SLIDING WINDOW ROUGHT MEASURABLE ON $I$–CORE OF TRIPLE SEQUENCES OF BERNSTEIN OPERATOR

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Abstract. We introduce sliding window rough $I$–core and study some basic properties of Bernstein polynomials of rough $I$–convergent of triple sequence spaces. Also, we study the set of all Bernstein polynomials of sliding window of rough $I$–limits of a triple sequence spaces and relation between analytic ness and Bernstein polynomials of sliding window of rough $I$–core of a triple sequence spaces.

Key words and Phrases: Ideal, triple sequences, rough convergence, closed and convex, cluster points and rough limit points, Bernstein operator.

1. INTRODUCTION

The idea of rough convergence was first introduced by Phu [9-11] in finite dimensional normed spaces. He showed that the set $\text{LIM}_r^I$ is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of $\text{LIM}_r^I$ on the roughness of degree $r$. 

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Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] studied that the $r$– limit set of the sequence is equal to intersection of these sets and that $r$– core of the sequence is equal to the union of these sets. Düntar and Cakan [9] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence The notion of $I$– convergence of a triple sequence spaces which is based on the structure of the ideal $I$ of subsets of $\mathbb{N}^3$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

In this paper we investigate some basic properties of rough $I$– convergence of a triple sequence spaces in three dimensional matrix spaces which are not earlier. We study the set of all rough $I$– limits of a triple sequence spaces and also the relation between analytic ness and rough $I$– core of a triple sequence spaces.

Let $K$ be a subset of the set of positive integers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and let us denote the set $K_{ijk} = \{(m, n, k) \in K : m \leq i, n \leq j, k \leq \ell \}$. Then the natural density of $K$ is given by

$$
\delta (K) = \lim_{i,j,\ell \to \infty} \frac{|K_{ij\ell}|}{ij\ell},
$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$. The Bernstein operator of order $(r,s,t)$ is given by

$$
B_{rst}(f,x) = \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} f \left( \frac{mnk}{rst} \right) \left( \frac{m}{r} \right) \left( \frac{s}{n} \right) \left( \frac{t}{k} \right) x^{m+n+k} (1-x)^{(m-r)+(n-s)+(k-t)}
$$

where $f$ is a continuous (real or complex valued) function defined on $[0,1]$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric $(X,d)$. Consider a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ such that $(B_{mnk}(f,x))$ is in $\mathbb{R}$, $m,n,k \in \mathbb{N}$. Let $f$ be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{rst}(f,x))$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st \lim x = 0$, provided that the set

$$
K_\varepsilon := \{(m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f,x) - f(x)| \geq \varepsilon \}
$$

has natural density zero for any $\varepsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta (K_\varepsilon) = 0$. That is,

$$
\lim_{rst \to \infty} \frac{1}{rst} |\{(m, n, k) \leq (r, s, t) : |B_{mnk}(f,x) - f(x)| \geq \varepsilon \}| = 0.
$$

In this case, we write $\delta \lim B_{mnk}(f,x) = f(x)$ or $B_{mnk}(f,x) \to \mathcal{S}^n f(x)$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers , $\chi_A$– the characteristic function of $A \subset \mathbb{N}, \mathbb{R}$ the set of all real numbers. A subset $A$ of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$
d(A) = \lim_{ij\ell \to \infty} \frac{1}{ij\ell} \sum_{m=1}^{i} \sum_{n=1}^{j} \sum_{k=1}^{t} \chi_A (K).
$$
A triple sequence (real or complex) can be defined as a function
\[ x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}(\mathbb{C}), \]
where \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \) denote the set of natural numbers, real
numbers and complex numbers respectively. The different types of notions of triple
sequence was introduced and investigated at the initial by Sahiner et al. [13,14],
Esi et al. [3-6], Datta et al. [7], Subramanian et al. [15-17], Debnath et al. [8] and
many others.

A triple sequence \( x = (x_{mnk}) \) is said to be triple analytic if
\[ \sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty. \]

The space of all triple analytic sequences are usually denoted by \( \Lambda^3 \). In this paper
we denote \((\gamma, \eta)\) as a sliding window pair provided:

(i) \( \gamma \) and \( \eta \) are both nondecreasing nonnegative real valued measurable
functions defined on \([0, \infty)\),
(ii) \( \gamma(\alpha) < \eta(\alpha) \) for every positive real number \( \alpha \), and \( \eta(\alpha) \to \infty \) as \( \alpha \to \infty \),
(iii) \( \liminfty_{abc}(\eta(\alpha) - \gamma(\alpha)) \geq 0 \) and
(iv) \( (0, \infty] = \bigcup \{ (s, \eta) : s \leq \alpha \} \) for all \( \alpha > 0 \).

Suppose \( I_{abc} = (\gamma(\alpha), \eta(\alpha)) \) and \( \eta(\alpha) - \gamma(\alpha) = \mu(I_{abc}) \), where \( \mu(A) \) denotes the
Lebesgue measure of the set \( A \).

**2. DEFINITIONS AND PRELIMINARIES**

Throughout the paper \( \mathbb{R}^3 \) denotes the real three dimensional case with the
metric. Consider a triple sequence \( x = (x_{mnk}) \) such that \( x_{mnk} \in \mathbb{R}^3; m,n,k \in \mathbb{N}^3 \).

The following definitions are obtained:

**Definition 2.1.** The function \( q = N(\gamma, \eta, f, q) \) summable to 0 and write \( N(\gamma, \eta, f, q) = \)
\[ \lim_{n \to \infty} \int_{I_{abc}} f(|g(t)|, o|^q|) dt \]
equals to 0.

**Definition 2.2.** Let \( f \) be a continuous function defined on the closed interval
\([0, 1] \). A triple sequence of Bernstein polynomials \( (B_{mnk}(f, x(t))) \) is said to be
sliding window measurable function of statistically convergent to \( (f, x(t)) \) denoted
by \( B_{mnk}(f, x(t)) \to st^{-limx(t)} (f, x(t)), \) if for any \( \epsilon > 0 \) we have \( d(A(\epsilon)) = 0, \)
where
\[ A(\epsilon) = \{ (m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t)) - (f, x(t))| \geq \epsilon \}. \]

**Definition 2.3.** Let \( f \) be a continuous function defined on the closed interval
\([0, 1] \). A triple sequence of Bernstein polynomials \( (B_{mnk}(f, x(t))) \) is said to be
sliding window measurable function of statistically convergent to \( (f, x(t)) \) denoted
by \( B_{mnk}(f, x(t)) \to st^{-limx(t)} (f, x(t)), \) provided that the set
\[ \{ (m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t)) - (f, x(t))| \geq \epsilon \}, \]
has natural density zero for every \( \epsilon > 0 \). In this case, \( (f, x(t)) \) is called the sliding
window measurable function of statistical limit of the sequence of Bernstein poly-
nomials.
Definition 2.4. Let \( f \) be a continuous function defined on the closed interval \([0,1]\). A triple sequence of Bernstein polynomials \( (B_{mnk}(f,x(t))) \) in a metric space \((X,|.|)\) and \( r \) be a non-negative real number is said to be sliding window measurable function of \( r \)-convergent to \( (f,x(t)) \), denoted by \( B_{mnk}(f,x(t)) \to^r (f,x(t)) \), if for any \( \epsilon > 0 \) there exists \( N_\epsilon \in \mathbb{N}^3 \) such that for all \( m,n,k \geq N_\epsilon \) we have
\[
|B_{mnk}(f,x(t)) - (f,x(t))| < r + \epsilon
\]
In this case \( B_{mnk}(f,x(t)) \) is called sliding window measurable function on \( r \)-limit of \((f,x(t))\).

Remark 2.5. We consider sliding window measurable function on \( r \)-limit set \( B_{mnk}(f,x(t)) \) which is denoted by \( \operatorname{LIM}^r B_{mnk}(f,x(t)) \) and is defined by
\[
\operatorname{LIM}^r B_{mnk}(f,x(t)) = \{ f : B_{mnk}(f,x(t)) \to^r (f,x(t)) \}.
\]

Definition 2.6. Let \( f \) be a continuous function defined on the closed interval \([0,1]\). A triple sequence of Bernstein polynomials \( (B_{mnk}(f,x(t))) \) in a metric space \((X,|.|)\) and \( r \) be a non-negative real number is said to be measurable function of \( r \)-convergent if \( \operatorname{LIM}^r B_{mnk}(f,x(t)) \neq \emptyset \) and \( r \) is called a rough convergence of measurable function of degree of \( B_{mnk}(f,x(t)) \). If \( r = 0 \) then it is ordinary convergence of triple sequence of Bernstein polynomials.

Definition 2.7. Let \( f \) be a continuous function defined on the closed interval \([0,1]\). A triple sequence of Bernstein polynomials \( (B_{mnk}(f,x(t))) \) in a metric space \((X,|.|)\) and \( r \) be a non-negative real number is said to be measurable function of \( r \)-statistically convergent to \((f,x(t))\), denoted by \( B_{mnk}(f,x(t)) \to^{r-sts} (f,x(t)) \), if for any \( \epsilon > 0 \) we have \( d(A(\epsilon)) = 0 \), where
\[
A(\epsilon) = \{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x(t)) - (f,x(t))| \geq r + \epsilon\}.
\]
In this case \((f,x(t))\) is called sliding window measurable function of \( r \)-statistical limit of \( B_{mnk}(f,x(t)) \). If \( r = 0 \) then it is ordinary statistical convergent of triple sequence of Bernstein polynomials.

Definition 2.8. A class \( I \) of subsets of a nonempty set \( X \) is said to be an ideal in \( X \) provided

(i) \( \phi \in I \)
(ii) \( A, B \in I \) implies \( A \cup B \in I \).
(iii) \( A \in I, B \subset A \) implies \( B \in I \).

\( I \) is called a nontrivial ideal if \( X \notin I \).

Definition 2.9. A nonempty class \( F \) of subsets of a nonempty set \( X \) is said to be a filter in \( X \). Provided

(i) \( \phi \in F \).
(ii) \( A, B \in F \) implies \( A \cap B \in F \).
(iii) \( A \in F, A \subset B \) implies \( B \in F \).

Definition 2.10. \( I \) is a non trivial ideal in \( X \), \( X \neq \phi \), then the class
\[
F(I) = \{ M \subset X : M = X \setminus A \text{ for some } A \in I \}
\]
is a filter on \( X \), called the filter associated with \( I \).
Definition 2.11. A non trivial ideal $I$ in $X$ is called admissible if \( \{x\} \in I \) for each $x \in X$.

Remark 2.12. If $I$ is an admissible ideal, then usual convergence in $X$ implies $I$-convergence in $X$.

Remark 2.13. If $I$ is an admissible ideal, then usual rough convergence implies rough $I$-convergence.

Definition 2.14. Let $f$ be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $\{B_{mnk}(f,x(t))\}$ in a metric space $(X,|.|)$ and $r$ be a non-negative real number is said to be rough measurable function of ideal convergent or $rI$-convergent to $(f,x(t))$, denoted by

\[
B_{mnk}(f,x(t)) \xrightarrow{rI} (f,x(t)) ,
\]

if for any $\epsilon > 0$ we have

\[
\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x(t)) - (f,x(t))| \geq r + \epsilon\} \in I.
\]

In this case $\{B_{mnk}(f,x(t))\}$ is called sliding window measurable function of $rI$-convergent to $(f,x(t))$ and a triple sequence of Bernstein polynomials $\{B_{mnk}(f,x(t))\}$ is called rough sliding window measurable function of $I$-convergent to $(f,x(t))$ with $r$ as roughness of degree. If $r = 0$ then it is ordinary $I$-convergent.

Generally, let $f$ be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $\{B_{mnk}(g,x(t))\}$ is not $I$-convergent in usual sense and $|B_{mnk}(f,x(t)) - B_{mnk}(g,x(t))| \leq r$ for all $(m,n,k) \in \mathbb{N}^3$ or

\[
\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x(t)) - B_{mnk}(g,x(t))| \geq r\} \in I.
\]

for some $r > 0$. Then the triple sequence of Bernstein polynomials of sliding window measurable function of $(B_{mnk}(f,x(t)))$ is $rI$-convergent. Also, it is clear that $rI$-limit of a sequence $B_{mnk}(f,x(t))$ of Bernstein polynomial is not necessarily unique.

Definition 2.15. Consider $rI$-limit set of $f(x)$, which is denoted by

\[
I - \text{LIM}^r B_{mnk}(f,x(t)) = \{ f : B_{mnk}(f,x(t)) \to I^r (f,x(t)) \},
\]

then the triple sequence of Bernstein polynomials $\{B_{mnk}(f,x(t))\}$ is said to be sliding window measurable function of $rI$-convergent if $I - \text{LIM}^r B_{mnk}(f,x(t)) \neq \emptyset$ and $r$ is called a rough sliding window measurable function of $I$-convergent degree of $B_{mnk}(f,x(t))$.

Definition 2.16. Let $f$ be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $\{B_{mnk}(f,x(t))\}$ is said to be sliding window measurable function of $I$-analytic if there exists a positive real number $M$ such that

\[
\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x(t))|^{1/m+n+k} \geq M\} \in I.
\]
Definition 2.17. A point of the function \( (f, x(t)) \in X \) is said to be an sliding window measurable function of \( I \) accumulation point and Let \( f \) be a continuous function defined on the closed interval \([0,1] \). A Bernstein polynomials \( (B_{mnk}(f,x(t))) \) is a metric space \( (X,d) \) if and only if for each \( \epsilon > 0 \) the set
\[
\{(m,n,k) \in \mathbb{N}^3 : d\left(B_{mnk}(f,x(t)),f(x(t))\right) = |B_{mnk}(f,x(t)) - f(x(t))| < \epsilon\} \notin I.
\]

We denote the set of all \( I \) - accumulation points of \( (B_{mnk}(f,x(t))) \) by \( I \left(\Gamma(B_{mnk}(f,x(t)))\right) \).

Definition 2.18. Let \( f \) be a continuous function defined on the closed interval \([0,1] \). A triple sequence of Bernstein polynomials \( (B_{mnk}(f,x(t))) \) is said to be rough sliding window measurable function of \( I \) convergent if
\[
I - \text{LIM}^r B_{mnk}(f,x(t)) \neq \varnothing.
\]

It is clear that if \( I - \text{LIM}^r B_{mnk}(f,x(t)) \neq \varnothing \) for a triple sequence of Bernstein polynomials \( (B_{mnk}(f,x(t))) \) of real numbers, then we have \( I - \text{LIM}^r B_{mnk}(f,x(t)) = [I - \limsup B_{mnk}(f,x(t)) - r, I - \liminf B_{mnk}(f,x(t)) + r] \).

Definition 2.19. Let \( f \) be a continuous function defined on the closed interval \([0,1] \). A triple sequence of Bernstein polynomials \( (B_{mnk}(f,x(t))) \) is said to be rough sliding window measurable function of \( I \) core \( B_{mnk}(f,x(t)) \) is defined to the closed interval \([+\infty,-\infty] \).

3. MAIN RESULTS

Theorem 3.1. Let \( f \) be a continuous function defined on the closed interval \([0,1] \). A triple sequence of Bernstein polynomials of rough sliding window measurable function of \( (B_{mnk}(f,x(t))) \) of real numbers and \( I \subset 2^\mathbb{N} \) be an admissible ideal, we have \( \dim \ (I - \text{LIM}^r B_{mnk}(f,x(t))) \leq 2r \). In general, \( \dim \ (I - \text{LIM}^r B_{mnk}(f,x(t))) \) has an upper bound.

Proof. Assume that \( \text{diam} \ (\text{LIM}^r B_{mnk}(f,x(t))) \). Then, \( \exists B_{mnk}(p,x(t)), B_{mnk}(q,x(t)) \) in \( \text{LIM}^r B_{mnk}(f,x(t)) \) such that
\[
|B_{mnk}(p,x(t)) - B_{mnk}(q,x(t))| > 2r.
\]

Take \( \epsilon \in \left(0, \frac{|B_{mnk}(p,x(t)) - B_{mnk}(q,x(t))|}{2} - r\right) \). Because \( B_{mnk}(p,x(t)) \) and \( B_{mnk}(q,x(t)) \) in \( I - \text{LIM}^r B_{mnk}(f,x(t)) \), we have \( A_1(\epsilon) \in I \) and \( A_2(\epsilon) \in I \) for every \( \epsilon > 0 \), where
\[
A_1(\epsilon) = \{(i,j,k) \in \mathbb{N}^3 : |B_{mnk}(f,x(t)) - B_{mnk}(p,x(t))| \geq r + \epsilon\}
\]
and
\[
A_2(\epsilon) = \{(i,j,k) \in \mathbb{N}^3 : |B_{mnk}(f,x(t)) - B_{mnk}(q,x(t))| \geq r + \epsilon\}.
\]

Using the properties \( F(I) \), we get
\[
(A_1(\epsilon) \cap A_2(\epsilon)) \in F(I).
\]
Thus we write,

$$ |B_{mnk}(p, x(t)) - B_{mnk}(q, x(t))| \leq |B_{mnk}(f, x(t)) - B_{mnk}(p, x(t))| + |B_{mnk}(f, x(t)) - B_{mnk}(q, x(t))| $$

$$ < (r + \epsilon) + (r + \epsilon) < 2(r + \epsilon), $$

for all \((m, n, k) \in A_1(\epsilon) \cap A_2(\epsilon),\) which is a contradiction. Hence

$$ \text{diam} \left( \text{LIM}^* B_{mnk}(f, x(t)) \right) \leq 2r. $$

Now, consider a triple sequence of Bernstein polynomials of rough sliding window measurable function of \((B_{mnk}(f, x(t)))\) of real numbers such that

$$ I - \lim_{mnk \to \infty} B_{mnk}(f, x(t)) = f(x(t)). $$

Let \(\epsilon > 0\). Then we can write

$$ \{(m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t)) - (f, x(t))| \geq \epsilon \} \in I $$

Thus, we have

$$ |B_{mnk}(f, x(t)) - B_{mnk}(p, x(t))| \leq |B_{mnk}(f, x(t)) - (f, x(t))| + |(f, x(t)) - B_{mnk}(p, x(t))| $$

$$ \leq |B_{mnk}(f, x(t)) - (f, x(t))| + r $$

$$ \leq r + \epsilon, $$

for each \(B_{mnk}(p, x(t))\) in

$$ B_r((f, x(t))) := \{B_{mnk}(p, x(t)) \in \mathbb{R}^3 : |B_{mnk}(p, x(t)) - (f, x(t))| \leq r \}. $$

Then, we get

$$ |B_{mnk}(f, x(t)) - B_{mnk}(p, x(t))| < r + \epsilon $$

for each \((m, n, k) \in \{(m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t)) - (f, x(t))| < \epsilon \}.\) Because the triple sequence of Bernstein polynomials of rough sliding window measurable function of \(B_{mnk}(f, x(t))\) is \(I-\) convergent to \((f, x(t)),\) we have

$$ \{(m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f, x(t)) - (f, x(t))| < \epsilon \} \in F(I). $$

Therefore, we get \(p \in I - \text{LIM}^* B_{mnk}(f, x(t))\). Consequently, we can write

$$ I - \text{LIM}^* B_{mnk}(f, x(t)) = B_r((f, x(t))). $$

(1)

Because \(\text{dim} \left( B_r((f, x(t))) \right) = 2r,\) this shows that in general, the upper bound \(2r\) of the diameter of the set \(I - \text{LIM}^* B_{mnk}(f, x(t))\) is not lower bound.

\(\Box\)

**Theorem 3.2.** Let \(f\) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials of sliding window measurable function of \((B_{mnk}(f, x(t)))\) of real numbers, \(I \subset \mathbb{R}^1\) be an admissible ideal. For an arbitrary \((f, c) \in I(\Gamma_x),\) we have \(|B_{mnk}(f, x(t)) - (f, c)| \leq r\) for all \(B_{mnk}(f, x(t))\) in \(I - \text{LIM}^* B_{mnk}(f, x(t)).\)
Proof. Assume on the contrary that there exist a point \((f, c) \in I (\Gamma_x)\) and \(B_{mnk} (f, x (t))\) in \(I - \lim^* B_{mnk} (f, x (t))\) such that \(|B_{mnk} (f, x (t)) - (f, c)| > r\). Define
\[
\epsilon := \frac{|B_{mnk} (f, x (t)) - (f, c)| - r}{3}.
\]

Then
\[
\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk} (f, x (t)) - (f, c)| < \epsilon\} \subseteq \{(m, n, k) \in \mathbb{N}^3 : |B_{mnk} (f, x (t)) - (f, x (t))| \geq r + \epsilon\}.
\]

So by (3.2) we have
\[
\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk} (f, x (t)) - (f, c)| < \epsilon\} \subseteq \{(m, n, k) \in \mathbb{N}^3 : |B_{mnk} (f, x (t)) - (f, x (t))| \geq r + \epsilon\}.
\]

Since \((f, c) \in I (\Gamma_x)\), we have
\[
\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk} (f, x (t)) - (f, c)| < \epsilon\} \not\subseteq I.
\]

But from definition of \(I\) convergence, since
\[
\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk} (f, x (t)) - (f, x (t))| \geq r + \epsilon\} \subseteq I,
\]
so by (3.2) we have
\[
\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk} (f, x (t)) - (f, c)| < \epsilon\} \subseteq I,
\]
which contradicts the fact \((f, c) \in I (\Gamma_x)\). On the other hand, if \((f, c) \in I (\Gamma_x)\) i.e.,
\[
\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk} (f, x (t)) - (f, c)| < \epsilon\} \not\subseteq I,
\]
then
\[
\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk} (f, x (t)) - (f, x (t))| \geq r + \epsilon\} \not\subseteq I,
\]
which contradicts the fact \((f, x (t)) \in I - \lim^* B_{mnk} (f, x (t))\). \(\square\)

**Theorem 3.3.** Let \(f\) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials of rough sliding window measurable function of \((B_{mnk} (f, x (t))) \Rightarrow (f, x (t))\) \(\iff I - \lim^* B_{mnk} (f, x (t)) = B_r ((f, x (t)))\).

**Proof. Necessity:** By Theorem 3.1.

**Sufficiency:** Let \(I - \lim^* B_{mnk} (f, x (t)) = B_r ((f, x (t))) (\neq \phi)\). Thus the triple sequence spaces of Bernstein polynomials of rough sliding window measurable function of \((B_{mnk} (f, x (t)))\) is \(I\) analytic. Suppose that \((f, x (t))\) has another \(I\) cluster point \((f', x (t))\) different from \((f, x (t))\). The point
\[
(f, x (t)) = (f, x (t)) + \frac{r}{2(|f, x (t)| - |f', x (t)|)} ([f, x (t)] - (f', x (t)))
\]
and
\[
(f, x (t)) - (f', x (t)) = (f, x (t)) - (f', x (t)) + \frac{r}{2(|f, x (t)| - |f', x (t)|)} ([f, x (t)] - (f', x (t))) - (f', x (t)) - (f', x (t)).
\]
and consequently, we have

\[(f, x(t)) - (f', x(t)) = |(f, x(t)) - (f', x(t))| \]

\[\left| \frac{r}{|f(x(t)) - (f', x(t))|} (f, x(t)) - (f', x(t)) \right| \]

\[\left| (f, x(t)) - (f', x(t)) \right| = |(f, x(t)) - (f', x(t))| + r > r.\]

Since \((f, x(t)) \in I(\Gamma_x)\), by Theorem 3.2, \((f, x(t)) \notin I - LIM^r B_{mnk}(f, x(t))\).

It is not possible as

\[|\left( f, x(t) \right) - (f, x(t)) | = r\]

and

\[I - LIM^r B_{mnk}(f, x(t)) = \bar{B}_r ((f, x(t)) \).\]

Since \((f, x(t))\) is the unique-I– cluster point of \((f, x(t))\). Hence

\[B_{mnk}(f, x(t)) \rightarrow I \frac{y_1 + y_2}{2} f(x(t)).\]

**Corollary 3.4.** If \((X, |.|)\) is a strictly convex spaces and Let \(f\) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials of rough sliding window measurable function of \((B_{mnk}(f, x(t))) \in X\), there exists \(y_1, y_2 \in I - LIM^r B_{mnk}(f, x(t))\) such that \(|y_1 - y_2| = 2r\), then this triple sequence \((f, x(t)) \rightarrow I \frac{y_1 + y_2}{2} f(x(t)).\)

**Theorem 3.5.** If \(I - LIM^r \neq \phi\), then \(I - \lim sup B_{mnk}(f, x(t))\) and \(I - \lim \inf B_{mnk}(f, x(t))\) belong to the set \(I - LIM^r B_{mnk}(f, x(t))\).

**Proof.** We know that \(I - LIM^r B_{mnk}(f, x(t)) \neq \phi\), a triple sequence of Bernstein polynomials of rough sliding window measurable function of \((B_{mnk}(f, x(t))) \in I–\) analytic. The number \(I - \lim \inf B_{mnk}(f, x(t))\) is an \(I–\) cluster point of \((f, x(t))\) and consequently, we have

\[\left| (f, x(t)) - I - \lim \inf B_{mnk}(f, x(t)) \right| \leq r \forall (f, x(t)) \in I - LIM^r (f, x(t)).\]

Let \(A = \{(m, n, k) \in \mathbb{N}^3 : |(f, x(t)) - B_{mnk}(f, x(t))| \geq r + \epsilon\}\). Now if \((m, n, k)\) is not in \(A\), then

\[|B_{mnk}(f, x(t)) - (I - \lim \inf B_{mnk}(f, x(t)))| \leq |B_{mnk}(f, x(t)) - (f, x(t))| + |(f, x(t)) - (I - \lim \inf B_{mnk}(f, x(t)))| < 2r + \epsilon.\]

Thus

\[I - \lim \inf B_{mnk}(f, x(t)) \in I - LIM^r B_{mnk}(f, x(t)).\]

Similarly it can be shown that \(I - \lim sup x_{mnk}(t) \in I - LIM^r x_{mnk}(t).\)

**Corollary 3.6.** Let \(f\) be a continuous function defined on the closed interval \([0, 1]\). A triple sequence of Bernstein polynomials of rough sliding window measurable function of \((B_{mnk}(f, x(t)))\) of real numbers, if \(I - LIM^r B_{mnk}(f, x(t)) \neq \phi\), then
Proof. We have
\[ I - \liminf B_{mnk} (f, x(t)) = [I - \limsup B_{mnk} (f, x(t)) - 2r, I - \liminf B_{mnk} (f, x(t)) + 2r] . \]

Then the result follows from Theorem 3.5. \(\square\)

**Theorem 3.7.** Let \( f \) be a continuous function defined on the closed interval \([0,1]\).
A triple sequence of Bernstein polynomials of rough sliding window measurable function of \( (B_{mnk} (f, x(t))) \) of real numbers. Then the dimension \( \dim (I - \text{core} \{ B_{mnk} (f, x(t)) \}) \)
of the set
\[ I - \text{core} \{ B_{mnk} (f, x(t)) \} = r \iff I - \text{core} \{ f (x(t)) \} = I - \lim B_{mnk} (f, x(t)) \]

Proof. Consider
\[
\begin{align*}
\dim (I - \text{core} \{ B_{mnk} (f, x(t)) \}) & = r \\
\iff (I - \limsup B_{mnk} (f, x(t))) - (I - \liminf B_{mnk} (t)) & = r \\
\iff I - \text{core} \{ x_{mnk} (t) \} & = I - \liminf B_{mnk} (f, x(t)) \\
& = I - \limsup B_{mnk} (f, x(t)) \\
& = I - \lim B_{mnk} (f, x(t)) + r \\
& = I - \lim B_{mnk} (f, x(t)) .
\end{align*}
\]

Also it is easy to see that
(i) \( r > \text{diam} (I - \text{core} \{ B_{mnk} (f, x(t)) \}) \iff I - \text{core} \{ B_{mnk} (f, x(t)) \} \subset I - \lim B_{mnk} (f, x(t)) \),
(ii) \( r < \text{diam} (I - \text{core} \{ B_{mnk} (f, x(t)) \}) \iff I - \lim B_{mnk} (f, x(t)) \subset I - \text{core} \{ B_{mnk} (f, x(t)) \} . \)

\(\square\)

**Theorem 3.8.** Let \( f \) be a continuous function defined on the closed interval \([0,1]\).
A triple sequence of Bernstein polynomials of rough sliding window measurable function of \( (B_{mnk} (f, x(t))) \) of real numbers, if \( \bar{r} = \inf \{ r \geq 0 : I - \lim B_{mnk} (f, x(t)) \neq \emptyset \} \),
then \( \bar{r} = \text{radius} (I - \text{core} \{ B_{mnk} (f, x(t)) \}) \).

Proof. If the set \( I - \text{core} \{ B_{mnk} (f, x(t)) \} \) is singleton, then \( \text{radius} (I - \text{core} \{ B_{mnk} (f, x(t)) \}) \)
is 0 and the triple sequence of Bernstein polynomials of sliding window measurable function is \( I - \) convergent, i.e., \( I - \lim B_{mnk} (f, x(t)) \neq \emptyset \). Hence we get \( \bar{r} = \text{radius} (I - \text{core} \{ B_{mnk} (f, x(t)) \}) = 0 \).

Now assume that the set \( I - \text{core} \{ B_{mnk} (f, x(t)) \} \) is not a single tone. We can write \( I - \text{core} \{ B_{mnk} (f, x(t)) \} = [a, b] \) where \( a = I - \lim \inf B_{mnk} (f, x(t)) \) and \( b = I - \lim \sup B_{mnk} (f, x(t)) \).

Now let us assume that \( \bar{r} \neq \text{radius} (I - \text{core} \{ B_{mnk} (f, x(t)) \}) \).
If \( \bar{r} < \text{radius} (I - \text{core} \{ x_{mnk} (t) \}) \), then define \( \bar{\epsilon} = \frac{b-a}{3} - \bar{r} \). Now, by definition of \( \bar{r} \) implies that \( I - \lim B_{mnk} (f, x(t)) \neq \emptyset \), given \( \epsilon > 0 \exists l \in \mathbb{R} : A = \{(m,n,k) \in \mathbb{N} : |B_{mnk} (f, x(t)) - f (x(t))| \geq (\bar{r} + \bar{\epsilon}) + \epsilon \} \in I \). Since \( \bar{r} + \bar{\epsilon} < \frac{b-a}{3} \)
which is a contradiction of the definition of \( a \) and \( b \).

If \( \bar{r} > \text{radius} (I - \text{core} \{ B_{mnk} (f, x(t)) \}) \), then define \( \bar{\epsilon} = \frac{b-a}{3} \) and \( \bar{r} = \frac{b-a}{3} \). Then we get
\[
\begin{align*}
\inf B_{mnk} (f, x(t)) & \leq I - \lim B_{mnk} (f, x(t)) \leq \sup B_{mnk} (f, x(t)) \\
& \leq I - \text{core} \{ B_{mnk} (f, x(t)) \} \\
& \leq I - \text{core} \{ f (x(t)) \} \\
& \leq I - \lim B_{mnk} (f, x(t)) .
\end{align*}
\]

Hence
\[
\begin{align*}
\inf B_{mnk} (f, x(t)) & \leq I - \lim B_{mnk} (f, x(t)) \leq \sup B_{mnk} (f, x(t)) \\
& \leq I - \text{core} \{ B_{mnk} (f, x(t)) \} \\
& \leq I - \text{core} \{ f (x(t)) \} \\
& \leq I - \lim B_{mnk} (f, x(t)) .
\end{align*}
\]
$r - 2\bar{r}$. It is clear that $0 \leq r' \leq \bar{r}$ and by definitions of $a$ and $b$, the number $b - a \in I - LIM^r B_{mnk}(f, x(t))$. Then we get

$$\bar{r} \in \{r \geq 0 : I - LIM^r B_{mnk}(f, x(t)) \neq \emptyset\},$$

which contradicts the equality

$$\bar{r} = \inf\{r \geq 0 : I - LIM^r B_{mnk}(f, x(t)) \neq \emptyset\} \text{ as } r' < r.$$

\hfill $\Box$

**Corollary 3.9.** Let $f$ be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials of rough sliding window measurable function of $(B_{mnk}(f, x(t)))$ of real numbers, then $I - \text{core}\{B_{mnk}(f, x(t))\} = I - \text{LIM}^{2\bar{r}} B_{mnk}(f, x(t))$

**Proof.** It follows that Theorem 3.7 and Theorem 3.8. \hfill $\Box$

**REFERENCES**


