J. Indones. Math. Soc.
Vol. 25, No. 3 (2019), pp. 171–182.

NUMERICAL SOLUTION FOR A CLASS OF FRACTIONAL VARIATIONAL PROBLEM VIA SECOND ORDER B-SPLINE FUNCTION

Noratiqah Farhana binti Ismail[1], Chang Phang[2]

[1,2]**Department of Mathematics and Statistics, Universiti Tun Hussein Onn Malaysia,**[1]**noratiqah.farhana@gmail.com**
[2]**pchang@uthm.edu.my**

Abstract. In this paper, we solve a class of fractional variational problems (FVPs) by using operational matrix of fractional integration which derived from second order spline (B-spline) basis function. The fractional derivative is defined in the Caputo and Riemann-Liouville fractional integral operator. The B-spline function with unknown coefficients and B-spline operational matrix of integration are used to replace the fractional derivative which is in the performance index. Next, we applied the method of constrained extremum which involved a set of Lagrange multipliers. As a result, we get a system of algebraic equations which can be solve easily. Hence, the value for unknown coefficients of B-spline function is obtained as well as the solution for the FVPs. Finally, the illustrative examples shown the validity and applicability of this method to solve FVPs.

Key words and Phrases: fractional variational problems, B-spline function, operational matrix of integration, Riemann-Liouville fractional integration, Lagrange multiplier

2000 Mathematics Subject Classification: 49M05, 49M30
Received: 30-08-2017, revised: 14-09-2018, accepted: 18-09-2018.

Abstrak. Dalam paper ini, diselesaikan suatu kelas masalah variasional fraksional (FVPs) dengan menggunakan matriks operasional dari integral fraksional yang diperoleh dari basis fungsi spline orde-2 (B-spline). Derivatif fraksional didefinisikan dalam operator integral fraksional Caputo dan Riemann-Liouville. Fungsi B-spline dengan koefisien tak-tentu dan matriks operasional B-spline dari integral digunakan untuk menggantikan derivatif fraksional yang berada dalam *performance index*. Selanjutnya, diaplikasikan metode *constrained extremum* yang melibatkan sekumpulan pengali Lagrange. Sebagai hasilnya, diperoleh suatu sistem persamaan aljabar yang dapat diselesaikan dengan mudah. Dengan demikian, nilai dari koefisien-koefisien tak-tentu dari fungsi B-spline dapat ditentukan dan soluso dari FVPs dapat diperoleh. Akhirnya, beberapa contoh diberikan untuk menunjukkan kevalidan dan aplikasi dari metode dalam paper ini untuk menyelesaikan FVPs.

Kata kunci dan frasa: masalah variasional fraksional, fungsi B-spline, matriks operasional integral, fraksional integral Riemann-Liouville, pengali Lagrange

1. INTRODUCTION

Fractional calculus is a branch of mathematical analysis that studies the non-integer order of differentiation and integration [14]. It is an old mathematical topic with history as long as ordinary calculus because it has been developed since the year of 1695 due to some theories discovered by G.W. Leibniz and L Hopital [13]. In the past few decades, fractional calculus has attracted the attention of many mathematicians and physicists as they realized that the fractional calculus has various fascinating applications in solving real life problems which include numerous fields of science, physics, as well as biology. For instance, electrical circuit [3], complex dynamic modeling in biological tissues [16], economics and finance ([15],[25]), the model of viscoelastic ([12],[17],[18]), chemistry [19], and thermodynamics [24]. Hence, fractional calculus problem has become a significant topic to be studied by researchers, as example in [20, 21]. In most cases, there is no exact solution for fractional calculus problem. Therefore, the numerical methods of fractional calculus problem are needed to find the approximate solution. Among the methods are including spectral tau method [5], finite different method [28], and others.

On the other hand, the variational problems are the problems that involved finding the maximum or minimum value of a certain function. Once it contains fractional order derivative in the performance index, it will become fractional varitonal problems (FVPs). In this paper, we consider FVPs as follow:

$$\text{Min J} = \int_0^T L(t, x(t), D^\alpha x(t))dt, \tag{1}$$

subject to the boundary conditions

$$y^{(j)}(0) = a_j; \qquad y^{(j)}(T) = b_j; \qquad j = 0, 1, \ldots, n-1 \tag{2}$$

where $0 \leq t \leq T$, $n-1 < \alpha \leq n$, D^α is a fractional operator and α is a positive fractional number. We assume that $L(t, x(t), D^\alpha x(t))$ is continuous functions.

This FVPs was introduced by Riewe during 1996 where the non-integer order derivatives were used in non-conservative systems of mechanics [23]. In recent years, many mathematicians show their interest toward FVPs and solve it using various type of methods such as Euler Lagrange equation ([1],[2]), Jacobi polynomials [4], Legendre polynomials [6], Rayleigh-Ritz method [11] and many more. However, all these methods are based on smooth function (i.e. polynomials are smooth).

Hence, the main aim for this paper is to propose a new numerical scheme based on operational matrix via piecewise function, which is second order of B-spline function to solve the FVPs. The numerical scheme is through second order B-spline operational matrix of integration where the fractional derivative is in the sense of Riemann-Liouville. This B-spline operational matrix of integration was successfully used in [9] to solve the fractional integro-differential equations and fractional partial differential equations. Here, we extend it to solve the FVPs. The approach used in [9] for obtaining fractional integration of B-spline functions is via Laplace transform. This method is used as approximation of function due to it properties that have explicit formula, symmetric and also compact support of B-spline.

This present paper is arranged as follow. In section 2, some necessary preliminaries of fractional integration and derivatives are investigated as well as brief explanation for B-spline function. In section 3, the derivations for B-spline operational matrix of integration are discussed here. In section 4, the numerical scheme is presented to show how the operational matrix of fractional integration derive from B-spline function and be used to solve FVPs. In section 5, the proposed method is applied to solve numerical examples of FVPs. Finally, the conclusion for this paper is given in section 6.

2. PRELIMINARIES

In this section, we briefly describe some important definitions that will be using in this paper.

2.1. **Fractional integration and derivative.**

The Riemann-Liouville fractional integration of order $\alpha > 0$ of $f(t)$ is described as :

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \ t > 0 \tag{3}$$

where $\Gamma(\cdot)$ is gamma function. Below shows several important properties of Riemann Liouville integral:

$$I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t), \ \alpha > 0, \ \beta > 0 \tag{4}$$

$$I^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha} \tag{5}$$

Meanwhile the Caputo fractional derivative of order $\alpha > 0$ of $f(t)$ is described as :

$$^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-x)^{n-\alpha-1}f^n(x)dx,\ t > 0 \tag{6}$$

where $n-1 < \alpha \leq n$, $n \,\epsilon\, \mathbb{N}$ and $n = \lceil \alpha \rceil$.

2.2. **B-spline function.** B-spline is a piecewise polynomial function and possesses a high degree of smoothness at the knots which is known as the places where the pieces of polynomial are join together.

Definition 2.1. *The cardinal B-splines, $N_n(t)$ of order n are described via this convolution product:*

$$N_1(t) = \chi_{[0,1]}(t), \tag{7}$$

$$N_n(t) = N_1(t) * N_{n-1}(t),\ \ (n \geq 2) \tag{8}$$

Then, the two-scale relationship of order n for the B-spline function is :

$$N_n(t) = \sum_{k=0}^{n} 2^{1-n}\binom{n}{k}N_n(2t-k) \tag{9}$$

In this paper, we will using second order B-spline function, hence, $n = 2$, so from (9), we have

$$N_2(t) = \sum_{k=0}^{2}\frac{1}{2}\binom{2}{k}N_2(2t-k) \tag{10}$$

We use the second order B-spline function to solve FVP as describe in problem (1). We only focused on domain $[0,1]$. Therefore, we define second order B-spline scaling function, $\varphi_{2,j,k}(t)$ as

$$\varphi_{2,j,k}(t) = N_{2,j,k}(t)\chi_{[0,1]}(t). \tag{11}$$

where $N_{2,j,k} = N_2(2^jt-k)$ and j is involving in discretization step, which j must be $2^j \geq 3$. Hence, we have the B-spline function of order 2 as in [9].

$$\varphi_{2,j,k}(t) = \begin{cases} 2^jt-k & , \quad \frac{k}{2^j} \leq t < \frac{k+1}{2^j} \\ 2-(2^jt-k) & , \quad \frac{k+1}{2^j} \leq t < \frac{k+2}{2^j} \\ 0 & , \quad \text{otherwise} \end{cases} \tag{12}$$

where $k = 0, 1, \ldots, 2^j - 2$.

2.3. **Function Approximation.** For a fixed $j = J$, a function of $f(t)$ which is defined on interval $[0,1]$ can be approximated by the second order B-spline function as follow,

$$f(t) \approx \sum_{k=r}^{2j-1} c_k\varphi_{2,j,k(t)} = C^T\Phi_J(t) \tag{13}$$

where the vectors of coefficient, C and $\Phi_J(t)$ are shown below with $r = -1$:

$$C = [c_r, c_{r+1}, \dots, c_{2^j-1}]^T \tag{14}$$

$$\Phi_J(t) = [\varphi_{2,j,r}(t), \varphi_{2,j,r+1}(t), \dots, \varphi_{2,j,2^j-1}(t)]^T. \tag{15}$$

Then, c_k can be find using

$$c_k = \int_0^1 f(t)\tilde{\varphi}_{2,j,k}(t)dt \tag{16}$$

where $\tilde{\varphi}_{2,j,k}(t)$ is a vector of dual function of $\varphi_{2,j,k}(t)$ for $k = r, r+1, \dots, 2^j - 1$. Practically, we can also find the dual function, $\tilde{\varphi}_{2,j,k}(t)$ by following equation.

$$\tilde{\Phi}_J(t) = P^{-1}\Phi_J(t) \tag{17}$$

where $\tilde{\Phi}_J(t) = [\tilde{\varphi}_{2,j,r}(t), \tilde{\varphi}_{2,j,r+1}(t), \dots, \tilde{\varphi}_{2,j,2^j-1}(t)]^T$ and P is given by $\int_0^1 \Phi_J(t)\Phi_J^T(t)dt$.

3. B-SPLINE OPERATIONAL MATRIX OF FRACTIONAL INTEGRATION

In this section, we will briefly discuss the derivation of B-spline operational matrix of fractional integration. For this part, we follow the work in [9]. We use Laplace transform to calculate the fractional integration for second order B-spline function with $k = 0, 1, \dots, 2^j - 2$. In this process, we first change $\varphi_{n,j,k}(t)$ to unit step function, we get

$$\begin{aligned}\varphi_{2,j,k}(t) = 2^j\left(t - \frac{k}{2^j}\right)u\left(t - \frac{k}{2^j}\right) - 2^{j+1}\left(t - \frac{k+1}{2^j}\right)u\left(t - \frac{k+1}{2^j}\right)\\ + 2^j\left(t - \frac{k+2}{2^j}\right)u\left(t - \frac{k+2}{2^j}\right).\end{aligned} \tag{18}$$

From (18), we convert it into Laplace transform:

$$\mathcal{L}\{\varphi_{2,j,k}(t)\} = \frac{2^j}{s^2}\left(e^{-\frac{k}{2^j}s} - 2e^{-\frac{k+1}{2^j}s} + e^{-\frac{k+2}{2^j}s}\right)$$

By using definition from Riemann-Liouville fractional integration (3), we have the Laplace transform of $I^\alpha\varphi_{2,j,k}(t)$ as

$$\begin{aligned}\mathcal{L}\{I^\alpha\varphi_{2,j,k}(t)\} &= \frac{1}{\Gamma(\alpha)}\mathcal{L}\{t^{\alpha-1}\}\mathcal{L}\{\varphi_{2,j,k}(t)\}\\ &= 2^j\left(\frac{1}{s^{\alpha+2}}e^{-\frac{k}{2^j}s} - \frac{2}{s^{\alpha+2}}e^{-\frac{k+1}{2^j}s} + \frac{1}{s^{\alpha+2}}e^{-\frac{k+2}{2^j}s}\right)\end{aligned} \tag{19}$$

Then, take the inverse Laplace for both sides of equation (19), we will get

$$\begin{aligned} I^{\alpha}\varphi_{2,j,k}(t) = \frac{2^j}{\Gamma(\alpha+2)}\Bigg[\left(t-\frac{k}{2^j}\right)^{\alpha+1} u\left(t-\frac{k}{2^j}\right) - 2\left(t-\frac{k+1}{2^j}\right)^{\alpha+1} \\ u\left(t-\frac{k+1}{2^j}\right) + \left(t-\frac{k+2}{2^j}\right)^{\alpha+1} u\left(t-\frac{k+2}{2^j}\right)\Bigg] \end{aligned} \tag{20}$$

The unit step function in (20) can be easily rewrite in piecewise function. Hence, by letting $a = \dfrac{1}{2^{j\alpha}\Gamma(\alpha+2)}$, the fractional integration of second order B-spline function is achieved as follow:

$$I^{\alpha}\varphi_{2,j,k(t)} = a\begin{cases} 0 & , \quad t < \dfrac{k}{2^j} \\ (2^j t - k)^{\alpha+1} & , \quad \dfrac{k}{2^j} \leq t < \dfrac{k+1}{2^j} \\ (2^j t - k)^{\alpha+1} - 2(2^j t - k - 1)^{\alpha+1} & , \quad \dfrac{k+1}{2^j} \leq t < \dfrac{k+2}{2^j} \\ (2^j t - k)^{\alpha+1} - 2(2^j t - k - 1)^{\alpha+1} & \\ +(2^j t - k - 2)^{\alpha+1} & , \quad \dfrac{k+2}{2^j} \leq t \end{cases} \tag{21}$$

3.1. **Operational Matrices.** Here we will explain more on operational matrix of fractional integration for second order B-spline function. First, let $\tilde{\Phi}_J$ be the vector of dual function of Φ_J where order 2 and $r = -1$.

$$\tilde{\Phi}_J = [\tilde{\varphi}_{2,j,r}(t), \tilde{\varphi}_{2,j,r+1}(t), \ldots, \tilde{\varphi}_{2,j,2^j-1}(t)]^T$$

By using duality principle as in [10], we obtain

$$\int_0^1 \tilde{\Phi}_J(t)\Phi_J^T(t)dt = I \tag{22}$$

where I represent identity matrix.

Theorem 3.1. *If the matrix $P_2 = [p_{i,r}]$ is describe as :*

$$P_2 = \int_0^1 \Phi_J(t)\Phi_J^T(t)dt, \tag{23}$$

then we have :
(a) P_2 which is a $(2^j+1)(2^j+1)$ symmetric matrix :

$$P_2 = \frac{1}{6(2^j)}\begin{bmatrix} 2 & 1 & 0 & \ldots & 0 \\ 1 & 4 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & 4 & 1 \\ 0 & \ldots & 0 & 1 & 2 \end{bmatrix}$$

(b) The vector of dual function for $\tilde{\Phi}_J(t)$ *can be obtained by using linear combination of* $\Phi_J(t)$ [27] *which interpret as :*

$$\tilde{\Phi}_J(t) = (P_2^{-1}\Phi_J(t))$$

PROOF. The following formula is simple to be applied for proving the theorem [9] :

$$p_{i,k} = \int_0^1 \varphi_{n,j,i}(t)\varphi_{n,j,k}(t)dt,$$

where $j \geq n, i, k = r, \ldots, 2^j - 1$ and $r = 1 - n$.

Theorem 3.2. *Consider vector of second-order B-spline,* $\Phi_J(t)$ *and let* $\alpha > 0$*, then*

$$I^\alpha \Phi_J(t) = F^\alpha \Phi_J(t) \tag{24}$$

where F^α *is the B-spline operational matrix of fractional integration and is defined as follow:*

$$F^\alpha = E_2 P_2^{-1} \tag{25}$$

where element of E_2 *is given as* $e_{i,k}$*,*

$$e_{i,k} = \int_0^1 (I^\alpha \varphi_{2,j,k}(t))\varphi_{2,j,i}(t)dt \tag{26}$$

and $P_2 = \int_0^1 \Phi_J(t)\Phi_J^T(t)dt$

PROOF. Multiply both sides of equation (24) with $\Phi_J^T(t)$ and integrate from 0 to 1, we have

$$\int_0^1 F^\alpha \Phi_J(t)\Phi_J^T(t)dt = \int_0^1 I^\alpha \Phi_J(t)\Phi_J^T(t)dt$$

By using equation (23), we have

$$F^\alpha P_2 = \int_0^1 I^\alpha \Phi_J(t)\Phi_J^T(t)dt$$

Then, $F^\alpha = \int_0^1 I^\alpha \Phi_J^T(t)\Phi_J(t)dtP_2^{-1}$. Hence, let $E_2 = \int_0^1 I^\alpha \Phi_J(t)\Phi_J^T(t)dt$, we have

$$F^\alpha = E_2 P_2^{-1}$$

where each of element E_2 is $e_{i,k} = \int_0^1 (I^\alpha \varphi_{2,j,k}(t))\varphi_{2,j,i}(t)dt$.

4. THE NUMERICAL SCHEME

In this section, we describe in details on how to solve the FVPs in (1) and (2) by applying the operational matrix of fractional integration which is derived from second order B-spline function.

Firstly, we approximate $D^\alpha x(t)$ with B-spline function, $\Phi_J(t)$,

$$D^\alpha x(t) \simeq C^T \Phi_J(t) \tag{27}$$

where C is the unknown coefficients of B-spline function and both of C and $\Phi_J(t)$ are given in (14) and (15) respectively. After that, by doing integration for (27), we get

$$\begin{aligned} I^\alpha D^\alpha x(t) &\simeq C^T I^\alpha \Phi_J(t) \\ x(t) &\simeq C^T F^\alpha \Phi_J(t) + \sum_{i=0}^{n-1} x_i \frac{t^i}{i!} \end{aligned} \tag{28}$$

where F^α is the second order B-spline operational matrix of integration. From equation (27) and (28), we can approximate the performance index of FVPs in (1) as follow:

$$J = \int_0^1 L\left[t, C^T F^\alpha \Phi_J(t) + \sum_{i=0}^{n-1} x_i \frac{t^i}{i!}, C^T \Phi_J(t)\right] dt \tag{29}$$

The integral in (29) may be solved by any numerical integration such as Gauss quadrature. Alternative, we can solve the integral by using Maple as it is more convenient to be use and computer oriented. Then, let assume

$$J^* = J + \lambda(x(t) - b_j) \tag{30}$$

where λ is refer as unknown Lagrange multiplier and b_j is the initial boundary condition given in (2).

$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]^T \tag{31}$$

In this way, the J^* we get in (30) is expanded by unknown coefficient of B-spline function, $c_1, c_2, \ldots, c_n$ and also unknown Lagrange multiplier as in (31). So, by employing the necessary conditions of optimality (32) for the performance index and the boundary condition (2), we can reduce the problem of FVPs to a system of algebraic equations which can easily solved using any numerical methods.

$$\frac{\delta J^*}{\delta c_1} = 0, \frac{\delta J^*}{\delta c_2} = 0, \ldots, \frac{\delta J^*}{\delta c_n} = 0, \frac{\delta J^*}{\delta \lambda_n} = 0 \tag{32}$$

Once we get the value for coefficients, we can simply calculate the value for J using equation (29).

5. NUMERICAL EXAMPLE

In this section, we will provide two problems for FVPs and solve it by using our proposed scheme which is via second order B-spline function.

Example 1: We consider the problem based on example given in paper by Mohammed, O. H. [22].

$$J = \int_0^1 \left((D^\alpha x(t))^2 + D^\alpha x(t)\right) dt \tag{33}$$

subject to the boundary condition

$$x(0) = x_0,\ x(1) = \text{unspecified } 0 < \alpha \leq 1, \tag{34}$$

The exact solution for $\alpha = 1$ is

$$x(t) = \frac{-x^2}{4} \tag{35}$$

In paper [22], a scheme via hat basis function is applied to obtain the numerical solution for FVPs. Here, we compare our present result and result in [22] with the exact solution as shown in TABLE (1). From this comparison, we can see that our new numerical scheme gives more accurate result. Figure (1) shows the comparison of our numerical solution of equation (33) for $\alpha = 1, 0.5$ and 0.6 with the exact solution. The numerical solution is comparable with those in [22].

Table 1. Comparison of numerical solution of equation (33) using our present method and hat basis function with the exact solution for $\alpha = 1$.

t	Our method, $N = 9$	Hat basis function [22]	Exact solution
0.000	-0.0000000000	0	0.0000000000
0.125	-0.0040716105	-0.003909	-0.0039062500
0.250	-0.0156096076	-0.016	-0.0156250000
0.375	-0.0352224584	-0.035	-0.0351562500
0.500	-0.0625209798	-0.063	-0.0625000000
0.625	-0.0976837660	-0.098	-0.0976562500
0.750	-0.1406403837	-0.141	-0.1406250000
0.875	-0.1914149097	-0.191	-0.1914062500
1.000	-0.2500000000	-0.250	-0.2500000000

Example 2: We consider the following problem as in paper by Ezz-Eldien, S. S.[7].

$$\text{Min J} = \int_0^1 (D^{\frac{1}{2}}x(t) + D^{\frac{3}{2}}x(t) + x(t) - g(t))^2 dt \tag{36}$$

where $g(t) = t^3 + \dfrac{6}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{8}{\sqrt{\pi}}t^{\frac{3}{2}} + 1$ and $x(0) = 1$, $x'(0) = 0$, $x(1) = 2$, $x'(1) = 3$. The exact solution is $x(t) = t^3 + 1$.

In TABLE (2), the absolute error of $x(t)$ is listed with different value of N using our present method. Meanwhile, TABLE (3) shows the absolute error for the performance index by using the proposed method which is B-spline function.

6. CONCLUSION

In this paper, we have constructed a new numerical scheme based on second order B-spline function for a type of Fractional Variational Problems, FVPs. In our

FIGURE 1. Comparison of our numerical solutions for Example 1 when $\alpha = 1,\ 0.5$ and 0.6 with the exact solution

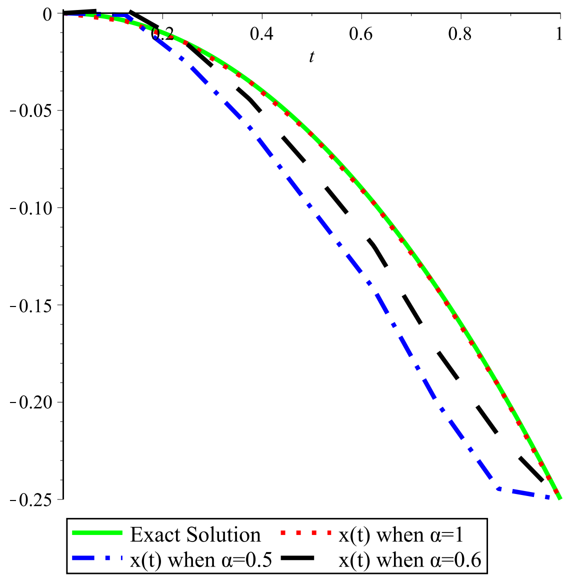

TABLE 2. Comparison of absolute errors for $x(t)$ with different value of N for Example 2.

t	$N = 5$	$N = 9$	$N = 17$
0.0	4.64396E-12	1.97000E-12	3.85518E-13
0.1	7.17682E-03	1.05002E-03	3.04161E-04
0.2	8.35365E-03	2.61284E-03	4.10309E-04
0.3	1.48077E-02	3.85609E-03	5.43256E-04
0.4	2.05389E-02	3.62480E-03	1.15415E-03
0.5	2.27014E-03	4.41441E-04	4.04515E-05
0.6	2.95628E-02	4.76539E-03	1.74005E-03
0.7	2.08555E-02	8.20811E-03	1.41740E-03
0.8	2.64014E-02	9.40113E-03	1.55819E-03
0.9	4.02007E-02	7.06316E-03	2.59205E-03
1.0	3.43952E-10	-5.70000E-10	-4.45600E-11

approach, we used operational matrix of fractional integration of B-spline function which derive via Laplace transform to reduce the FVPs into the system of algebraic equation. The two numerical examples show that the scheme able to give high accuracy result.

TABLE 3. The abolute error for the performance index by the present method for Example 2.

Method	$N = 5$	$N = 9$	$N = 17$
Present method	0.0051313	0.0003663	0.0000249
Exact	0		

REFERENCES

[1] Agrawal, O. P., "Generalized variational problems and EulerLagrange equations", *Comput. Math. Appl.*, **59** (2010), 1852 - 1864.

[2] Atanackovi, T.M., Konjik, S. and Pilipovi, S., "Variational problems with fractional derivatives: Euler Lagrange equations",*J. Phys. A*, **41** (2008), 095201.

[3] Debnath, L., "Recent applications of fractional calculus to science and engineering", *Int. J. Math. Math. Sci.* **54** (2003), 3413 - 3442.

[4] Dehghan, M., Hamedi, E.A. and Khosravian-Arab, H., "A numerical scheme for the solution of a class of fractional variational and optimal control problems using the modified Jacobi polynomials", *J. Vib. Control*, **22** (2016), 1547 - 1559.

[5] Doha, E.H., Bhrawy, A.H. and Ezz-Eldien, S.S., "An efficient Legendre spectral tau matrix formulation for solving fractional subdiffusion and reaction subdiffusion equations", *J. Comput. Dyn.*, **10** (2015), 021019.

[6] Ezz-Eldien, S. S., *et al.*, "New numerical approach for fractional variational problems using shifted Legendre orthonormal polynomials", *J. Optim. Theory Appl.*, (2016), 1 - 26.

[7] Ezz-Eldien, S.S., "New quadrature approach based on operational matrix for solving a class of fractional variational problems", *J. Comput. Phys.*, **317** (2016), 362 - 381.

[8] Goswami, J. C., Chan, A. K., and Chui, C. K., "On solving first-kind integral equations using wavelets on a bounded interval", *IEEE Trans. Antennas and Propagation*, **43** (1995), 614 - 622.

[9] Ismaeelpour, T., Hemmat, A. A., and Saeedi, H., "B-spline operational matrix of fractional integration", *Optik-International Journal for Light and Electron Optics*, **130** (2017), 291 - 305.

[10] Jafari, H., Tajadodi, H. and Baleanu, D., "A numerical approach for fractional order Riccati differential equation using B-spline operational matrix", *Fract. Calc. Appl. Anal.*, **18** (2015), 387 - 399.

[11] Khader, M.M. and Hendy, A.S., "A numerical technique for solving fractional variational problems", *Math. Methods Appl. Sci.*, **36** (2013), 1281 - 1289.

[12] Lewandowski, R., and Choryczewski, B., "Identification of the parameters of the KelvinVoigt and the Maxwell fractional models, used to modeling of viscoelastic dampers", *Computers & structures*, **88** (2010), 1 - 17.

[13] Loverro, A., "Fractional calculus: history, definitions and applications for the engineer", Rapport technique, Univeristy of Notre Dame: Department of Aerospace and Mechanical Engineering (2004), 1 - 28.

[14] Machado, J.T., Kiryakova, V. and Mainardi, F., "Recent history of fractional calculus", *Commun. Nonlinear Sci. Numer. Simul.*, **16** (2011), 1140 - 1153.

[15] Machado, J. T., and Mata, M. E., "Pseudo phase plane and fractional calculus modeling of western global economic downturn", *Commun. Nonlinear Sci. Numer. Simul*, **22** (2015), 396 - 406.

[16] Magin, R. L., "Fractional calculus models of complex dynamics in biological tissues", *Comput. Math. Appl.*, **59** (2010), 1586 - 1593.

[17] Mainardi, F., "An historical perspective on fractional calculus in linear viscoelasticity", *Fract. Calc. Appl. Anal.*, **15** (2012), 712 - 717.

[18] Meral, F. C., Royston, T. J., and Magin, R., "Fractional calculus in viscoelasticity: an experimental study", *Commun. Nonlinear Sci. Numer. Simul.*, **15** (2010), 939 - 945.

[19] Metzler, R., Schick, W., Kilian, H.G. and Nonnenmacher, T.F., "Relaxation in filled polymers: A fractional calculus approach." *J. Chem. Phys.*, **103** (1995), 7180 - 7186.
[20] Mishra, L. N., Agarwal, R. P. and Sen, M., "Solvability and asymptotic behavior for some nonlinear quadratic integral equation involving Erdlyi-Kober fractional integrals on the unbounded interval." *Progress in Frac. Diff. Appl.*, **2(3)** (2016), 153-168.
[21] Mishra, L. N., and Sen, M., "On the concept of existence and local attractivity of solutions for some quadratic Volterra integral equation of fractional order." *Appl. Math. Comput.*, **285** (2016), 174-183.
[22] Mohammed, O. H., "A direct method for solving fractional order variational problems by hat basis functions", *Ain Shams Eng. J.*, (2016).
[23] Riewe, F., "Nonconservative lagrangian and hamiltonian mechanics",*Phys. Rev. E*, **53** (1996), 1890 - 1899.
[24] Saxena, R. K., Mathai, A. M., and Haubold, H. J., "On generalized fractional kinetic equations", *Phys. A*, **344** (2004), 657 - 664.
[25] Scalas, E., Gorenflo, R., and Mainardi, F., "Fractional calculus and continuous-time finance." *Phys. A*, **284** (2000), 376 - 384.
[26] West, B. J., "Fractional calculus in bioengineering", *J. Stat. Phys.*, **126** (2007), 1285 - 1286.
[27] Wony, P., "Construction of dual B-spline functions", *J. Comput. Appl. Math.*, **260** (2014), 301 - 311.
[28] Zhuang, P., Liu, F., Turner, I. and Gu, Y., "Finite volume and finite element methods for solving a one-dimensional space-fractional Boussinesq equation", *Appl. Math. Model.*, **38** (2014), 3860 - 3870.