The Probability That an Ordered Pair of Elements is an Engel Pair

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Abstract. Let $G$ be a finite group. We denote by $ep(G)$ the probability that $[x, y] = 1$ for two randomly chosen elements $x$ and $y$ of $G$ and some positive integer $n$. For $x \in G$ we denote by $E_G(x)$ the subset $\{y \in G : [y, n] x = 1 \text{ for some integer } n\}$. $G$ is called an $E$-group if $E_G(x)$ is a subgroup of $G$ for all $x \in G$. Among other results, we prove that if $G$ is a non-abelian $E$-group with $ep(G) > \frac{1}{6}$, then $G$ is not simple and minimal non-solvable.

Key words: finite group, $E$-group, Engel element.

1. INTRODUCTION

Let $G$ be any group and $x_1, \cdots, x_n \in G$. We define $[x_1, x_2] = x_2^{-1}x_1^{-1}x_1x_2$ and for $n \geq 2$ inductively $[x_1, \cdots, x_n]$ as follows:

$[x_1, \cdots, x_n] = [x_1, \cdots, x_{n-1}]^{-1}x_n^{-1}[x_1, \cdots, x_{n-1}]x_n$. 

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If \( x_2 = \cdots = x_n \), then we denote \([x_1, \cdots, x_n] \) by \([x_1, n-1, x_2] \). If there are two elements \( x \) and \( y \) in \( G \) such that \([y, n, x] = 1 \) for some integer \( n > 0 \), then \( x \) is called a left Engel element with respect to \( y \). An element \( x \) of \( G \) is called a left Engel element if for all \( y \in G \) there exists an integer \( n = n(x, y) > 0 \) such that \([y, n, x] = 1 \). The engelizer of \( x \) in \( G \) is defined to be the subset \( E_G(x) = \{ y \in G : [y, n, x] = 1 \text{ for some } n \} \). We notice that \( E_G(x) \) is not necessary a subgroup of \( G \). A group \( G \) is called an E-group if \( E_G(x) \) is a subgroup of \( G \) for all \( x \in G \).

In particular, \( G \) is Engel if \( E_G(x) = G \) for all \( x \in G \). In [15] Peng introduced and investigated the structure of finite solvable E-groups and also he generalized the previous results in [14]. After him other group theorists such as Heineken and Casolo studied E-groups (see [5], [6] and [12]). They determined simple E-groups.

In this paper we focus on E-groups with a different viewpoint. For this reason, we consider some generalization of the commutativity degree of a finite group \( G \), which means the probability that two randomly chosen elements of \( G \) commute. In other words

\[
ep(G) = \frac{|\{(x, y) \in G \times G : [x, y] = 1\}|}{|G|^2}.
\]

In the literature, the commutativity degree of a group \( G \) denoted variously by \( d(G) \), \( pr(G) \) or \( ep(G) \) and are studied and generalized by some authors (see [11], [8], [10]). Note that \( ep(G) = \frac{\sum_{x \in G} |E_G(x)|}{|G|^2} \) where \( C_G(x) = \{ y \in G : [y, x] = 1 \} \) and we know that \( C_G(x) \) is a subgroup of \( G \) for every \( x \in G \).

Since \( ep(A_3) = \frac{1}{12} \), J. Dixon observed that \( ep(G) \leq \frac{1}{12} \) for any finite non-abelian simple group \( G \) (see [9]) and Guralnick and Robinson [9] extended and proved that every group \( G \) with \( ep(G) > \frac{1}{12} \) is solvable. Now by considering \( ep(G) \) and what are said in above, for all finite groups \( G \), we define \( ep(G) \) to be the probability of being left Engel element in a group \( G \), i.e.,

\[
ep(G) = \frac{|\{(x, y) \in G \times G : [x, n, y] = 1 \text{ for some natural number } n\}|}{|G|^2}.
\]

Also it is evident that \( ep(G) = \frac{\sum_{x \in G} |E_G(x)|}{|G|^2} \). But \( E_G(x) \) is not necessary a subgroup of \( G \) and so it is difficult to obtain information about a group \( G \) from \( ep(G) \). Next since \( ep(A_3) = \frac{1}{6} \) (see Proposition 2.6), it is natural that we propose the following question:

**Question 1.1.** If \( G \) is a finite group with \( ep(G) \geq \frac{1}{6} \), then is \( G \) solvable?

In the present paper, we will show that if \( G \) is an E-group, then it is not simple and minimal non-solvable.

**Theorem 1.2.** Let \( G \) be a non-abelian E-group with \( ep(G) \geq \frac{1}{6} \). Then \( G \) is not simple and minimal non-solvable.

In this article all groups are finite and an engelizer in a group means the engelizer of some element of the group. We denote by \( PSL(2, q) \) and \( Sz(q) \) the projective special linear group of degree two over the finite field of size \( q \) and the Suzuki group over the finite field of size \( q = 2^{2m+1} \) with \( m > 0 \), respectively. Also
$C_n$, $C_2^n$ and $D_{2n}$ denote the cyclic group of order $n$, the elementary abelian 2-group of rank $n$ and the dihedral group of order $2n$, respectively. Other notation is standard and can be found in [16].

2. Main Results

The following result has been proved by Heincken and Casolo in [12] and [5]. This will be used in proof of Theorem 1.2.

Theorem 2.1. A non-abelian simple group is an E-group if and only if it is one of the following groups:

- $PSL(2, 2^n)$, $n \geq 2$;
- $Sz(2^{2m+1})$ with $m > 0$.

Recall that a collection $S$ of proper subgroups of a group $G$ is called a partition if every nonidentity element of $G$ belongs to a unique subgroup in $S$. In what follows we determine the structure of all engelizers in a Suzuki group.

Lemma 2.2. Let $G = Sz(q)$ and $r = \sqrt{2}$. If $1 \neq x \in G$, then $E_G(x)$ has one of the following structures.

1. The Frobenius group $P \rtimes C_{q-1}$ where $P$ is Sylow 2-subgroup of $G$;
2. The dihedral group $D_{2(q-1)}$;
3. The Frobenius group $C_{q-2r+1} \rtimes C_4$;
4. The Frobenius group $C_{q+2r+1} \rtimes C_4$.

Proof. By Theorem 2.1, $G$ is an $E$-group and so $E_G(x)$ is a subgroup of $G$. If $E_G(x) = G$, then $x$ is a left Engel element of $G$ and therefore since $G$ is finite, by Corollary 3.17 of [3], $x$ belongs to the Hirsch-Plotkin radical of $G$, a contradiction. Therefore $E_G(x)$ is a proper subgroup of $G$. By Theorems 3.10 and 3.11 of [13], $\Gamma = \{P^g, A^g, B^g, C^g : g \in G\}$ is a partition of $G$ where $P$ is a Sylow 2-subgroup of $G$, $A$ is cyclic of order $q-1$, $B$ is cyclic of order $q - 2r + 1$ and $C$ is cyclic of order $q + 2r + 1$. It follows that $x \in M$ for some $M \in \Gamma$. Since $A, B$ and $C$ are cyclic and $P$ is nilpotent, then for all $y \in M$ we have $M = E_M(y)$, hence $M \leq E_G(x)$.

Now we consider the following three cases.

Case 1. If $M$ is a Sylow 2-subgroup of $G$, then $K = M \rtimes A^g$ for some $g \in G$ and also $M' = Z(M)$ by Claim 3.2 of [4]. Since $\frac{M'}{M}$ is a cyclic group, $K' \leq M$. Now if $y \in K$, then $[y, x] \in M'$ which implies $[y, x] = 1$. Consequently $E_G(x) = K$, as desired.

Case 2. Suppose that $M$ is cyclic of order $q - 1$. By page 137 of [18], we see that $K = M \rtimes \langle a \rangle \cong D_{2(q-1)}$ for some involution $a \in G$. It follows that $[a, x] \in M$ and since $M$ is cyclic, we have $[a, x] = 1$ which yields that $a \in E_G(x)$. Consequently $E_G(x) = K$, as wanted.

Case 3. Suppose that $M$ is a cyclic group of order either $q - 2r + 1$ or $q + 2r + 1$. Then $K = M \rtimes H$ where $H$ is a cyclic group of order 4. Similarly if $a \in H$, then $[a, x] = 1$ and so $E_G(x) = K$. This completes the proof.
In the following lemma, we compute $ep(Sz(q))$.

**Proposition 2.3.** $ep(Sz(q)) = \frac{q^2 + 3q - 2}{q^2(q+1)(q-1)}$. Where $q = 2^{2n+1}$, $n > 0$. In particular $ep(Sz(q)) \leq \frac{1}{6}$.

**Proof.** Let $G = Sz(q)$. For a given subgroup $H$ of $G$, we denote the set of all conjugates of $H$ in $G$ by $Cl(H)$. Let $\Gamma$ be the partition of $G$ described in the proof of Lemma 2.2. If $1 \neq x \in G$, then $x \in M$ for some $M \in \Gamma$. If $M = P^g$ for some $g \in G$, then by using Lemma 2.2 and simple calculation one can see that $|EG(x)| = q^2(q-1)$. But the number of Sylow $2$-subgroups of $G$ is $q^2 + 1$ (see Theorems 3.10 and 3.11 of chapter XI in [13] or proof of Theorem 1.2 of [2]). It follows that

$$\sum_{X \in Cl(P)} \sum_{1 \neq x \in X} |EG(x)| = q^2(q-1)(q^2 + 1)(q^2 - 1).$$

If $M = A^g$ for some $g \in G$, then $|EG(x)| = 2(q-1)$. The number of conjugates of $A$ in $G$ is $\frac{q^2(q^2 + 1)}{2}$ (see Theorems 3.10 and 3.11 of chapter XI in [13] or proof of Theorem 1.2 of [2]), we have

$$\sum_{X \in Cl(A)} \sum_{1 \neq x \in X} |EG(x)| = q^2(q^2 + 1)(q-2)(q-1).$$

If $M = B^g$ for some $g \in G$, then $|EG(x)| = 4(q-2r + 1)$. The number of conjugates of $B$ in $G$ is $\frac{2(q-1)(q^2 + 1)}{4(q-2r + 1)}$ (see Theorems 3.10 and 3.11 of chapter XI in [13] or proof of Theorem 1.2 of [2]). It follows that

$$\sum_{X \in Cl(B)} \sum_{1 \neq x \in X} |EG(x)| = q^2(q-1)(q^2 + 1)(q-2r).$$

If $M = C^g$ for some $g \in G$, then $|EG(x)| = 4(q + 2r + 1)$. The number of conjugates of $C$ in $G$ is $\frac{4(q-1)(q^2 + 1)}{4(q+2r + 1)}$ (see Theorems 3.10 and 3.11 of chapter XI in [13] or proof of Theorem 1.2 of [2]). So

$$\sum_{X \in Cl(C)} \sum_{1 \neq x \in X} |EG(x)| = q^2(q-1)(q^2 + 1)(q+2r).$$

Consequently

$$ep(G) = \frac{\sum_{x \in G} |EG(x)|}{|G|^2} = \frac{|G| + \sum_{1 \neq x \in G} |EG(x)|}{(q^2(q^2 + 1)(q-1))^2} = \frac{q^2 + 3q - 2}{q^2(q^2 + 1)(q-1)}. \quad \square$$

According to a well known theorem of [7] (see chapter 12 pages 260-287), the maximal subgroups of $PSL(2, 2^n)$, $(n > 1)$ fall into four families as follows.

**Theorem 2.4.** Every maximal subgroup of $G = PSL(2, q)$ with $q = 2^n > 3$ is isomorphic to one of the following.

1. $C_2 \times C_{q-1}$, that is, the stabilizer of a point of the projective line;
2. Dihedral group $D_{2(q-1)}$;
3. Dihedral group $D_{2(q+1)}$;
4. Projective general linear group $PGL(2, q_0)$ for $q = q_0$ with $r$ a prime and $q_0 \neq 2$.

Lemma 2.5. Let $G = PSL(2, 2^n)$ such that $n > 1$ and $1 \neq x \in G$. Then $E_G(x)$ is isomorphic to $D_{2(q-1)}, D_{2(q+1)}$ or $C_2^n \rtimes C_{q-1}$.

Proof. It follows from Theorem 2.1 that $E_G(x)$ is a subgroup of $G$. By Proposition 3.21 of [1], $G$ has a partition $\Gamma = \{ P^g, A^g, B^g | g \in G \}$ such that $P$ is an elementary abelian 2-group of order $q$, $A$ is cyclic of order $q - 1$ and $B$ is cyclic of order $q + 1$. Therefore $x \in M$ where $M \in \Gamma$ and since $P, A$ and $B$ are abelian groups, we have $M \leq E_G(x)$ (Note that since $G$ is finite, $E_G(x) \neq G$ by exercise 12.3.2 of [16] (see also Theorem 3.14 and Corollary 3.17 of [3]).

Case 1. Let $M = A^g$ for some $g \in G$. By Proposition 3.21 of [1], we have normalizer of $\langle x \rangle$ in $G$; $N_G(x)$ is a dihedral group of order $2(q - 1)$ and so by Theorem 2.4, it is a maximal subgroup of $G$. Therefore $N_G(x) = M \rtimes \langle a \rangle$ for some involution $a \in G$. It follows that $a \in E_G(x)$ and hence $E_G(x) = N_G(x) \cong D_{2(q-1)}$, as wanted.

Case 2. Let $M = B^g$ for some $g \in G$. Similarly to case 1, we get $E_G(x) \cong D_{2(q+1)}$.

Case 3. Let $M = P^g$ for some $g \in G$. Then $M$ is contained in a maximal subgroup $T$ such that $T = M \rtimes L$ and $L$ is cyclic of order $q - 1$ by Theorem 2.4. Since $x \in M$, we have $E_G(x) = T$. This completes the proof.

In the following we compute $ep(PSL(2, 2^n))$ for $n > 1$.

Proposition 2.6. If $q > 3$ is even, then $ep(PSL(2, q)) = \frac{3q^2 - 2}{q(q-1)(q+1)}$. In particular $ep(PSL(2, 2^n)) \leq ep(A_5) = \frac{1}{2}$.

Proof. Let $G = PSL(2, q)$. Then $G$ has a partition
$$\Gamma = \{ P^g, A^g, B^g | g \in G \}$$
which is the same as in the proof of Lemma 2.5. Also by Lemma 2.5, $E_G(x)$ is a maximal subgroup of $G$ for any $1 \neq x \in G$. But by Proposition 2.4 of [2], the numbers of conjugates of $A, B$ and $P$ in $G$ are $\frac{q(q+1)}{2}, \frac{q(q-1)}{2}$ and $q+1$, respectively. Therefore we have
$$\sum_{X \in Cl(A)} \sum_{1 \neq x \in X} \left| E_G(x) \right| = 2(q-1)(q-2)\frac{q(q+1)}{2} = (q-1)q(q+1)(q-2),$$
$$\sum_{X \in Cl(B)} \sum_{1 \neq x \in X} \left| E_G(x) \right| = 2(q+1)q\frac{q(q-1)}{2} = (q-1)q^2(q+1)$$
and also
$$\sum_{X \in Cl(P)} \sum_{1 \neq x \in X} \left| E_G(x) \right| = q(q-1)(q-1)(q+1) = (q-1)^2q(q+1).$$
It follows that $ep(G) = \frac{\sum_{x \in G} |E_G(x)|}{|G|^2} = \frac{|G| + \sum_{x \in G} |E_G(x)|}{(q-1)q(q+1)^2} = \frac{3q-2}{q^2(q+1)},$ as desired. \hfill \square

The following lemma due to Heineken, will be used in the proof of the main result.

**Lemma 2.7.** Let $G$ be an $E$-group and $N \leq G$. Then both of $N$ and $G/N$ are $E$-groups.

**Proof.** Theorem 1 of [12] and its proof give the result. \hfill \square

**Lemma 2.8.** Let $G$ be a group and $N \triangleleft G$. Then $ep(G) \leq ep(G/N)$.

**Proof.** Let $S = \{(a, b) \in G \times G; (a, b) is an Engel pair\}$ and $\overline{S} = \{(A, B) \in G/N \times G/N; (A, B) is an Engel pair\}$. Now the subsets $A \times B$ for $(A, B) \in \overline{S}$ are pairwise disjoint subsets of $G \times G$ where each has size $|N||N|$. Clearly $S \subseteq \bigcup_{(A, B) \in \overline{S}} A \times B$ and thus $|S| \leq |\overline{S}||N|^2$ from which it follows that

$$ep(G) = \frac{|S|}{|G|^2} \leq \frac{|\overline{S}||N|^2}{|G|^2} = ep(G/N).$$

\hfill \square

**Proof of Theorem 1.2.**

Let $G$ be a non-abelian simple $E$-group. Then by Theorem 2.1, $G \cong PSL(2, 2^n), \ n \geq 2$ or $G \cong Sz(2^{2m+1})$ with $m > 0$. Now assuming $ep(G) > \frac{1}{6}$ is on the contrary to Lemmas 2.3 and 2.6.

Now let $G$ be a minimal non-solvable $E$- group with $ep(G) > \frac{1}{6}$. By Corollary 1 of [17] we know that if $G$ is a minimal non-solvable group, then for some normal subgroup $N$ of $G$, $G/N$ is isomorphic to one of the following groups:

1. $PSL(2, 2^p)$, $p$ a prime.
2. $PSL(3^p)$, $p$ an odd prime.
3. $PSL(2, p)$, $p > 3$ a prime congruent to 2 or 3 mod 5.
4. $Sz(2^p)$, $p$ an odd prime.
5. $PSL(3, 3)$.

Also Lemma 2.7 shows that if $G$ is an $E$-group $G/N$ is an $E$-group too where $N$ is a normal subgroup of $G$. Next since by Theorem 2.1, only simple groups which are $E$- groups are $PSL(2, 2^n)$, $n \geq 2$; $Sz(2^{2m+1})$ with $m > 0$, $G/N$ can be only isomorphic to one of the cases 1 or 4 in above. But for these cases, from Lemmas 2.3 and 2.6 we have $ep(G/N) \leq \frac{1}{6}$. Next, since by Lemma 2.8, $ep(G) \leq ep(G/N)$, we have also that $ep(G) \leq \frac{1}{6}$. Therefore we get a contradiction by our assumption and proof is complete.
REFERENCES