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DIFFERENTIAL SANDWICH THEOREMS FOR SOME SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH LINEAR OPERATORS

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Abstract. Let q_1 and q_2 be univalent in $\Delta := \{z : |z| < 1\}$ with $q_1(0) = q_2(0) = 1$. We give some applications of first order differential subordination and superordination to obtain sufficient conditions for a normalized analytic functions f with $f(0) = 0$, $f'(0) = 1$ to satisfy

$$q_1(z) \prec \left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \prec q_2(z).$$

1. INTRODUCTION

Let $\mathcal{H}$ be the class of functions analytic in $\Delta := \{z : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = z + a_2z^2 + \cdots$. Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1}$$

then p is a solution of the differential superordination (1). (If f is subordinate to F, then F is called a superordinate of f.) An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (1). An univalent subordinant $\bar{q}$ that satisfies $q \prec \bar{q}$ for all subordinants q of (1) is said to be the best subordinant. Recently Miller and

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Mocanu [10] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu [10], Bulboaca [3] considered certain classes of first order differential superordinations as well as superordination-preserving operators [2]. Using the results of [3], Shanmugam et al. [12] obtained sufficient conditions for a normalized analytic function $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z),$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z)\ ,$$

respectively where q_1 and q_2 are given univalent functions in Δ.
For $\alpha_j \in \mathbb{C} \quad (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}, j = 1, 2, \ldots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}\ .$$

$$(l \leq m+1; l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathbb{N}). \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Dziok-Srivastava operator [5] (see also [13]) $H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the Hadamard product

$$\begin{aligned} H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)f(z) &:= h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{a_n z^n}{(n-1)!}. \end{aligned} \tag{2}$$

It is well known [5] that

$$\begin{aligned} &\alpha_1 H_m^l(\alpha_1 + 1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)f(z) \\ &\quad = z[H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)f(z)]' \\ &\quad\quad +(\alpha_1 - 1)H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)f(z). \end{aligned} \tag{3}$$

To make the notation simple, we write

$$H_m^l[\alpha_1]f(z) := H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)f(z).$$

Special cases of the Dziok-Srivastava linear operator includes the Hohlov linear operator [6] , the Carlson-Shaffer linear operator[4], the Ruscheweyh derivative operator [11], the generalized Bernardi-Libera-Livingston linear integral operator (*cf.* [1], [7], [8]).

2. PRELIMINARIES

In our present investigation, we shall need the following definition and results. In this paper unless otherwise mentioned α and β are complex numbers.

Definition 2.1: [10, Definition 2, p. 817] *Let Q be the set of all functions f that are analytic and injective on $\bar{\Delta} - E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial\Delta : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Theorem 2.1 : [9, Theorem 3.4h , p. 132] *Let q be univalent in the unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(\omega) \neq 0$ when $\omega \in q(\Delta)$.*

Set $\xi(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + \xi(z)$. Suppose that,

1. $\xi(z)$ is starlike univalent in Δ and
2. $\Re \frac{zh'(z)}{\xi(z)} > 0$ for $z \in \Delta$.

If p is analytic in Δ with $p(\Delta) \subseteq D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{4}$$

then $p \prec q$ and q is the best dominant.

Lemma 2.1 : [12] *Let q be univalent in Δ with $q(0) = 1$. Further assuming that*

$$\Re \left[\frac{\alpha}{\beta} + 1 + \frac{zq''(z)}{q'(z)} \right] > 0.$$

If p is analytic in Δ, with $p(\Delta) \subseteq D$ and

$$\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z),$$

then $p \prec q$ and q is the best dominant.

Theorem 2.2 : [3] *Let q be univalent in the unit disk Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*

1. $\Re\left[\frac{\vartheta'(q(z))}{\varphi(q(z))}\right] > 0$ *for $z \in \Delta$, and*
2. $\xi(z) = zq'(z)\varphi(q(z))$ *is starlike univalent function in Δ.*

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\Delta) \subset D$ and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{5}$$

then $q \prec p$ and q is the best subordinant.

Lemma 2.2 : [12] *Let q be univalent in Δ , $q(0) = 1$. Further assuming that* $\Re\left[\frac{\alpha}{\beta}q'(z)\right] > 0$.
If $p \in \mathcal{H}[q(0), 1] \cap Q$, and $\alpha p + \beta zp'$ is univalent in Δ, and

$$\alpha q(z) + \beta zq'(z) \prec \alpha p(z) + \beta zp'(z) \ ,$$

then $q \prec p$ and q is the best subordinant.

3. SUBORDINATION RESULTS FOR ANALYTIC FUNCTIONS

By making use of Lemma 2.3, we prove the following results.

Theorem 3.1 : *Let q be univalent in Δ with $q(0) = 1$ and satisfying*

$$\Re\left[\frac{\alpha}{\beta} + 1 + \frac{zq''(z)}{q'(z)}\right] > 0. \tag{6}$$

Let

$$\Psi(\alpha, \beta, \lambda; z) := \alpha\left(\frac{zf'(z)}{f(z)}\right)^{\lambda} + \beta\lambda\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\}. \tag{7}$$

If $f \in \mathcal{A}$ satisfies

$$\Psi(\alpha, \beta, \lambda; z) \prec \alpha q(z) + \beta zq'(z), \tag{8}$$

then

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \prec q(z) \ ,$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{zf'(z)}{f(z)}\right)^{\lambda}.$$

Then by means of simple computation we can show that

$$\Psi(\alpha, \beta, \lambda; z) = \alpha p(z) + \beta z p'(z).$$

Now (8) becomes

$$\alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z),$$

and Theorem 3.1 follows by an application of Lemma 2.1.

By taking $q(z) = \dfrac{1 + Az}{1 + Bz}$ $(-1 \leq B < A \leq 1)$ we have the following Example.

Example 3.1 : Let $q(z) = \dfrac{1 + Az}{1 + Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 3.1. Further assuming that (6) holds. If $f \in \mathcal{A}$, then

$$\Psi(\alpha, \beta, \lambda; z) \prec \alpha\left(\frac{1 + Az}{1 + Bz}\right) + \beta\frac{(A - B)z}{(1 + Bz)^2},$$

$$\Rightarrow \quad \left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \prec \frac{1 + Az}{1 + Bz}\ ,$$

and $\dfrac{1 + Az}{1 + Bz}$ is the best dominant.

Also if $q(z) = \dfrac{1 + z}{1 - z}$, then for $f \in \mathcal{A}$ we have

$$\Psi(\alpha, \beta, \lambda; z) \prec \alpha\left(\frac{1 + z}{1 - z}\right) + \frac{2\beta z}{(1 - z)^2},$$

$$\Rightarrow \quad \left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \prec \frac{1 + z}{1 - z}\ ,$$

and $\dfrac{1 + z}{1 - z}$ is the best dominant.

4. SUPERORDINATION RESULTS FOR ANALYTIC FUNCTIONS

Theorem 4.1 : *Let q be convex univalent in Δ with $q(0) = 1$. Let $f \in \mathcal{A}$, $\left(\dfrac{zf'(z)}{f(z)}\right)^{\lambda} \in \mathcal{H}[1, 1] \cap Q$, with*

$$\Re\left[\frac{\alpha}{\beta} q'(z)\right] > 0. \tag{9}$$

If $\Psi(\alpha,\beta,\lambda;z)$ *as defined by (7) is univalent in* Δ, *with*

$$\alpha q(z)+\beta zq'(z)\prec\Psi(\alpha,\beta,\lambda;z)\ ,$$

then

$$q(z)\prec\left(\frac{zf'(z)}{f(z)}\right)^{\lambda},$$

and q *is the best subordinant.*

Proof. Theorem 4.1 follows by an application of Lemma 2.2.

By taking $q(z)=\dfrac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 4.1, we have the following Example.

Example 4.1 : Let q be convex univalent in Δ.
Also let $f\in\mathcal{A}$, $\left(\dfrac{zf'(z)}{f(z)}\right)^{\lambda}\in\mathcal{H}[1,1]\cap Q$. Further assuming that (9) holds. If $\Psi(\alpha,\beta,\lambda;z)$ as defined by (7) is univalent in Δ, and

$$\alpha\left(\frac{1+Az}{1+Bz}\right)+\frac{\beta(A-B)z}{(1+Bz)^2}\prec\Psi(\alpha,\beta,\lambda;z),$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\ ,$$

and $\dfrac{1+Az}{1+Bz}$ is the best subordinant.

Inparticular, we have

$$\alpha\left(\frac{1+z}{1-z}\right)+\frac{2\beta z}{(1-z)^2}\prec\Psi(\alpha,\beta,\lambda;z),$$

implies

$$\frac{1+z}{1-z}\prec\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\ ,$$

and $\dfrac{1+z}{1-z}$ is the best subordinant.

5. SANDWICH THEOREMS

By combining the results of subordination and superordination, we get the following "Sandwich theorems".

Theorem 5.1 : *Let q_1 and q_2 be convex univalent in Δ and satisfying (9) and (6) respectively.*
Let $f \in \mathcal{A}$, $\left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \in \mathcal{H}[1,1] \cap Q$ and $\Psi(\alpha, \beta, \lambda; z)$ as defined by (7) is univalent in Δ. Further if

$$\alpha q_1(z) + \beta z q_1'(z) \prec \Psi(\alpha, \beta, \lambda; z) \prec \alpha q_2(z) + \beta q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \prec q_2(z),$$

and q_1 and q_2 are respectively the best subordinant and best dominant.

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}, q_2(z) = \frac{1+A_2 z}{1+B_2 z}$ $(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1)$, we have the following Example.

Example 5.1 : If $f \in \mathcal{A}$, $\left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \in \mathcal{H}[1,1] \cap Q$ and $\Psi(\alpha, \beta, \lambda; z)$ as defined by (7) is univalent in Δ , and

$$\Psi_1(A_1, B_1, \alpha, \beta, \lambda; z) \prec \Psi(\alpha, \beta, \lambda; z) \prec \Psi_2(A_2, B_2, \alpha, \beta, \lambda; z) \ ,$$

then

$$\frac{1+A_1 z}{1+B_1 z} \prec \left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \prec \frac{1+A_2 z}{1+B_2 z} \ ,$$

where

$$\Psi_1(A_1, B_1, \alpha, \beta, \lambda; z) := \alpha \left(\frac{1+A_1 z}{1+B_1 z}\right) + \frac{\beta(A_1 - B_1)z}{(1+B_1 z)^2},$$

$$\Psi_2(A_2, B_2, \alpha, \beta, \lambda; z) := \alpha \left(\frac{1+A_2 z}{1+B_2 z}\right) + \frac{\beta(A_2 - B_2)z}{(1+B_2 z)^2} \ .$$

The functions $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are respectively the best subordinant and best dominant.

6. APPLICATION TO DZIOK-SRIVASTAVA OPERATOR

Theorem 6.1 : *Let q be univalent in Δ with $q(0) = 1$. Let*
$\eta(\alpha, \beta, \lambda, l, m; z) := \left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right)^{\lambda} \times$

$$\left[(\alpha + \beta\lambda)\left\{\frac{(\alpha_1+1)\left(H_m^l[\alpha_1+2]f(z)\right)}{H_m^l[\alpha_1+1]f(z)} - \alpha_1\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right) - 1\right\}\right]. \qquad (10)$$

If $f \in \mathcal{A}$ satisfies

$$\eta(\alpha,\beta,\lambda,l,m;z) \prec \alpha p(z) + \beta z q'(z) \ ,$$

then

$$\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right)^{\lambda} \prec q(z) \ ,$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right)^{\lambda}. \tag{11}$$

By taking logarithmic derivative of (11) we get

$$\frac{zp'(z)}{p(z)} = \lambda\left[\frac{z\left(H_m^l[\alpha_1+1]f(z)\right)'}{H_m^l[\alpha_1+1]f(z)} - \frac{z\left(H_m^l[\alpha_1]f(z)\right)'}{H_m^l[\alpha_1]f(z)}\right]. \tag{12}$$

By using identity

$$z\left(H_m^l[\alpha_1]f(z)\right)' = \alpha_1 H_m^l[\alpha_1+1]f(z) - (\alpha_1-1)H_m^l[\alpha_1]f(z),$$

and (11) in (12) we get

$$\alpha p(z) + \beta z p'(z) = \left(\tfrac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right)^{\lambda} \times$$

$$\left[(\alpha+\beta\lambda)\left\{\frac{(\alpha_1+1)H_m^l[\alpha_1+2]f(z)}{H_m^l[\alpha_1+1]f(z)} - \alpha_1\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right) - 1\right\}\right].$$

Now Theorem 6.1 follows as an application of Lemma 2.1.

By taking $l = 2$, $m = 1$ and $\alpha_2 = 1$ in Theorem 6.1 we have the following corollary.

Corollary 6.1 : Let q be univalent in Δ with $q(0) = 1$. Let

$$\phi(a,c,\alpha,\beta,\lambda : z) := \left(\tfrac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda} \times$$

$$\left[\alpha + \beta\lambda\left\{\frac{(a+1)L(a+2,c)f(z)}{L(a+1,c)f(z)} - \frac{aL(a+1,c)f(z)}{L(a,c)f(z)} - 1\right\}\right].$$

If $f \in \mathcal{A}$ satisfies

$$\phi(a,c,\alpha,\beta,\lambda : z) \prec \alpha q(z) + \beta z q'(z) \ ,$$

then

$$\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda} \prec q(z) \ ,$$

and q is the best dominant.

Taking $a = 1$ and $c = 1$ in corollary 6.1 we get the following corollary.

Corollary 6.2 : Let q be univalent in Δ with $q(0) = 1$. If $f \in \mathcal{A}$ and

$$\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^{\lambda}\left[\alpha+\beta\lambda\left\{\frac{(a+1)D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{aD^{n+1}f(z)}{D^n f(z)}-1\right\}\right]\prec \alpha q(z)+\beta z q'(z),$$

then

$$\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^{\lambda}\prec q(z)\ ,$$

and q is the best dominant.

Since the superordination results are a dual of the subordination here we state only the results pertaining to the superordination.

Theorem 6.2 : *Let q be convex univalent in Δ with $q(0) = 1$. Let $f \in \mathcal{A}$, $\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right)^{\lambda} \in \mathcal{H}[1,1]\cap Q$, with $\Re\left[\frac{\alpha}{\beta}q'(z)\right] > 0$. Further if $\eta(\alpha,\beta,\lambda,l,m;z)$ as defined by (10) is univalent in Δ, with*

$$\alpha q(z)+\beta z q'(z)\prec \eta(\alpha,\beta,\lambda,l,m;z)\ ,$$

then

$$q(z)\prec\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right)^{\lambda}\ ,$$

and q is the best subordinant.

Theorem 6.3 : *Let q be convex univalent in Δ.*
Let $f \in \mathcal{A}$, $\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda} \in \mathcal{H}[1,1]\cap Q$ and $\phi(a,c,\alpha,\beta,\lambda : z)$ as defined by (13) is univalent in Δ. If

$$\alpha q(z)+\beta z q'(z)\prec \phi(a,c,\alpha,\beta,\lambda : z)\ ,$$

then

$$q(z)\prec\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\lambda}\ ,$$

and q is the best subordinant.

Taking $q(z) = \frac{1+Az}{1+Bz}\ ,\ \frac{1+z}{1-z}$ in Theorem 6.1 we can get more results and we omit the details involved.

Combining the results of subordination and superordination, we state the following Sandwich Theorems.

Theorem 6.4 : *Let q_1 and q_2 be convex univalent in Δ satisfying (9) and (6) respectively. If $f \in \mathcal{A}$, $\left(\frac{H^l_m[\alpha_1+1]f(z)}{H^l_m[\alpha_1]f(z)}\right)^\lambda \in \mathcal{H}[1,1] \cap Q$ and $\eta(\alpha,\beta,\lambda,l,m;z)$ as defined by (10) is univalent in Δ, and*

$$\alpha q_1(z) + \beta z q_1'(z) \prec \eta(\alpha,\beta,\lambda,l,m;z) \prec \alpha q_2(z) + \beta z q_2'(z) \ ,$$

then

$$q_1(z) \prec \left(\frac{H^l_m[\alpha_1+1]f(z)}{H^l_m[\alpha_1]f(z)}\right)^\lambda \prec q_2(z) \ ,$$

and $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.

For $q_1(z) = \frac{1+A_1z}{1+B_1z}$, $q_2(z) = \frac{1+A_2z}{1+B_2z}(-1 \le B_2 < B_1 < A_1 < A_2 \le 1)$, we have the following corollary.

Corollary 6.3 : If $f \in \mathcal{A}$, $\left(\frac{H^l_m[\alpha_1+1]f(z)}{H^l_m[\alpha_1]f(z)}\right)^\lambda \in \mathcal{H}[1,1] \cap Q$ and $\eta(\alpha,\beta,\lambda,l,m;z)$ as defined by (10) is univalent in Δ, and

$$\Phi_1(A_1,B_1,\alpha,\beta;z) \prec \eta(\alpha,\beta,\lambda,l,m;z) \prec \Phi_2(A_2,B_2,\alpha,\beta;z),$$

where

$$\Phi_1(A_1,B_1,\alpha,\beta;z) := \alpha\left(\frac{1+A_1z}{1+B_1z}\right) + \frac{\beta(A_1-B_1)z}{(1+B_1z)^2},$$

$$\Phi_2(A_2,B_2,\alpha,\beta;z) := \alpha\left(\frac{1+A_1z}{1+B_1z}\right) + \frac{\beta(A_2-B_2)z}{(1+B_2z)^2},$$

$$\Rightarrow \quad \frac{1+A_1z}{1+B_1z} \prec \left(\frac{H^l_m[\alpha_1+1]f(z)}{H^l_m[\alpha_1]f(z)}\right)^\lambda \prec \frac{1+A_2z}{1+B_2z} \ .$$

The functions $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are respectively the best subordinant and best dominant.

Theorem 6.5 : *Let q_1 and q_2 be convex univalent in Δ and satisfing (9) and (6) respectively. If $f \in \mathcal{A}$, $\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^\lambda \in \mathcal{H}[1,1] \cap Q$ and $\phi(a,c,\alpha,\beta,\lambda:z)$ as defined by (13) is univalent in Δ, and*

$$\alpha q_1(z) + \beta z q_1'(z) \prec \phi(a,c,\alpha,\beta,\lambda:z) \prec \alpha q_2(z) + \beta z q_2'(z) \ ,$$

then

$$q_1(z) \prec \left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^\lambda \prec q_2(z) \ ,$$

and $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.

Theorem 6.6 : *Let $q_1(z)$ and $q_2(z)$ be convex univalent in Δ and satisfing (9) and (6) respectively. Let $f \in \mathcal{A}$, $\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^{\lambda} \in \mathcal{H}[1,1] \cap Q$,*

$$\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^{\lambda}\left[\alpha+\beta\lambda\left\{\frac{(a+1)D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{aD^{n+1}f(z)}{D^n f(z)}-1\right\}\right],$$

is univalent in Δ. Further if
$\alpha q_1(z)+\beta z q_1'(z) \prec \left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^{\lambda} \times$

$$\left[\alpha+\beta\lambda\left\{\frac{(a+1)D^{n+2}f(z)}{D^{n+1}f(z)}-\frac{aD^{n+1}f(z)}{D^n f(z)}-1\right\}\right] \prec \alpha q_2(z)+\beta z q_2'(z)\ ,$$

then

$$q_1(z) \prec \left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^{\lambda} \prec q_2(z),$$

and q_1 and q_2 are respectively the best subordinant and best dominant.

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