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The Probability That an Ordered Pair of Elements is an Engel Pair

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Abstract. Let G be a finite group. We denote by $ep(G)$ the probability that $[x,_n y] = 1$ for two randomly chosen elements x and y of G and some positive integer n. For $x \in G$ we denote by $E_G(x)$ the subset $\{y \in G : [y,_n x] = 1 \ for\ some\ integer\ n\}$. G is called an E-group if $E_G(x)$ is a subgroup of G for all $x \in G$. Among other results, we prove that if G is an non-abelian E-group with $ep(G) > \frac{1}{6}$, then G is not simple and minimal non-solvable.

Key words : finite group, E-group, Engel element.

Abstrak. Misalkan G meruapakan suatu grup hingga. Misalkan juga $ep(G)$ merupakan peluang dari $[x,_n y] = 1$ untuk dua unsur yang dipilih secara random x dan y di G dan suatu bilangan bulat positif n. Untuk $x \in G$, misalkan $E_G(x)$ merupakan suatu subset $\{y \in G : [y,_n x] = 1 \ for\ some\ integer\ n\}$. Grup G disebut suatu E-group jika $E_G(x)$ merupakan suatu subgrup dari G untuk semua $x \in G$. Salah satu hasil dalam artikel ini adalah, dibuktikan bahwa jika G merupakan suatu non-abelian E-group dengan $ep(G) > \frac{1}{6}$, maka G bukan merupakan suatu grup simpel dan minimal non-solvable.

Kata kunci : grup hingga, E-group, unsur Engel.

1. INTRODUCTION

Let G be any group and $x_1, \cdots, x_n \in G$. We define $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ and for $n \geq 2$ inductively $[x_1, \cdots, x_n]$ as follows:

$$[x_1, \cdots, x_n] = [x_1, \cdots, x_{n-1}]^{-1}x_n^{-1}[x_1, \cdots, x_{n-1}]x_n.$$

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If $x_2 = \cdots = x_n$, then we denote $[x_1, \cdots, x_n]$ by $[x_{1,n-1} x_2]$. If there are two elements x and y in G such that $[y,_n x] = 1$ for some integer $n > 0$, then x is called a left Engel element with respect to y. An element x of G is called a left Engel element if for all $y \in G$ there exists an integer $n = n(x, y) > 0$ such that $[y,_n x] = 1$. The engelizer of x in G is defined to be the subset $E_G(x) = \{y \in G : [y,_n x] = 1 \; for\ some\ n\}$. We notice that $E_G(x)$ is not necessary a subgroup of G. A group G is called an *E-group* if $E_G(x)$ is a subgroup of G for all $x \in G$. In particular, G is *Engel* if $E_G(x) = G$ for all $x \in G$. In [15] Peng introduced and investigated the structure of finite solvable E-groups and also he generalized the previous results in [14]. After him other group theorists such as Heineken and Casolo studied E-groups (see [5], [6] and [12]). They determined simple E-groups. In this paper we focus on E-groups with a different viewpoint. For this reason, we conider some generalization of the commutativity degree of a finite group G, which means the probability that two randomly chosen elements of G commute. In other words

$$cp(G) = \frac{|\{(x, y) \in G \times G : [x, y] = 1\}|}{|G|^2}.$$

In the literature, the commutativity degree of a group G denoted variously by $d(G)$, $pr(G)$ or $cp(G)$ and are studied and generalized by some authors (see [11], [8], [10]). Note that $cp(G) = \frac{\sum_{x \in G} |C_G(x)|}{|G|^2}$ where $C_G(x) = \{y \in G : [y, x] = 1\}$ and we know that $C_G(x)$ is a subgroup of G for every $x \in G$.

Since $cp(A_5) = \frac{1}{12}$, J. Dixon observed that $cp(G) \leq \frac{1}{12}$ for any finite non-abelian simple group G (see [9]) and Guralnick and Robinson [9] extended and proved that every group G with $cp(G) > \frac{1}{12}$ is solvable. Now by considering $cp(G)$ and what are said in above, for all finite groups G, we define $ep(G)$ to be the probability of being left Engel element in a group G,i.e.,

$$ep(G) = \frac{|\{(x, y) \in G \times G; [x,_n y] = 1 \; for\ some\ \ natural\ number\ n\}|}{|G|^2}.$$

Also it is evident that $ep(G) = \frac{\sum_{x \in G} |E_G(x)|}{|G|^2}$. But $E_G(x)$ is not necessary a subgroup of G and so it is difficult to obtain information about a group G from $ep(G)$. Next since $ep(A_5) = \frac{1}{6}$ (see Proposition 2.6), it is natural that we propose the following question:

Question 1.1. *If G is a finite group with $ep(G) > \frac{1}{6}$, then is G solvable?*

In the present paper, we will show that if G is an E-group, then it is not simple and minimal non-solvable.

Theorem 1.2. *Let G be a non-abelian E-group with $ep(G) > \frac{1}{6}$. Then G is not simple and minimal non-solvable.*

In this article all groups are finite and an engelizer in a group means the engelizer of some element of the group. We denote by $PSL(2, q)$ and $Sz(q)$ the projective special linear group of degree two over the finite field of size q and the Suzuki group over the finite field of size $q = 2^{2m+1}$ with $m > 0$, respectively. Also

C_n, C_2^n and D_{2n} denote the cyclic group of order n, the elementary abelian 2-group of rank n and the dihedral group of order $2n$, respectively. Other notation is standard and can be found in [16].

2. Main Results

The following result has been proved by Heineken and Casolo in [12] and [5]. This will be used in proof of Theorem 1.2.

Theorem 2.1. *A non-abelian simple group is an E-group if and only if it is one of the following groups:*

$PSL(2, 2^n)$, $n \geq 2$; $Sz(2^{2m+1})$ *with* $m > 0$.

Recall that a collection S of proper subgroups of a group G is called a partition if every nonidentity element of G belongs to a unique subgroup in S. In what follows we determine the structure of all engelizers in a Suzuki group.

Lemma 2.2. *Let $G = Sz(q)$ and $r = \sqrt{\frac{q}{2}}$. If $1 \neq x \in G$, then $E_G(x)$ has one of the following structures.*

1. *The Frobenius group $P \rtimes C_{q-1}$ where P is Sylow 2-subgroup of G;*
2. *The dihedral group $D_{2(q-1)}$;*
3. *The Frobenius group $C_{q-2r+1} \rtimes C_4$;*
4. *The Frobenius group $C_{q+2r+1} \rtimes C_4$.*

Proof. By Theorem 2.1, G is an E-group and so $E_G(x)$ is a subgroup of G. If $E_G(x) = G$, then x is a left Engel element of G and therefore since G is finite, by Corollary 3.17 of [3], x belongs to the Hirsch-Plotkin radical of G, a contradiction. Therefore $E_G(x)$ is a proper subgroup of G. By Theorems 3.10 and 3.11 of [13], $\Gamma = \{P^g, A^g, B^g, C^g : g \in G\}$ is a partition of G where P is a Sylow 2-subgroup G, A is cyclic of order $q-1$, B is cyclic of order $q-2r+1$ and C is cyclic of order $q+2r+1$. It follows that $x \in M$ for some $M \in \Gamma$. Since A, B and C are cyclic and P is nilpotent, then for all $y \in M$ we have $M = E_M(y)$, hence $M \leq E_G(x)$. Let K be a maximal subgroup of G containing $E_G(x)$. Then K is isomorphic to $P \rtimes C_{q-1}, C_{q-2r+1} \rtimes C_4, C_{q+2r+1} \rtimes C_4, D_{2(q-1)}$ or $Sz(q_0)$ where $q_0^l = q$, l is prime and $q_0 > 2$ (see for example page 343 of [4]). Now we consider the following three cases.

Case 1. If M is a Sylow 2-subgroup of G, then $K = M \rtimes A^g$ for some $g \in G$ and also $M' = Z(M)$ by Claim 3.2 of [4]. Since $\frac{K}{M}$ is a cyclic group, $K' \leq M$. Now if $y \in K$, then $[y,_2 x] \in M'$ which implies $[y,_3 x] = 1$. Consequently $E_G(x) = K$, as a desired.

Case 2. Suppose that M is cyclic of order $q-1$. By page 137 of [18], we see that $K = M \rtimes \langle a \rangle \cong D_{2(q-1)}$ for some involution $a \in G$. It follows that $[a, x] \in M$ and since M is cyclic, we have $[a,_2 x] = 1$ which yields that $a \in E_G(x)$. Consequently $E_G(x) = K$, as wanted.

Case 3. Suppose that M is a cyclic group of order either $q-2r+1$ or $q+2r+1$. Then $K = M \rtimes H$ where H is a cyclic group of order 4. Similarly if $a \in H$, then $[a,_2 x] = 1$ and so $E_G(x) = K$. This completes the proof.

□

In the following lemma, we compute $ep(Sz(q))$.

Proposition 2.3. $ep(Sz(q)) = \frac{q^2+3q-2}{q^2(q^2+1)(q-1)}$ *where* $q = 2^{2n+1}, n > 0$. *In particular* $ep(Sz(q)) \leq \frac{1}{6}$.

Proof. Let $G = Sz(q)$. For a given subgroup H of G, we denote the set of all conjugates of H in G by $Cl(H)$. Let Γ be the partition of G described in the proof of Lemma 2.2. If $1 \neq x \in G$, then $x \in M$ for some $M \in \Gamma$. If $M = P^g$ for some $g \in G$, then by using Lemma 2.2 and simple calculation one can see that $|E_G(x)| = q^2(q-1)$. But the number of Sylow 2-subgroups of G is q^2+1(see Theorems 3.10 and 3.11 of chapter XI in [13] or proof of Theorem 1.2 of [2]). It follows that

$$\sum_{X\in Cl(P)} \sum_{1\neq x\in X} |E_G(x)| = q^2(q-1)(q^2+1)(q^2-1).$$

If $M = A^g$ for some $g \in G$, then $|E_G(x)| = 2(q-1)$. Since the number of conjugates of A in G is $\frac{q^2(q^2+1)}{2}$(see Theorems 3.10 and 3.11 of chapter XI in [13] or proof of Theorem 1.2 of [2]), we have

$$\sum_{X\in Cl(A)} \sum_{1\neq x\in X} |E_G(x)| = \frac{q^2(q^2+1)}{2}(q-2)2(q-1).$$

If $M = B^g$ for some $g \in G$, then $|E_G(x)| = 4(q-2r+1)$. But the number of conjugates of B in G is $\frac{q^2(q-1)(q^2+1)}{4(q-2r+1)}$(see Theorems 3.10 and 3.11 of chapter XI in [13] or proof of Theorem 1.2 of [2]). It follows that

$$\sum_{X\in Cl(B)} \sum_{1\neq x\in X} |E_G(x)| = q^2(q-1)(q^2+1)(q-2r).$$

If $M = C^g$ for some $g \in G$, then $|E_G(x)| = 4(q+2r+1)$. But the number of conjugates of C in G is $\frac{q^2(q-1)(q^2+1)}{4(q+2r+1)}$(see Theorems 3.10 and 3.11 of chapter XI in [13] or proof of Theorem 1.2 of [2]). So

$$\sum_{X\in Cl(C)} \sum_{1\neq x\in X} |E_G(x)| = q^2(q-1)(q^2+1)(q+2r).$$

Consequently

$$ep(G) = \frac{\sum_{x\in G} |E_G(x)|}{|G|^2} = \frac{|G| + \sum_{1\neq x\in G} |E_G(x)|}{(q^2(q^2+1)(q-1))^2} = \frac{q^2+3q-2}{q^2(q^2+1)(q-1)}.$$

□

According to a well known theorem of [7](see chapter 12 pages 260-287), the maximal subgroups of $PSL(2, 2^n)$, $(n > 1)$ fall into four families as follows.

Theorem 2.4. *Every maximal subgroup of* $G = PSL(2, q)$ *with* $q = 2^n > 3$ *is isomorphic to one of the following.*

1. $C_2^n \rtimes C_{q-1}$, *that is, the stabilizer of a point of the projective line;*

2. *Dihedral group* $D_{2(q-1)}$*;*
3. *Dihedral group* $D_{2(q+1)}$*;*
4. *Projective general linear group* $PGL(2, q_0)$ *for* $q = q_0^r$ *with* r *a prime and* $q_0 \neq 2$.

Lemma 2.5. *Let* $G = PSL(2, 2^n)$ *such that* $n > 1$ *and* $1 \neq x \in G$*. Then* $E_G(x)$ *is isomorphic to* $D_{2(q-1)}, D_{2(q+1)}$ *or* $C_2^n \rtimes C_{q-1}$.

Proof. It follows from Theorem 2.1 that $E_G(x)$ is a subgroup of G. By Proposition 3.21 of [1], G has a partition $\Gamma = \{P^g, A^g, B^g | g \in G\}$ such that P is an elementary abelian 2- group of order q, A is cyclic of order $q-1$ and B is cyclic of order $q+1$. Therefore $x \in M$ where $M \in \Gamma$ and since P, A and B are abelian groups, we have $M \leq E_G(x)$(Note that since G is finite, $E_G(x) \neq G$ by exercise 12.3.2 of [16](see also Theorem 3.14 and Corollary 3.17 of [3]).

Case 1. Let $M = A^g$ for some $g \in G$. By Proposition 3.21 of [1], we have normalizer of $\langle x \rangle$ in G; $N_G(x)$ is a dihedral group of order $2(q-1)$ and so by Theorem 2.4, it is a maximal subgroup of G. Therefore $N_G(x) = M \rtimes \langle a \rangle$ for some involution $a \in G$. It follows that $a \in E_G(x)$ and hence $E_G(x) = N_G(x) \cong D_{2(q-1)}$, as wanted.

Case 2. Let $M = B^g$ for some $g \in G$. Similarly to case 1, we get $E_G(x) \cong D_{2(q+1)}$.

Case 3. Let $M = P^g$ for some $g \in G$. Then M is contained in a maximal subgroup T such that $T = M \rtimes L$ and L is cyclic of order $q-1$ by Theorem 2.4. Since $x \in M$, we have $E_G(x) = T$. This completes the proof. □

In the following we compute $ep(PSL(2, 2^n))$ for $n > 1$.

Proposition 2.6. *If* $q > 3$ *is even, then* $ep(PSL(2, q)) = \frac{3q-2}{q(q-1)(q+1)}$*. In particular* $ep(PSL(2, 2^n)) \leq ep(A_5) = \frac{1}{6}$.

Proof. Let $G = PSL(2, q)$. Then G has a partition

$$\Gamma = \{P^g, A^g, B^g | g \in G\}$$

which is the same as in the proof of Lemma 2.5. Also by Lemma 2.5, $E_G(x)$ is a maximal subgroup of G for any $1 \neq x \in G$. But by Proposition 2.4 of [2], the numbers of conjugates of A, B and P in G are $\frac{q(q+1)}{2}$, $\frac{q(q-1)}{2}$ and $q+1$, respectively. Therefore we have

$$\sum_{X \in Cl(A)} \sum_{1 \neq x \in X} |E_G(x)| = 2(q-1)(q-2)\frac{q(q+1)}{2} = (q-1)q(q+1)(q-2),$$

$$\sum_{X \in Cl(B)} \sum_{1 \neq x \in X} |E_G(x)| = 2(q+1)q\frac{q(q-1)}{2} = (q-1)q^2(q+1)$$

and also

$$\sum_{X \in Cl(P)} \sum_{1 \neq x \in X} |E_G(x)| = q(q-1)(q-1)(q+1) = (q-1)^2 q(q+1).$$

It follows that $ep(G) = \frac{\sum_{x\in G}|E_G(x)|}{|G|^2} = \frac{|G|+\sum_{1\neq x\in G}|E_G(x)|}{((q-1)q(q+1))^2} = \frac{3q-2}{q(q-1)(q+1)}$, as desired. □

The following lemma due to Heineken, will be used in the proof of the main result.

Lemma 2.7. *Let G be an E-group and $N \trianglelefteq G$. Then both of N and G/N are E-groups.*

Proof. Theorem 1 of [12] and its proof give the result. □

Lemma 2.8. *Let G be a group and $N \trianglelefteq G$. Then $ep(G) \leq ep(G/N)$.*

Proof. Let $S = \{(a,b) \in G \times G; (a,b)\, is\, an\, Engel\, pair\}$ and $\overline{S} = \{(A,B) \in G/N \times G/N; (A,B)\, is\, an\, Engel\, pair\}$. Now the subsets $A \times B$ for $(A,B) \in \overline{S}$ are pairwise disjoint subsets of $G \times G$ where each has size $|N|.|N|$. Clearly $S \subseteq \bigcup_{(A,B)\in\overline{S}} A \times B$ and thus $|S| \leq |\overline{S}|.|N|^2$ from which it follows that

$$ep(G) = \frac{|S|}{|G|^2} \leq \frac{|\overline{S}|.|N|^2}{|G|^2} = ep(G/N).$$

□

Proof of Theorem 1.2.

Let G be a non-abelian simple E-group. Then by Theorem 2.1, $G \cong PSL(2, 2^n)$, $n \geq 2$ or $G \cong Sz(2^{2m+1})$ with $m > 0$. Now assuming $ep(G) > \frac{1}{6}$ is on the contrary to Lemmas 2.3 and 2.6.

Now let G be a minimal non-solvable E- group with $ep(G) > \frac{1}{6}$. By Corollary 1 of [17] we know that if G is a minimal non-solvable group, then for some normal subgroup N of G, G/N is isomorphic to one of the following groups:

1. $PSL(2, 2^p), p$ a prime.
2. $PSL2(3^p)$, p an odd prime.
3. $PSL(2, p)$, $p > 3$ a prime congruent to 2 or $3\, mod\, 5$.
4. $Sz(2^p)$, p an odd prime.
5. $PSL(3, 3)$.

Also Lemma 2.7 shows that if G is an E-group G/N is an E-group too where N is a normal subgroup of G. Next since by Theorem 2.1, only simple groups which are E- groups are $PSL(2, 2^n)$, $n \geq 2$; $Sz(2^{2m+1})$ with $m > 0$, G/N can be only isomorphic to one of the cases 1 or 4 in above. But for these cases, from Lemmas 2.3 and 2.6 we have $ep(G/N) \leq \frac{1}{6}$. Next, since by Lemma 2.8, $ep(G) \leq ep(G/N)$, we have also that $ep(G) \leq \frac{1}{6}$. Therefore we get a contradiction by our assumption and proof is complete.

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