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SLIDING WINDOW ROUGHT MEASURABLE ON $I-$ CORE OF TRIPLE SEQUENCES OF BERNSTEIN OPERATOR

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Abstract. We introduce sliding window rough $I-$ core and study some basic properties of Bernstein polynomials of rough $I-$ convergent of triple sequence spaces. Also, we study the set of all Bernstein polynomials of sliding window of rough $I-$ limits of a triple sequence spaces and relation between analytic ness and Bernstein polynomials of sliding window of rough $I-$ core of a triple sequence spaces.

Key words and Phrases: Ideal, triple sequences, rough convergence, closed and convex, cluster points and rough limit points, Bernstein operator.

Abstrak. Dalam makalah ini, diperkenalkan konsep *sliding window rough $I-$ core* dan dikaji beberapa properti dasar dari polinomial Bernstein dari *rough $I-$ convergent* di ruang barisan *triple*. Dikaji juga himpunan semua polinomial Bernstein dari *sliding window rough $I-$ limit* di ruang barisan *triple* dan kaitan antara *analytic ness* dengan polinomial Bernstein dari *sliding window rough $I-$ core* di ruang barisan *triple*.

Kata kunci: Ideal, barisan *triple*, *rough convergence*, tertutup dan konveks, titik klaster dan titik *rough limit*, operator Bernstein.

1. INTRODUCTION

The idea of rough convergence was first introduced by Phu [9-11] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r.

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Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] studied that the $r-$ limit set of the sequence is equal to intersection of these sets and that $r-$ core of the sequence is equal to the union of these sets. D$\ddot{u}$ndar and Cakan [9] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence The notion of $I-$ convergence of a triple sequence spaces which is based on the structure of the ideal I of subsets of $\mathbb{N}^3$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

In this paper we investigate some basic properties of rough $I-$ convergence of a triple sequence spaces in three dimensional matrix spaces which are not earlier. We study the set of all rough $I-$ limits of a triple sequence spaces and also the relation between analytic ness and rough $I-$ core of a triple sequence spaces.

Let K be a subset of the set of positive integers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and let us denote the set $K_{ik\ell} = \{(m,n,k) \in K : m \leq i, n \leq j, k \leq \ell\}$. Then the natural density of K is given by

$$\delta(K) = lim_{i,j,\ell \to \infty} \frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$. The Bernstein operator of order (r,s,t) is given by

$$B_{rst}(f,x) = \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right) \binom{r}{m} \binom{s}{n} \binom{t}{k} x^{m+n+k} (1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ such that $(B_{mnk}(f,x))$ is in $\mathbb{R}$, $m,n,k \in \mathbb{N}$. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{rst}(f,x))$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st - lim\ x = 0$, provided that the set

$$K_\epsilon := \left\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x) - f(x)| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta(K_\epsilon) = 0$. That is,

$$lim_{rst \to \infty} \frac{1}{rst} |\{(m,n,k) \leq (r,s,t) : |B_{mnk}(f,x) - (f,x)| \geq \epsilon\}| = 0.$$

In this case, we write $\delta - limB_{mnk}(f,x) = f(x)$ or $B_{mnk}(f,x) \to^{S_B} f(x)$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers , χ_A- the characteristic function of $A \subset \mathbb{N}, \mathbb{R}$ the set of all real numbers. A subset A of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d(A) = lim_{ij\ell \to \infty} \frac{1}{ij\ell} \sum_{m=1}^{i} \sum_{n=1}^{j} \sum_{k=1}^{\ell} \chi_A(K).$$

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al. [13,14]*, *Esi et al. [3-6]*, *Datta et al. [7]*,*Subramanian et al. [15-17]*, *Debnath et al. [8]* and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3. In this paper we denote (γ, η) as a sliding window pair provided:

(i) γ and η are both nondecreasing nonnegative real valued measurable functions defined on $[0, \infty)$,
(ii) $\gamma(\alpha) < \eta(\alpha)$ for every positive real number α, and $\eta(\alpha) \to \infty$ as $\alpha \to \infty$,
(iii) $liminf_{abc} (\eta(\alpha) - \gamma(\alpha)) > 0$ and
(iv) $(0, \infty] = \bigcup \{(\gamma(s) - \eta(s)] : s \leq \alpha\}$ for all $\alpha > 0$.

Suppose $I_{abc} = (\gamma(\alpha), \eta(\alpha)]$ and $\eta(\alpha) - \gamma(\alpha) = \mu(I_{abc})$, where $\mu(A)$ denotes the Lebesgue measure of the set A.

2. DEFINITIONS AND PRELIMINARIES

Throughout the paper $\mathbb{R}^3$ denotes the real three dimensional case with the metric. Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}^3; m, n, k \in \mathbb{N}^3$. The following definitions are obtained:

Definition 2.1. *The function g is $N(\gamma, \eta, f, q)$ summable to $\bar{0}$ and write $N(\gamma, \eta, f, q)-$ $limg = \bar{0}$ (or $g \to \bar{0}$ $N(\gamma, \eta, f, q)$) if and only if $lim_{abc \to \infty} \frac{1}{\mu(I_{abc})} \int_{I_{abc}} f(|g(t), \bar{o}|^q)\, dt$ equals to 0.*

Definition 2.2. *Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x(t)))$ is said to be sliding window measurable function of statistically convergent to $(f, x(t))$ denoted by $B_{mnk}(f, x(t)) \to^{st-limx(t)} (f, x(t))$, if for any $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where*

$$A(\epsilon) = \left\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t)) - (f, x(t))| \geq \epsilon\right\}.$$

Definition 2.3. *Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x(t)))$ is said to be sliding window measurable function of statistically convergent to $(f, x(t))$ denoted by $B_{mnk}(f, x(t)) \to^{st-limx(t)} (f, x(t))$, provided that the set*

$$\left\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t)) - (f, x(t))| \geq \epsilon\right\},$$

has natural density zero for every $\epsilon > 0$. In this case, $(f, x(t))$ is called the sliding window measurable function of statistical limit of the sequence of Berstein polynomials.

Definition 2.4. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x(t)))$ in a metric space $(X,|.,.|)$ and r be a non-negative real number is said to be sliding window measurable function of $r-$convergent to $(f,x(t))$, denoted by $B_{mnk}(f,x(t)) \to^{r} (f,x(t))$, if for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}^3$ such that for all $m,n,k \geq N_\epsilon$ we have*

$$|B_{mnk}(f,x(t)) - (f,x(t))| < r + \epsilon$$

In this case $B_{mnk}(f,x(t))$ is called sliding window measurable function on $r-$ limit of $(f,x(t))$.

Remark 2.5. *We consider sliding window measurable function on $r-$ limit set $B_{mnk}(f,x(t))$ which is denoted by $LIM^{r}B_{mnk}(f,x(t))$ and is defined by*

$$LIM^{r}B_{mnk}(f,x(t)) = \{f : B_{mnk}(f,x(t)) \to^{r} (f,x(t))\}.$$

Definition 2.6. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x(t)))$ is said to be measurable function of $r-$ convergent if $LIM^{r}B_{mnk}(f,x(t)) \neq \phi$ and r is called a rough convergence of measurable function of degree of $B_{mnk}(f,x(t))$. If $r=0$ then it is ordinary convergence of triple sequence of Bernstein polynomials.*

Definition 2.7. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x(t)))$ in a metric space $(X,|.,.|)$ and r be a non-negative real number is said to be measurable function of $r-$ statistically convergent to $(f,x(t))$, denoted by $B_{mnk}(f,x(t)) \to^{r-st_3} (f,x(t))$, if for any $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where*

$$A(\epsilon) = \left\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x(t)) - (f,x(t))| \geq r + \epsilon\right\}.$$

In this case $(f,x(t))$ is called sliding window measurable function of $r-$ statistical limit of $B_{mnk}(f,x(t))$. If $r=0$ then it is ordinary statistical convergent of triple sequence of Bernstein polynomials.

Definition 2.8. *A class I of subsets of a nonempty set X is said to be an ideal in X provided*

(i) $\phi \in I$
(ii) $A, B \in I$ *implies* $A \bigcup B \in I$.
(iii) $A \in I, B \subset A$ *implies* $B \in I$.

I is called a nontrivial ideal if $X \notin I$.

Definition 2.9. *A nonempty class F of subsets of a nonempty set X is said to be a filter in X. Provided*

(i) $\phi \in F$.
(ii) $A, B \in F$ *implies* $A \bigcap B \in F$.
(iii) $A \in F, A \subset B$ *implies* $B \in F$.

Definition 2.10. *I is a non trivial ideal in X, $X \neq \phi$, then the class*

$$F(I) = \{M \subset X : M = X \backslash A \; for \; some \; A \in I\}$$

is a filter on X, called the filter associated with I.

Definition 2.11. *A non trivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.*

Remark 2.12. *If I is an admissible ideal, then usual convergence in X implies I convergence in X.*

Remark 2.13. *If I is an admissible ideal, then usual rough convergence implies rough $I-$ convergence.*

Definition 2.14. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x(t)))$ in a metric space $(X, |.,.|)$ and r be a non-negative real number is said to be rough measurable function of ideal convergent or $rI-$ convergent to $(f, x(t))$, denoted by*

$$B_{mnk}(f, x(t)) \xrightarrow{rI} (f, x(t)),$$

if for any $\epsilon > 0$ we have

$$\left\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t)) - (f, x(t))| \geq r + \epsilon\right\} \in I.$$

In this case $(B_{mnk}(f, x(t)))$ is called sliding window measurable function of $rI-$ convergent to $(f, x(t))$ and a triple sequence of Bernstein polynomials $(B_{mnk}(f, x(t)))$ is called rough sliding window measurable function of $I-$ convergent to $(f, x(t))$ with r as roughness of degree. If $r = 0$ then it is ordinary $I-$ convergent.

Generally, let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(g, x(t)))$ is not $I-$ convergent in usual sense and $|B_{mnk}(f, x(t)) - B_{mnk}(g, x(t))| \leq r$ for all $(m,n,k) \in \mathbb{N}^3$ or

$$\left\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t)) - B_{mnk}(g, x(t))| \geq r\right\} \in I.$$

for some $r > 0$. Then the triple sequence of Bernstein polynomials of sliding window measurable function of $(B_{mnk}(f, x(t)))$ is $rI-$ convergent. Also, it is clear that $rI-$ limit of a sequence $B_{mnk}(f, x(t))$ of Bernstein polynomial is not necessarily unique.

Definition 2.15. *Consider $rI-$ limit set of $f(x)$, which is denoted by*

$$I - LIM^r B_{mnk}(f, x(t)) = \left\{f : B_{mnk}(f, x(t)) \to^{rI} (f, x(t))\right\},$$

then the triple sequence of Bernstein polynomials $(B_{mnk}(f, x(t)))$ is said to be sliding window measurable function of $rI-$ convergent if $I - LIM^r B_{mnk}(f, x(t)) \neq \phi$ and r is called a rough sliding window measurable function of $I-$ convergence degree of $B_{mnk}(f, x(t))$.

Definition 2.16. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f, x(t)))$ is said to be sliding window measurable function of $I-$ analytic if there exists a positive real number M such that*

$$\left\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t))|^{1/m+n+k} \geq M\right\} \in I.$$

Definition 2.17. *A point of the function* $(f, x(t)) \in X$ *is said to be an sliding window measurable function of* $I-$ *accumulation point and Let* f *be a continuous function defined on the closed interval* $[0, 1]$. *A Bernstein polynomials* $(B_{mnk}(f, x(t)))$ *is a metric space* (X, d) *if and only if for each* $\epsilon > 0$ *the set*

$$\left\{(m, n, k) \in \mathbb{N}^3 : d(B_{mnk}(f, x(t)), f(x(t))) = |B_{mnk}(f, x(t)) - f(x(t))| < \epsilon\right\} \notin I.$$

We denote the set of all $I-$ accumulation points of $(B_{mnk}(f, x(t)))$ by $I(\Gamma(B_{mnk}(f, x(t))))$.

Definition 2.18. *Let* f *be a continuous function defined on the closed interval* $[0, 1]$. *A triple sequence of Bernstein polynomials* $(B_{mnk}(f, x(t)))$ *is said to be rough sliding window measurable function of* $I-$ *convergent if*

$$I - LIM^r B_{mnk}(f, x(t)) \neq \phi.$$

It is clear that if $I - LIM^r B_{mnk}(f, x(t)) \neq \phi$ for a triple sequence of Bernstein polynomials $(B_{mnk}(f, x(t)))$ of real numbers, then we have $I - LIM^r B_{mnk}(f, x(t)) = [I - limsup\ B_{mnk}(f, x(t)) - r, I - liminf\ B_{mnk}(f, x(t)) + r]$.

Definition 2.19. *Let* f *be a continuous function defined on the closed interval* $[0, 1]$. *A triple sequence of Bernstein polynomials* $(B_{mnk}(f, x(t)))$ *is said to be rough sliding window measurable function of* $I - core\ B_{mnk}(f, x(t))$ *is defined to the closed interval* $[+\infty, -\infty]$.

3. MAIN RESULTS

Theorem 3.1. *Let* f *be a continuous function defined on the closed interval* $[0, 1]$. *A triple sequence of Bernstein polynomials of rough sliding window measurable function of* $(B_{mnk}(f, x(t)))$ *of real numbers and* $I \subset 2^{\mathbb{N}}$ *be an admissible ideal, we have dim* $(I - LIM^r B_{mnk}(f, x(t))) \leq 2r$. *In general, dim* $(I - LIM^r B_{mnk}(f, x(t)))$ *has an upper bound.*

Proof. Assume that diam $(LIM^r B_{mnk}(f, x(t)))$. Then, $\exists B_{mnk}(p, x(t))$, $B_{mnk}(q, x(t))$ in $LIM^r B_{mnk}(f, x(t))$ such that

$$|B_{mnk}(p, x(t)) - B_{mnk}(q, x(t))| > 2r.$$

Take $\epsilon \in \left(0, \frac{|B_{mnk}(p,x(t)) - B_{mnk}(q,x(t))|}{2} - r\right)$. Because $B_{mnk}(p, x(t))$ and $B_{mnk}(q, x(t))$ in $I - LIM^r B_{mnk}(f, x(t))$,
we have $A_1(\epsilon) \in I$ and $A_2(\epsilon) \in I$ for every $\epsilon > 0$, where

$$A_1(\epsilon) = \left\{(i, j, k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t)) - B_{mnk}(p, x(t))| \geq r + \epsilon\right\}$$

and

$$A_2(\epsilon) = \left\{(i, j, k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t)) - B_{mnk}(q, x(t))| \geq r + \epsilon\right\}.$$

Using the properties $F(I)$, we get

$$(A_1(\epsilon)^c \bigcap A_2(\epsilon)^c) \in F(I).$$

Thus we write,

$$\begin{aligned} \left|B_{mnk}\left(p,x\left(t\right)\right)-B_{mnk}\left(q,x\left(t\right)\right)\right| &\leq \left|B_{mnk}\left(f,x\left(t\right)\right)-B_{mnk}\left(p,x\left(t\right)\right)\right| \\ &\quad +\left|B_{mnk}\left(f,x\left(t\right)\right)-B_{mnk}\left(q,x\left(t\right)\right)\right| \\ &< (r+\epsilon)+(r+\epsilon)<2\left(r+\epsilon\right), \end{aligned}$$

for all $(m,n,k)\in A_1\left(\epsilon\right)^c \bigcap A_2\left(\epsilon\right)^c$, which is a contradiction. Hence

$$diam\left(LIM^r B_{mnk}\left(f,x\left(t\right)\right)\right)\leq 2r.$$

Now, consider a triple sequence of Bernstein polynomials of rough sliding window measurable function of $\left(B_{mnk}\left(f,x\left(t\right)\right)\right)$ of real numbers such that

$$I-lim_{mnk\to\infty}B_{mnk}\left(f,x\left(t\right)\right)=f\left(x\left(t\right)\right).$$

Let $\epsilon>0$. Then we can write

$$\left\{(m,n,k)\in\mathbb{N}^3:\left|B_{mnk}\left(f,x\left(t\right)\right)-\left(f,x\left(t\right)\right)\right|\geq\epsilon\right\}\in I$$

Thus, we have

$$\begin{aligned} \left|B_{mnk}\left(f,x\left(t\right)\right)-B_{mnk}\left(p,x\left(t\right)\right)\right| &\leq \left|B_{mnk}\left(f,x\left(t\right)\right)-\left(f,x\left(t\right)\right)\right| \\ &\quad +\left|\left(f,x\left(t\right)\right)-B_{mnk}\left(p,x\left(t\right)\right)\right| \\ &\leq \left|B_{mnk}\left(f,x\left(t\right)\right)-\left(f,x\left(t\right)\right)\right|+r \\ &\leq r+\epsilon, \end{aligned}$$

for each $B_{mnk}\left(p,x\left(t\right)\right)$ in

$$\bar{B}_r\left(\left(f,x\left(t\right)\right)\right):=\left\{B_{mnk}\left(p,x\left(t\right)\right)\in\mathbb{R}^3:\left|B_{mnk}\left(p,x\left(t\right)\right)-\left(f,x\left(t\right)\right)\right|\leq r\right\}.$$

Then, we get

$$\left|B_{mnk}\left(f,x\left(t\right)\right)-B_{mnk}\left(p,x\left(t\right)\right)\right|<r+\epsilon$$

for each $(m,n,k)\in\left\{(m,n,k)\in\mathbb{N}^3:\left|B_{mnk}\left(f,x\left(t\right)\right)-\left(f,x\left(t\right)\right)\right|<\epsilon\right\}$. Because the triple sequence of Bernstein polynomials of rough sliding window measurable function of $B_{mnk}\left(f,x\left(t\right)\right)$ is $I-$ convergent to $\left(f,x\left(t\right)\right)$, we have

$$\left\{(m,n,k)\in\mathbb{N}^3:\left|B_{mnk}\left(f,x\left(t\right)\right)-\left(f,x\left(t\right)\right)\right|<\epsilon\right\}\in F\left(I\right).$$

Therefore, we get $p\in I-LIM^r B_{mnk}\left(f,x\left(t\right)\right)$. Consequently, we can write

$$I-LIM^r B_{mnk}\left(f,x\left(t\right)\right)=\bar{B}_r\left(\left(f,x\left(t\right)\right)\right). \tag{1}$$

Because $dim\left(\bar{B}_r\left(\left(f,x\left(t\right)\right)\right)\right)=2r$, this shows that in general, the upper bound $2r$ of the diameter of the set $I-LIM^r B_{mnk}\left(f,x\left(t\right)\right)$ is not lower bound. □

Theorem 3.2. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of sliding window measurable function of $\left(B_{mnk}\left(f,x\left(t\right)\right)\right)$ of real numbers, $I\subset 3^{\mathbb{N}}$ be an admissible ideal. For an arbitrary $(f,c)\in I\left(\Gamma_x\right)$, we have $\left|B_{mnk}\left(f,x\left(t\right)\right)-(f,c)\right|\leq r$ for all $B_{mnk}\left(f,x\left(t\right)\right)$ in $I-LIM^r B_{mnk}\left(f,x\left(t\right)\right)$.*

Proof. Assume on the contrary that there exist a point $(f,c) \in I(\Gamma_x)$ and $B_{mnk}(f,x(t))$ in $I-LIM^r B_{mnk}(f,x(t))$ such that $|B_{mnk}(f,x(t))-(f,c)|>r$. Define

$$\epsilon := \frac{|B_{mnk}(f,x(t))-(f,c)|-r}{3}.$$

Then

$$\left\{(m,n,k)\in\mathbb{N}^3 : |B_{mnk}(f,x(t))-(f,c)|<\epsilon\right\} \subseteq \begin{array}{l}\{(m,n,k)\in\mathbb{N}^3 : \\ |B_{mnk}(f,x(t))-(f,x(t))| \geq r+\epsilon\}.\end{array} \quad (2)$$

Since $(f,c)\, c \in I(\Gamma_x)$, we have

$$\left\{(m,n,k)\in\mathbb{N}^3 : |B_{mnk}(f,x(t))-(f,c)|<\epsilon\right\} \notin I.$$

But from definition of $I-$ convergence, since

$$\left\{(m,n,k)\in\mathbb{N}^3 : |B_{mnk}(f,x(t))-f(x(t))| \geq r+\epsilon\right\} \in I,$$

so by (3.2) we have

$$\left\{(m,n,k)\in\mathbb{N}^3 : |B_{mnk}(f,x(t))-(f,c)|<\epsilon\right\} \in I,$$

which contradicts the fact $(f,c)\in I(\Gamma_x)$. On the other hand, if $(f,c)\in I(\Gamma_x)$ i.e.,

$$\left\{(m,n,k)\in\mathbb{N}^3 : |B_{mnk}(f,x(t))-(f,c)|<\epsilon\right\} \notin I,$$

then

$$\left\{(m,n,k)\in\mathbb{N}^3 : |B_{mnk}(f,x(t))-(f,x(t))| \geq r+\epsilon\right\} \notin I,$$

which contradicts the fact $(f,x(t)) \in I-LIM^r B_{mnk}(f,x(t))$. □

Theorem 3.3. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of rough sliding window measurable function of $(B_{mnk}(f,x(t))) \xrightarrow{I} (f,x(t)) \Longleftrightarrow I-LIM^r B_{mnk}(f,x(t)) = \bar{B}_r((f,x(t)))$.*

Proof. **Necessity:** By Theorem 3.1.

Sufficiency: Let $I-LIM^r B_{mnk}(f,x(t)) = \bar{B}_r((f,x(t)))\,(\neq \phi)$. Thus the triple sequence spaces of Bernstein polynomials of rough sliding window measurable function of $(B_{mnk}(f,x(t)))$ is $I-$ analytic. Suppose that $(f,x(t))$ has another $I-$ cluster point $\left(f',x(t)\right)$ different from $(f,x(t))$. The point

$$\left(\bar{f},x(t)\right) = (f,x(t)) + \frac{r}{\left|(f,x(t))-\left(f',x(t)\right)\right|}\left((f,x(t))-\left(f',x(t)\right)\right)$$

and

$$\left(\bar{f},x(t)\right)-\left(f',x(t)\right) = (f,x(t))-\left(f',x(t)\right) + \frac{r}{\left|(f,x(t))-\left(f',x(t)\right)\right|}\left[((f,x(t)) - \left(f',x(t)\right)\right) - \left(\left(f',x(t)\right)-\left(f',x(t)\right)\right)\Big].$$

$$\begin{aligned}\left|\left(\bar{f},x(t)\right)-\left(f',x(t)\right)\right| &= \left|(f,x(t))-\left(f',x(t)\right)\right| \\ &\quad +\frac{r}{\left|(f,x(t))-\left(f',x(t)\right)\right|}\left|(f,x(t))-\left(f',x(t)\right)\right|\end{aligned}$$

$$\begin{aligned}\left|\left(\bar{f},x(t)\right)-\left(f',x(t)\right)\right| &= \left|(f,x(t))-\left(f',x(t)\right)\right|+r \\ &> r.\end{aligned}$$

Since $\left(f',x(t)\right)\in I(\Gamma_x)$, by Theorem 3.2, $\left(\bar{f},x(t)\right)\notin I-LIM^r B_{mnk}(f,x(t))$. It is not possible as

$$\left|\left(\bar{f},x(t)\right)-(f,x(t))\right|=r$$

and

$$I-LIM^r B_{mnk}(f,x(t))=\bar{B}_r((f,x(t))).$$

Since $(f,x(t))$ is the unique-$I-$ cluster point of $(f,x(t))$. Hence $B_{mnk}(f,x(t))\xrightarrow{I} f(x(t))$. □

Corollary 3.4. *If $(X,|.,.|)$ is a strictly convex spaces and Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of rough sliding window measurable function of $(B_{mnk}(f,x(t)))\in X$, there exists $y_1,y_2\in I-LIM^r B_{mnk}(f,x(t))$ such that $|y_1-y_2|=2r$, then this triple sequence $(f,x(t))\to^I \frac{y_1+y_2}{2}$*

Theorem 3.5. *If $I-LIM^r\neq\phi$, then $I-lim\ sup\ B_{mnk}(f,x(t))$ and $I-lim\ inf\ B_{mnk}(f,x(t))$ belong to the set $I-LIM^{2r}B_{mnk}(f,x(t))$.*

Proof. We know that $I-LIM^r B_{mnk}(f,x(t))\neq\phi$, a triple sequence of Bernstein polynomials of rough sliding window measurable function of $(B_{mnk}(f,x(t)))$ is $I-$ analytic. The number $I-lim\ infB_{mnk}(f,x(t))$ is an $I-$ cluster point of $(f,x(t))$ and consequently, we have

$$|(f,x(t))-I-lim\ infB_{mnk}(f,x(t))|\leq r\ \forall\ (f,x(t))\in I-LIM^r(f,x(t)).$$

Let $A=\left\{(m,n,k)\in\mathbb{N}^3:|(f,x(t))-B_{mnk}(f,x(t))|\geq r+\epsilon\right\}$. Now if (m,n,k) is not in A, then

$$\begin{aligned}|B_{mnk}(f,x(t))-(I-lim\ infB_{mnk}(f,x(t)))| &\leq |B_{mnk}(f,x(t))-(f,x(t))| \\ &\quad +|(f,x(t))-(I-lim\ inf\ B_{mnk}(f,x(t)))| \\ &< 2r+\epsilon.\end{aligned}$$

Thus

$$I-lim\ infB_{mnk}(f,x(t))\in I-LIM^{2r}B_{mnk}(f,x(t)).$$

Similarly it can be shown that $I-lim\ sup\ x_{mnk}(t)\in I-LIM^{2r}x_{mnk}(t)$. □

Corollary 3.6. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of rough sliding window measurable function of $(B_{mnk}(f,x(t)))$ of real numbers, if $I-LIM^r B_{mnk}(f,x(t))\neq\phi$, then*

$$I - core\{f(x(t))\} \subseteq I - LIM^{2r} B_{mnk}(f, x(t)).$$

Proof. We have

$$I - LIM^r B_{mnk}(f, x(t)) = [I - lim\ sup\ B_{mnk}(f, x(t)) - 2r, I - lim\ inf B_{mnk}(f, x(t)) + 2r].$$

Then the result follows from Theorem 3.5. □

Theorem 3.7. *Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials of rough sliding window measurable function of $(B_{mnk}(f, x(t)))$ of real numbers. Then the $dim(I - core\{B_{mnk}(f, x(t))\})$ of the set*

$$I - core\{B_{mnk}(f, x(t))\} = r \Longleftrightarrow I - core\{f(x(t))\} = I - LIM^r B_{mnk}(f, x(t))$$

Proof. Consider

$$\begin{aligned} & dim(I - core\{B_{mnk}(f, x(t))\}) && = r \\ \Leftrightarrow\ & (I - lim\ sup B_{mnk}(f, x(t))) - (I - lim\ inf x_{mnk}(t)) && = r \\ \Leftrightarrow\ & I - core\{x_{mnk}(t)\} && = [I - lim\ inf x_{mnk}(t), \\ & && \quad I - lim\ sup\ B_{mnk}(f, x(t))] \\ & && = [I - I - lim\ sup\ B_{mnk}(f, x(t)) - r, \\ & && \quad I - lim\ inf B_{mnk}(f, x(t)) + r] \\ & && = I - LIM^r B_{mnk}(f, x(t)). \end{aligned}$$

Also it is easy to see that
(i) $r > diam(I - core\{B_{mnk}(f, x(t))\}) \Longleftrightarrow I - core\{B_{mnk}(f, x(t))\} \subset I - LIM^r B_{mnk}(f, x(t))$,
(ii) $r < diam(I - core\{B_{mnk}(f, x(t))\}) \Longleftrightarrow I - LIM^r B_{mnk}(f, x(t)) \subset I - core\{B_{mnk}(f, x(t))\}$. □

Theorem 3.8. *Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials of rough sliding window measurable function of $(B_{mnk}(f, x(t)))$ of real numbers, if $\bar{r} = inf\{r \geq 0 : I - LIM^r B_{mnk}(f, x(t)) \neq \phi\}$, then $\bar{r} = radius(I - core\{B_{mnk}(f, x(t))\})$.*

Proof. If the set $I - core\{B_{mnk}(f, x(t))\}$ is singleton, then $radius(I - core\{B_{mnk}(f, x(t))\})$ is 0 and the triple sequence of Bernstein polynomials of sliding window measurable function is $I-$ convergent, i.e., $I - LIM^0 B_{mnk}(f, x(t)) \neq \phi$. Hence we get $\bar{r} = radius(I - core\{B_{mnk}(f, x(t))\}) = 0$.

Now assume that the set $I - core\{B_{mnk}(f, x(t))\}$ is not a single ton. We can write $I - core\{B_{mnk}(f, x(t))\} = [a, b]$ where $a = I - lim\ inf B_{mnk}(f, x(t))$ and $b = I - lim\ sup\ B_{mnk}(f, x(t))$.

Now let us assume that $\bar{r} \neq radius(I - core\{B_{mnk}(f, x(t))\})$.

If $\bar{r} < radius(I - core\{x_{mnk}(t)\})$, then define $\bar{\epsilon} = \frac{\frac{b-a}{2} - \bar{r}}{3}$. Now, be definition of $\bar{r}$ implies that $I - LIM^{\bar{r}+\bar{\epsilon}} B_{mnk}(f, x(t)) \neq \phi$, given $\epsilon > 0\ \exists\ l \in \mathbb{R} : A = \left\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t)) - f(x(t))| \geq (\bar{r} + \bar{\epsilon}) + \epsilon\right\} \in I$. Since $\bar{r} + \bar{\epsilon} < \frac{b-a}{2}$ which is a contradiction of the definition of a and b.

If $\bar{r} > radius(I - core\{B_{mnk}(f, x(t))\})$, then define $\bar{\epsilon} = \frac{\bar{r} - \frac{b-a}{2}}{3}$ and $r' =$

$\bar{r}-2\bar{\epsilon}$. It is clear that $0 \leq r^{'} \leq \bar{r}$ and by definitions of a and b, the number $\frac{b-a}{2} \in I-LIM^{r^{'}} B_{mnk}\left(f, x\left(t\right)\right)$. Then we get

$$\bar{r} \in \left\{r \geq 0 : I-LIM^{r} B_{mnk}\left(f, x\left(t\right)\right) \neq \phi\right\},$$

which contradicts the equality

$$\bar{r} = inf\left\{r \geq 0 : I-LIM^{r} B_{mnk}\left(f, x\left(t\right)\right) \neq \phi\right\} \text{ as } r^{'} < r.$$

□

Corollary 3.9. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of rough sliding window measurable function of $\left(B_{mnk}\left(f, x\left(t\right)\right)\right)$ of real numbers, then $I-core\left\{B_{mnk}\left(f, x\left(t\right)\right)\right\} = I-LIM^{2\bar{r}} B_{mnk}\left(f, x\left(t\right)\right)$*

Proof. It follows that Theorem 3.7 and Theorem 3.8. □

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