mu)$.

On the other hand, we have that

$$d(\mu x + (1 - \mu)y, z) < \max\{d(x, z), d(y, z)\}$$

is satisfied for all $\mu \in (0, 1)$, and it follows that

$$\begin{aligned} M(S(x, y, \mu), z, t) &= H(t - d(S(x, y, \mu), z)) \\ &> H(t - \max\{d(x, z), d(y, z)\}) \\ &= \min\{H(t - d(x, z)), H(t - d(y, z))\} \\ &= \min\{M(x, z, t), M(y, z, t)\} \end{aligned}$$

holds, that is, condition (3) is satisfied.

Definition 4.15. *Let (X, M, T) be a fuzzy cone metric space with continuous t-norm T and let f be a self-mapping on X. We say that f is a non-expansive mapping if*

$$M(fx, fy, t) \geq M(x, y, t) \tag{5}$$

holds for all $x, y \in X$ and $t \gg \theta$.

Proposition 4.16. *Let (X, M, T) be a fuzzy cone metric space and $\{F_n\}_{n\in\mathbb{N}}$ a nested sequence of nonempty, closed subsets of X such that $\lim_{n\to\infty} \delta_{F_n} = 1$. Then there is exactly one point $x_0 \in F_n$, for every $n \in \mathbb{N}$.*

Lemma 4.17. *Let (X, M, T) be a fuzzy cone metric space and $\{F_n\}_{n\in\mathbb{N}}$ a nested sequence of nonempty, closed subsets of X. The sequence $\{F_n\}_{n\in\mathbb{N}}$ has fuzzy diameter zero, i.e., for each $\lambda \in (0,1)$ and $t \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $M(x, y, t) > 1 - \lambda$ for all $x, y \in F_{n_0}$ if and only if $\lim_{n\to\infty} \delta_{F_n} = 1$.*

5. Main Results

Definition 5.1. *Let Φ_W denote the class of all functions $\phi : P \to P$ satisfying the following condition: for each $t \gg \theta$, there exists $r \geq t$ such that $\lim_{n\to\infty} \phi^n(r) = 0$.*

Lemma 5.2. *Let $\phi \in \Phi_W$, then for each $t \gg \theta$ there exists $r \geq t$ such that $\phi(r) < t$.*

Definition 5.3. *A t-norm is said to be a Hadzić type (shortly H-type) t-norm if the family $\{T^m\}_{m\geq 1}$ of its iterates defined for each $t \in [0,1]$ by*

$$T^1(t) = T(t,t)$$

and, in general, for all $m > 1$, $T^m(t) = (t, T^{m-1}(t))$ is equi-continuous at $t = 1$, that is, given $\lambda > 0$ there exists $\eta(\lambda) \in (0,1)$ such that

$$\eta(\lambda) < t \leq 1 \Rightarrow T^{(m)}(\eta(\lambda)) \geq 1 - \lambda$$

for all $m > 0$.

Definition 5.4. *Let (X, M, T) be a fuzzy cone metric space with continuous t-norm T of H-type. A mapping $f : X \to X$ is said to be a fuzzy ϕ-contraction if there exists a function $\phi \in \Phi_W$ such that*

$$M(fx, fy, \phi(t)) \geq M(x, y, t) \tag{6}$$

for all $x, y \in X$ and $t \gg \theta$.

Lemma 5.5. *Let $\{x_n\}$ be a sequence in a fuzzy cone metric space with continuous t-norm T of H-type. If there exists a function $\phi \in \Phi_W$ such that*

(i) $\phi(t) > t$ for all $t \gg \theta$;
(ii) $M(x_n, x_{n+1}, \phi(t)) \geq M(x_{n-1}, x_n, t)$ for all $n \in \mathbb{N}$ and $t \gg \theta$,

then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 5.6. *Let (X, M, T) be a fuzzy cone metric space with continuous t-norm T of H-type. If there exists a function $\phi \in \Phi_W$ such that*

$$M(x, y, \phi(t)) \geq M(x, y, t) \tag{7}$$

for all $t \gg \theta$ and $x, y \in X$, then $x = y$.

Proof. Since M is monotonic, it is obvious that from (7) it follows $\phi(t) > 0$ for all $t \gg \theta$. Therefore we have $\phi^n(t) > 0$ for all $t \gg \theta$ and $n \geq 1$. By induction, we have from (7) that

$$M(x, y, \phi^n(t)) \geq M(x, y, t) \tag{8}$$

for all $t \gg \theta$ and $n \geq 1$.
To prove $x = y$, it is required that $M(x, y, t) = 1$ for all $t \gg \theta$. Suppose, to the contrary, that there exists some $t_0 \gg \theta$ such that $M(x, y, t_0) < 1$. Since $\lim_{t\to\infty} M(x, y, t) = 1$, there exists $t_1 > t_0$ such that

$$M(x, y, t) \geq M(x, y, t_0) \tag{9}$$

for all $t \geq t_1$.
Since $\phi \in \Phi_W$, there exists $t_2 \geq t_1$ such that $\lim_{n\to\infty} \phi^n(t_2) = 0$. Therefore, we can choose large enough $n_0 \geq 1$ such that $\phi^{n_0}(t_2) < t_0$. By the monotone property of M, using (8) and (9), we have

$$M(x, y, t_0) \geq M(x, y, \phi^{n_0}(t_2)) \geq M(x, y, t_2) > M(x, y, t_0),$$

which is a contradiction. Therefore, $M(x, y, t) = 1$ for all $t \gg \theta$ and so $x = y$. □

Theorem 5.7. *Let (X, M, T) be a complete fuzzy cone metric space with continuous t-norm T of H-type. If the mapping f is a fuzzy ϕ-contraction, then f has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary point X and the sequence $\{x_n\}$ be defined as follows: $x_{n+1} = f^n x_0$ for all $n \geq 1$. Since f is a fuzzy ϕ-contraction, by (6) for all $t \gg \theta$ we have

$$\begin{aligned} M(x_n, x_{n+1}, \phi(t)) &= M(fx_{n-1}, fx_n, \phi(t)) \\ &\geq M(x_{n-1}, x_n, t). \end{aligned}$$

By Lemma 5.5 we conclude that the sequence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$, that is, for all $t \gg \theta$,

$$\lim_{n\to\infty} M(x, x_n, t) = 1. \tag{10}$$

Now we prove that x is a fixed point of f. By (FCM-4) with $y = fx_n$ in Definition 3.1, we have

$$\begin{aligned} M(fx, x, t) &\geq T(M(fx, fx_n, \phi(t)), M(fx_n, x, t - \phi(t))) \\ &\geq T(M(fx, fx_n, \phi(t)), M(x_{n+1}, x, t - \phi(t))). \end{aligned}$$

Hence, by (6), we get

$$M(fx, x, t) \geq T(M(x, x_n, \phi(t)), M(x_{n+1}, x, t - \phi(t))) \tag{11}$$

Again by (FCM-3) and (FCM-4) with $y = x_{n=1}$ in Definition 3.1,

$$\begin{aligned} M(x, x_{n+1}, t - \phi(t)) &= M(x_{n+1}, x, t - \phi(t)) \\ &\geq T\left(M\left(x, x_{n+1}, \frac{t - \phi(t)}{2}\right), M\left(x_{n+1}, x, \frac{t - \phi(t)}{2}\right)\right) \end{aligned}$$

Hence, by (FCM-3) in Definition 3.1, we have

$$M(x_{n+1}, x, t-\phi(t)) \geq T\left(M\left(x, x_{n+1}, \frac{t-\phi(t)}{2}\right), M\left(x, x_{n+1}, \frac{t-\phi(t)}{2}\right)\right).$$

Now from (11) we get

$$M(fx,x,t) \geq T\left(M(x,x_n,t), T\left(M\left(x, x_{n+1}, \frac{t-\phi(t)}{2}\right), M\left(x, x_{n+1}, \frac{t-\phi(t)}{2}\right)\right)\right) \tag{12}$$

Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x$ and T is continuous, taking $n \to \infty$ in (12) we get, for all $t \gg \theta$,

$$M(fx, x, t) \geq T(1, T(1,1)) = T(1,1) = 1.$$

Hence $fx = x$, that is, x is a fixed point of f.

Next suppose that $y \neq x$ is another fixed point of f. Then, for all $t \gg \theta$, we have

$$\begin{aligned} M(x, y, \phi(t)) &= M(fx, fy, \phi(t)) \\ &\geq M(x, y, t) \end{aligned}$$

which implies, by Lemma 5.6, that $x = y$. Therefore, f has a unique fixed point in X. This achieves the proof of the theorem. □

Example 5.8. *Let $E = \mathbb{R}$. Then $P = \{x \in \mathbb{R} : x \geq 0\}$ is a normal cone with normal constant $K = 1$. Let $X = [0, \infty)$, $T(a, b) = \min\{a, b\}$ for all $a, b \in X$. Define $M : X^2 \times int(P) \to [0, 1]$ by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$, $d(x, y) = |x - y|$ and $t \gg \theta$. Then (X, M, T) is a fuzzy cone metric space.
Let $f : X \to X$ be a mapping defined by $fx = \frac{x}{2}$ and $\phi : P \to P$ be defined by

$$\phi(t) = \begin{cases} \frac{t}{2}, & \text{if } 0 \leq t < 1; \\ \frac{4-t}{3}, & \text{if } 1 \leq t \leq 3/2; \\ t - \frac{2}{3}, & \text{if } 3/2 < t < \infty. \end{cases}$$

It is easy to verify that $\phi \in \Phi_W$ and $(t) \geq \frac{t}{2}$ for all $t \geq 0$. Now we show that f satisfies (6). We have

$$\begin{aligned} M(fx, fy, \phi(t)) &= \frac{\phi(t)}{\phi(t) + |fx - fy|} \\ &= \frac{\phi(t)}{\phi(t) + |\frac{x}{2} - \frac{y}{2}|} \\ &\geq \frac{\frac{t}{2}}{\frac{t}{2} + |\frac{x}{2} - \frac{y}{2}|} \\ &\geq \frac{t}{t + d(x, y)} \\ &= M(x, y, t). \end{aligned}$$

This shows that (6) holds and f has a fixed point in X. The fixed point is 0.

Theorem 5.9. *Let (X, M, T) be a strictly convex, complete cone metric space with convex structure $S : X^2 \times [0,1] \to X$ satisfying (3). Let $f : C \to X$ be a non-self mapping satisfying*

$$M(fx, fy, \phi(t)) \geq M(x, y, t) \tag{13}$$

for all $x, y \in C$ and every $t \in int(P)$, where $\phi \in \Phi_W$ and C is a nonempty, closed and fuzzy bounded subset of X. Further, suppose that f has the property

$$f(\partial C) \subseteq C. \tag{14}$$

Then f has a unique fixed point in C.

Proof. Let $x \in \partial C$ be an arbitrary point. We shall construct the sequence $\{x_n\}$ as follows. Set $x_0 = x$. Since $x \in \partial C$, by (14) $fx_0 \in C$. Set $x_1 = fx_0$. Define $y_2 = fx_1$. If $y_2 \in C$, set $x_2 = y_2$. If $y_2 \notin C$ let us choose $x_2 \in \partial C$ so that $x_2 = S(x_1, y_2, \mu)$, $\mu \in (0,1)$. Continuing in this manner, we obtain a sequence $\{x_n\}$ satisfying

$$x_n = \begin{cases} fx_{n-1}, & \text{if } fx_{n-1} \in C; \\ S(x_{n-1}, fx_{n-1}, \mu), & \text{if } fx_{n-1} \notin C. \end{cases} \tag{15}$$

Notice that if $x_n = S(x_{n-1}, fx_{n-1}, \mu)$,$\mu \in (0,1)$, then obviously $x_{n+1} = fx_n$ and $x_{n-1} = fx_{n-2}$ for $n = 2, 2, \cdots$.

Let us consider nested sequence of nonempty closed sets defined by

$$G_n = \{x_n, x_{n+1}, \cdots\} \quad \text{and} \quad F_n = \overline{G_n},\ n \in \mathbb{N}.$$

We shall prove the family $\{F_n\}_{n\in\mathbb{N}}$ has fuzzy diameter zero.

Firstly, let us prove that:

$$\delta_{G_n}(\phi(t)) \geq \delta_{G_{n-2}}(t) \tag{16}$$

holds for every $t \gg \theta$. Hence, we will observe the following three cases that are all of the possibilities:

Case 1: $x_{n+p} = fx_{n+p-1}$ and $x_{n+q} = fx_{n+q-1}$ for arbitrary $p, q \in \mathbb{N} \cup \{0\}$.
Case 2: $x_{n+p} = f_{n+p-1}$ and $x_{n+q} = S(x_{n+q-1}, fx_{n+q-1}, \mu)$, $\mu \in (0,1)$ for arbitrary $p, q \in \mathbb{N} \cup \{0\}$.
Case 3: $x_{n+p} = S(x_{n+p-1}, fx_{n+p-1}, \mu_1)$, $\mu_1 \in (0,1)$ and

$$x_{n+q} = S(x_{n+q-1}, fx_{n+q-1}, \mu_2),$$

$\mu_2 \in (0,1)$ for arbitrary $p, q \in \mathbb{N} \cup \{0\}$.

Case 1: If $x_{n+p} = fx_{n+p-1}$ and $x_{n+q} = fx_{n+q-1}$ for arbitrary $p, q \in \mathbb{N} \cup \{0\}$, from (13) we have

$$\begin{aligned} M(x_{n+p}, x_{n+q}, \phi(t)) &= M(fx_{n+p-1}, fx_{n+q-1}, \phi(t)) \\ &\geq M(x_{n+p}, x_{n+q}, t) \\ &\geq \delta_{G_{n-2}}(t). \end{aligned} \tag{17}$$

Case 2: If $x_{n+p} = f_{n+p-1}$ and $x_{n+q} = S(x_{n+q-1}, fx_{n+q-1}, \mu)$, $\mu \in (0,1)$ for arbitrary $p, q \in \mathbb{N} \cup \{0\}$, then from (3) and (13) we have

$$\begin{aligned}
M(x_{n+p}, x_{n+q}, \phi(t)) &= M(fx_{n+p-1}, S(x_{n+q-1}, fx_{n+q-1}, \mu), \phi(t)) \\
&> \min\{M(fx_{n+p-1}, x_{n+q-1}, \phi(t)), M(fx_{n+p-1}, fx_{n+q-1}, \phi(t))\} \\
&= \min\{M(fx_{n+p-1}, fx_{n+q-2}, \phi(t)), M(fx_{n+p-1}, fx_{n+q-1}, \phi(t))\} \qquad (18) \\
&\geq \min\{M(x_{n+p-1}, x_{n+q-2}, t), M(x_{n+p-1}, x_{n+q-1}, t)\} \\
&\geq \delta_{G_{n-2}}(t).
\end{aligned}$$

Case 3: If $x_{n+p} = S(x_{n+p-1}, fx_{n+p-1}, \mu_1)$, $\mu_1 \in (0,1)$ and

$$x_{n+q} = S(x_{n+q-1}, fx_{n+q-1}, \mu_2)$$

, $\mu_2 \in (0,1)$ for arbitrary $p, q \in \mathbb{N} \cup \{0\}$, then from (3) and (13) we have

$$\begin{aligned}
M(x_{n+p}, x_{n+q}, \phi(t)) &= M(S(x_{n+p-1}, fx_{n+p-1}, \mu_1), S(x_{n+q-1}, fx_{n+q-1}, \mu_2), \phi(t)) \\
&> \min\{M(x_{n+p-1}, x_{n+q-1}, \phi(t)), M(x_{n+p-1}, x_{n+q-1}, \phi(t)), \\
&M(fx_{n+p-1}, x_{n+q-1}, \phi(t)), M(fx_{n+p-1}, fx_{n+q-1}, \phi(t))\} \\
&= \min\{M(fx_{n+p-2}, x_{n+q-2}, \phi(t)), M(fx_{n+p-2}, x_{n+q-1}, \phi(t)), \\
&M(fx_{n+p-1}, fx_{n+q-2}, \phi(t)), M(fx_{n+p-1}, fx_{n+q-1}, \phi(t))\} \qquad (19) \\
&\geq \min\{M(x_{n+p-2}, x_{n+q-2}, t), M(x_{n+p-2}, x_{n+q-1}, t), \\
&M(x_{n+p-1}, x_{n+q-2}, t), M(x_{n+p-1}, x_{n+q-1}, t)\} \\
&\geq \delta_{G_{n-2}}(t).
\end{aligned}$$

Since the inequalities (17), (18) and (19) are of all the possibilities we have that

$$\delta_{G_n}(\phi(T)) = \sup_{\epsilon < \phi(t)} \inf_{x,y \in G_n} M(x, y, \epsilon) = \sup_{\epsilon < \phi(t)} \inf_{p,q \in \mathbb{N} \cup \{0\}} M(x_{n+p}, x_{n+q}, \epsilon) \geq \delta_{G_{n-2}}(t),$$

i.e. it follows that (16) holds for every $t \gg \theta$.

Now, we shall prove that family $\{F_n\}_{n\in\mathbb{N}}$ has fuzzy diameter zero. Let $\lambda \in (0,1)$ and $t \gg \theta$ be arbitrary. From $G_k \subseteq K$, for arbitrary $k \in \mathbb{N}$, it follows that G_k is a fuzzy bounded set. Now, from Lemma 4.6 we have that for every $\lambda \in (0,1)$ there exist $t_0 \gg \theta$ such that

$$M(x,y,t_0) > 1-\lambda \tag{20}$$

for all $x,y \in G_k$. Hence, for every $\lambda \in (0,1)$ there exist $t_0 \gg \theta$ we get that

$$\delta_{G_k}(t_0) \geq 1-\lambda.$$

From the definition of $\phi \in \Phi_W$, for such t_0, there exists $s \geq t_0$ such that

$$\lim_{n\to\infty} \phi^n(s) = 0.$$

Hence, there exists $l \in \mathbb{N}$ such that $\phi^l(s) < s$. From the previous we can conclude that there exists an even number p, $p > l$, such that $\phi^p(s) < t$, i.e. $\phi^{2m}(s) < t$ where $m = \frac{p}{2}$.

Let $n = 2m+k$ and $x,y \in G_n$ be arbitrary. Applying induction in (16) we obtain

$$\delta_{G_n}(t) \geq \delta_{G_n}(\phi^{2m}(s)) \geq \delta_{n-2m}(s).$$

From the previous inequality it follows that

$$\delta_{G_n}(t) \geq \delta_{G_n}(\phi^{2m}(s)) \geq \delta_{n-2m}(s) \geq \delta_{G_k}(t_0) \geq 1-\lambda$$

that is,

$$\delta_{G_n}(t) \geq 1-\lambda.$$

Finally, since G_n and F_n have the same fuzzy diameter, we obtain that

$$\delta_{F_n}(t) \geq 1-\lambda$$

i.e.,we get that

$$M(x,y,t) \geq 1-\lambda$$

for all $x,y \in F_n$, i.e., the family $\{F_n\}_{n\in\mathbb{N}}$ has fuzzy diameter zero.

Applying Proposition 4.16 and Lemma 4.17 we conclude that family $\{F_n\}_{n\in\mathbb{N}}$ has nonempty intersection, which consists of exactly one point z i.e. $z \in F_n$, for all $n \in \mathbb{N}$. Since the family $\{F_n\}_{n\in\mathbb{N}}$ has fuzzy diameter zero, then for each $\lambda \in (0,1)$ and each $t \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ holds

$$M(x_n,z,t) > 1-\lambda.$$

From the last inequality it follows that

$$\lim_{n\to\infty} M(x_n,z,t) > 1-\lambda$$

holds for every $\lambda \in (0,1)$. Taking $\lambda \to 0$ we get

$$\lim_{n\to\infty} F(x_n,z,t) = 1, \quad i.e., \lim_{n\to\infty} x_n = z.$$

By the construction of sequence $\{x_n\}_{n\in\mathbb{N}}$ it follows that there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $x_{n_k+1} = fx_{n_k}$. It is obvious that $\lim_{n_k\to\infty} x_{n_k+1} = z$ and $\lim_{n_k\to\infty} fx_{n_k} = z$.

From the definition of $\phi \in \Phi_W$, since for arbitrary $t \gg \theta$, there exists $r \geq t$

such that $\lim_{n\to\infty} \phi^n(r) = 0$, it follows that there exists $l \in \mathbb{N}$ such that $\phi^l(r) < t$. Now, from inequality (13) and previous we get

$$M(fx_{n_k}, fz, t) \geq M(fx_{n_k}, fz, \phi^l(r)) \geq M(x_{n_k}, z, \phi^{l-1}(r)).$$

Taking the limit in previous inequality, applying Lemma , we obtain

$$\lim_{n\to\infty} M(fx_{n_k}, fz, t) \geq 1.$$

Hence, since previous inequality holds for arbitrary $t \gg \theta$, we get that

$$M(z, fz, t) \geq 1.$$

holds for every $t \gg \theta$, i.e. we get that $fz = z$, i.e. z is the fixed point of f. Furthermore, since set C is closed set, we conclude that $z \in C$.

Let us prove that z is a unique fixed point. For this purpose let us suppose that there exists another fixed point, denoted by u. From the condition (13) follows

$$M(fz, fu, t) \geq M(z, u, t)$$

for every $t \gg \theta$. Therefore we get that

$$M(z, u, \phi(t)) \geq M(z, u, t)$$

for every $t \gg \theta$. Finally, applying Lemma 5.6 it follows that $z = u$. This achieves the proof. □

Example 5.10. *Let $X = E = \mathbb{R}$ and $P = [0, \infty)$ and let $d : X^2 \to E$ defined by $d(x, y) = |x - y|$ and let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{3}{5} - \frac{x^2}{2}$ and $C = [-1/2, 1/2]$.*

Let us show that all conditions of Theorem 5.9 are satisfied. From Example 4.3 and Example 4.14 we have that $(X, M, \min)$ is a strictly convex fuzzy cone metric space satisfies condition (3) with a convex structure $S(x, y, \mu) = \mu x + (1-\mu)y$ for all $x, y \in X$ and $\mu \in (0, 1)$. The mapping $f : [-1/2, 1/2] \to [19/40, 3/5]$ is a non-self mapping and it satisfying the condition $f(\partial C) \subseteq C$ because $f(-1/2) = f(1/2) = 19/40 \in C$. It is obvious that C is nonempty and closed set. Furthermore, C is a metrically bounded set and hence it is a fuzzy bounded set.
Let us define function $\phi : int(P) \to int(P)$ by

$$\phi(t) = \begin{cases} \frac{t}{t+1}, & \text{if } 0 < t < 1; \\ -\frac{3}{4}t + \frac{7}{4}, & \text{if } 1 \leq t \leq \frac{4}{3}; \\ t - \frac{7}{12}, & \text{if } t > \frac{4}{3}. \end{cases}$$

For a function ϕ we have that $\lim_{n\to\infty} \phi^n(t) = \lim_{n\to\infty} \dfrac{t}{1+nt} = 0$ holds for every $t \in (0, 1)$, but does not satisfy the condition $\lim_{n\to\infty} \phi^n(t) = 0$, for every $t \geq 1$, because $\phi(1) = \phi(\frac{19}{12}) = 1$ we get

$$\lim_{n\to\infty} \phi^n(1) = \lim_{n\to\infty} \phi^n(\frac{19}{12}) = 1.$$

Now we will show that function ϕ satisfying the condition in Definition 5.1 for all $t \geq 1$, that is, we show by induction that

$$\lim_{n\to\infty} \phi^n \left(\frac{7k}{12}\right) = 0 \tag{21}$$

holds for $k = 2, 3, \cdots$. It is obvious that (21) holds for $k = 2$, because $\phi(\frac{7}{6}) = \frac{7}{8} \in (0,1)$. Let us assume that (21) holds for $k = l$. Then, for $k = l + 1$, we have that $\frac{7(l+1)}{12} > \frac{4}{3}$, and it follows

$$\lim_{n\to\infty} \phi^n \left(\frac{7(l+1)}{12}\right) = \lim_{n\to\infty} \phi^{n-1}\left(\frac{7l}{12}\right) = 0$$

which shows that (21) holds for $k = l + 1$, and by induction (21) holds, for every $k = 2, 3, \cdots$. Finally, we can conclude that for every $t \geq 1$, exists $r = \frac{7k_0}{12} > t$ for sufficiently large $k_0 = 2, 3, \cdots$, such that (21) holds. Hence the function ϕ satisfies the condition in Definition 5.1.
Notice that $\phi(t) > \frac{t}{2}$ holds for $t \gg \theta$ and $|x^2 - y^2| \leq |x - y|$ holds for every $x, y \in C$, because $|x + y| \leq 1$ holds for every $x, y \in C$. Then, we get

$$\begin{aligned} M(fx, fy, \phi(t)) &= H(\phi(t) - d(fx, fy)) = H(\phi(t) - |fx - fy|) \\ &= H(\phi(t) - \frac{1}{2}|x^2 - y^2|) \geq H\left(\frac{t}{2} - \frac{1}{2}|x - y|\right) \\ &= H(t - |x - y|) = H(t - d(x, y)) = M(x, y, t), \end{aligned}$$

that is, condition (13) is satisfied for every $x, y \in C$. Since all conditions of Theorem 5.9 are satisfied and so we obtain that f has a unique fixed point $x = \frac{-5+\sqrt{55}}{5} \in C$.

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