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FURTHER REMARKS ON n-DISTANCE-BALANCED GRAPHS

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Abstract. Throughout this paper, we present a new strong property of graph so-called nicely n-distance-balanced notably stronger than the concept of n-distance-balanced recently given by the authors. We also initially introduce a new graph invariant modifying Szeged index and is suitable to study n-distance-balanced graphs. Looking for the graphs extremal with respect to the modified Szeged index it turns out the n-distance-balanced graphs with odd integer n are the only bipartite graphs maximizing the modified Szeged index. This also disproves a conjecture proposed by Khalifeh et al. [Khalifeh, M.H., Yousefi-Azari, H., Ashrafi, A.R. and Wagner S.G., Some new results on distance-based graph invariants, *European J. Combin.* **30** (2009) 1149–1163]. Furthermore, we gather some facts concerning with the nicely n-distance-balanced graphs generated by some well-known graph products. To enlighten the reader some examples are provided. Moreover, a conjecture and a problem are presented within the results of this article.

Key words and Phrases: Nicely n-distance-balanced, Szeged index, lexicographic order, Cartesian and strong products.

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Abstrak. Dalam makalah ini, disajikan suatu *strong property* baru dari graf yang disebut dengan *nicely n-distance-balanced* yang 'lebih kuat' jika dibandingkan dengan konsep *n-distance-balanced.* Pertama, diperkenalkan invarian yang baru untuk graf yang diperoleh dari modifikasi indeks Szeged, dan digunakan untuk mengkaji graf yang *n-distance-balanced.* Dalam pencarian graf ekstremal terhadap indeks Szeged yang dimodifikasi, ditemukan bahwa graf yang *n-distance-balanced* dengan bilangan bulat ganjil n adalah satu-satunya graf bipartit yang memaksimalkan indeks Szeged yang dimodifikasi. Hal ini membantah dugaan yang diajukan oleh Khalifeh et al. [Khalifeh, M.H., Yousefi-Azari, H., Ashrafi, A.R. and Wagner S.G., Some new results on distance-based graph invariants, *European J. Combin.* **30** (2009) 1149–1163]. Selain itu, dikumpulkan juga beberapa fakta mengenai graf *nicely n-distance-balanced* yang dibangun oleh 'hasil kali' yang *well-known* dari graf. Kemudian, beberapa contoh disajikan untuk memperjelas konsep yang ada di dalam paper ini kepada pembaca. Lebih jauh, dugaan dan masalah terbuka terkait hasil-hasil dalam penelitian ini juga disajikan.

Kata kunci: *Nicely n-distance-balanced*, indeks Szeged, lexicografis, hasil kali Cartesian dan *strong product.*

1. INTRODUCTION

It is well-known that in graph theory, the distance-balanced graphs are considered as one of the important class of graphs. The significance of these graphs is evident from their applications in various areas, especially theoretical computer science (more precisely, balance in communication networks), and molecular analysis in chemical studies.

Let G be a finite, undirected and connected graph with diameter d, and let $V(G)$ and $E(G)$ indicate the vertex set and the edge set of G, respectively. For $a, b \in V(G)$, let $d(a,b) = d_G(a,b)$ denote the minimal path-length distance between a and b. For any pair of vertices a, b of G with $d(a,b) = n$, we denote

$$W^G_{a\underline{n}b} = \{x \in V(G) \mid d(x,a) < d(x,b)\},$$

and

$$\underset{a\ \underline{n}\ b}{W^G} = \{x \in V(G) \mid d(x,a) = d(x,b)\}.$$

Definition 1.1 (Faghani, Pourhadi and Kharazi [5]). *A graph G is called n-distance-balanced (n-DB) if for each $a, b \in V(G)$ with $d(a,b) = n$ we have $|W^G_{a\underline{n}b}| = |W^G_{b\underline{n}a}|$.*

For $n = 1$ the graph G is simply called distance-balanced (DB), which was initially introduced by Jerebic, Klavžar and Rall [9]. For recent results on DB graphs, see [8-15].

We say that G is a *nicely n-distance-balanced* (NnDB or nicely n-DB for short) whenever there exists a positive integer γ_G (or simply γ), such that for any arbitrary pair of vertices a, b of $V(G)$ with $d(a, b) = n$, we have

$$|W^G_{a\underline{n}b}| = |W^G_{b\underline{n}a}| = \gamma_G.$$

The concept of NnDB graphs appears quite naturally in the context of n-DB graphs. Besides, it is obvious to see that both N1DB and NDB graphs which were defined by Kutnar and Miklavič [13], are the same.

One of the objects of this paper is to explore purely metric properties of being NnDB using the graph invariant Szeged index, which was initially introduced by Gutman [6] in 1994, and since then investigated in several papers (see for example [1],[4]). Besides, we suggest a developed version of the Szeged index in the context of graphs, which is effective to proceed with the investigations on n-DB graphs. Furthermore, in Section 3, we discuss the NnDB graphs constructed by some well-known graph products with some illustrative examples.

2. MODIFIED SZEGED INDEX

The study on n-DB graphs motivates us to give a topological invariant as a generalized form of Szeged index. Using the notation $MSz_n(G)$, we define *modified Szeged index* of a graph G with respect to n is given as

$$MSz_n(G) = \sum_{\Gamma_n(G)} |W^G_{u\underline{n}v}| \cdot |W^G_{v\underline{n}u}|,$$

where $\Gamma_n(G)$ is the set of all pairs of vertices $u, v \in V(G)$ with $d(u, v) = n$. For an NnDB graph G with constant γ, we have

$$MSz_n(G) = \gamma^2 |\Gamma_n(G)|. \tag{1}$$

One can easily see that $MSz_1(G)$ coincides to the usual concept of Szeged index $Sz(G)$.

Assume that G is an arbitrary graph. By the simple inequality $2 \leq |W^G_{a\underline{n}b}| + |W^G_{b\underline{n}a}| \leq |V(G)|$ and applying the arithmetic geometric mean inequality, we get

$$|W^G_{a\underline{n}b}| \cdot |W^G_{b\underline{n}a}| \leq \left(\frac{|W^G_{a\underline{n}b}| + |W^G_{b\underline{n}a}|}{2} \right)^2 \leq \frac{|V(G)|^2}{4},$$

where $a, b \in \Gamma_n(G)$. Now, summing over all the vertices $a, b \in \Gamma_n(G)$, we derive that

$$MSz_n(G) \leq \frac{|V(G)|^2 |\Gamma_n(G)|}{4}. \tag{2}$$

Using (2) for $n = 1$ together with the fact that $\Gamma_1(G) = E(G)$, we have

$$Sz(G) = MSz_1(G) \le \frac{|V(G)|^2|E(G)|}{4}. \tag{3}$$

Now, we give a characterization of bipartite graphs extremal with respect to the modified Szeged index as follows.

Proposition 2.1. *Let G be a connected bipartite graph. Then, G is NnDB for some odd integer n if and only if $MSz_n(G) = \frac{|V(G)|^2|\Gamma_n(G)|}{4}$.*

Proof. As we already obtained that for any graph G, $MSz_n(G) \le \frac{|V(G)|^2|\Gamma_n(G)|}{4}$, let us consider

$$MSz_n(G) = \frac{|V(G)|^2|\Gamma_n(G)|}{4}.$$

Since G is bipartite and n is odd, then $\underset{a\ \underline{n}\ b}{W^G} = \emptyset$ and

$$|W^G_{a\underline{n}b}| = |W^G_{b\underline{n}a}| = \frac{|V(G)|}{2}$$

for $a, b \in \Gamma_n(G)$. Consequently, G is NnDB with constant $\gamma_G = \frac{|V(G)|}{2}$. For the converse, suppose G is NnDB. Then for any $a, b \in \Gamma_n(G)$, $|W^G_{a\underline{n}b}| = |W^G_{b\underline{n}a}|$. Since G is bipartite and n is odd, we also have $|W^G_{a\underline{n}b}| + |W^G_{b\underline{n}a}| = |V(G)|$. Hence, $Sz(G) = \frac{|V(G)|^2 \cdot |E(G)|}{4}$.

Now, considering the proof of Proposition 2.1, we have the following immediate consequence inspired by a result of Kutnar and Miklavič [13, Lemma 3.2].

Corollary 2.2. *Assume G is bipartite. Then,*

$$G \text{ is } n\text{-DB} \iff G \text{ is NnDB} \iff MSz_n(G) = \frac{|V(G)|^2|\Gamma_n(G)|}{4}, \tag{4}$$

for any odd integer n.

In Figure 1, we present a class of nonregular bipartite n-DB graphs with maximum modified Szeged index.

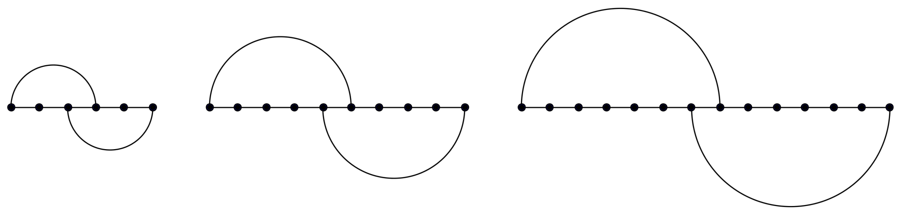

Figure 1. Nonregular bipartite graphs with 3-DB, 5-DB and 7-DB properties and maximum modified Szeged index.

Remark 2.3. *Comparing Lemma 3.2 in* [13] *and previous corollary, we observe that the former result is more advantageous than the latter. Moreover, for $n = 1$, relation* (4) *assures us that the conjecture proposed by Khalifeh, Yousefi-Azari, Ashrafi and Wagner* [10] *is not fulfilled by certain DB graphs. Indeed, for $n = 1$, Corollary 2.2 shows that any nonregular bipartite 1-DB graph, for instance Handa graph, cannot fulfill the Conjecture 2.4. Moreover, considering the counterexample presented by Aouchiche and Hansen (draw the Figure 2 in* [1]*), we see that our obtained result is satisfied by this example. Moreover, our result also answers the question asked by Chiniforooshan and Wu* [2] *related to the problem of existence and characterization of nonregular bipartite graphs with extreme Szeged index.*

Conjecture 2.4 ([10])**.** *For a connected graph G,*

$$Sz(G) = \frac{|V(G)|^2 \cdot |E(G)|}{4}, \tag{5}$$

if and only if G is bipartite and regular.

As a counterexample to this conjecture as above, Chiniforooshan and Wu [2] *presented a regular bipartite graph with 14 vertices not satisfying* (5)*, which is also a non-DB graph (draw Figure 1 in* [2]*).*

We know that if G is a connected graph without even cycles and contains k vertices, and $u, v \in \Gamma_n(G)$, then $|W^G_{u\underline{n}v}| + |W^G_{v\underline{n}u}| = k$ for an even integer n. Using this fact together with the next lemma, we find an equivalent formula for $MSz_2(G)$, where G is a connected graph without even cycles.

In the following we give a condition which will be used further on.

Remark 2.5. *For any $u, v \in \Gamma_2(G)$, the shortest path connecting x to v (or u) does not contain the shortest path connecting x to u (or v).*

Let us recall that $d_G(u)$ is the total distance of vertex u of G, that is,

$$d_G(u) = \sum_{v \in V(G)} d(u, v).$$

Then, we have the following results.

Lemma 2.6. *Let G be a connected graph without even cycles. Suppose that Remark 2.5 holds. Then,*

$$|W^G_{u\underline{2}v}| - |W^G_{v\underline{2}u}| = d_G(v) - d_G(u). \tag{6}$$

Proof. First, we notice that

$$W^G_{u\underline{2}v} \;\cup\; W^G_{v\underline{2}u} \;\cup\; {}_{u\,\underline{2}\,v}W^G = V(G).$$

Moreover,

$$d_G(u) = \sum_{x \in W^G_{u\underline{2}v}} d(x, u) + \sum_{x \in {}_{u\underline{2}v}W^G} d(x, u) + \sum_{x \in W^G_{v\underline{2}u}} d(x, u), \tag{7}$$

$$d_G(v) = \sum_{x \in W^G_{u\underline{2}v}} d(x,v) + \sum_{x \in \underset{u\underline{2}v}{W^G}} d(x,v) + \sum_{x \in W^G_{v\underline{2}u}} d(x,v). \tag{8}$$

Subtracting (7) from (8) and taking into account G has no even cycle, we get

$$\begin{aligned} d(x,v) &= d(x,u) + 1 \quad \text{if } x \in W^G_{u\underline{2}v}, \\ d(x,v) &= d(x,u) - 1 \quad \text{if } x \in W^G_{v\underline{2}u}, \\ d(x,v) &= d(x,u) \qquad\quad \text{if } x \in \underset{u\,\underline{2}\,v}{W^G}, \end{aligned}$$

we straightforwardly arrive at (6). We notice that Remark (2.5) implies that there is no $x \in W^G_{u\underline{2}v}$ with $d(x,v) = d(x,u) + 2$.

Remark 2.7. *Let us focus on Lemma 2.6. Suppose that w is a middle vertex for the pair $u, v \in \Gamma_2(G)$. Relation* (6) *shows for any connected graph G with no even cycle satisfying Remark* (2.5) *we have*

$$|W^G_{u\underline{2}v}| - |W^G_{v\underline{2}u}| = \Big(|W^G_{uw}| + |W^G_{wv}|\Big) - \Big(|W^G_{vw}| + |W^G_{wu}|\Big) \tag{9}$$

which automatically implies that any DB graph without even cycle and satisfying Remark (2.5) *is a 2-DB graph. Recall that for any arbitrary edge $ab \in E(G)$, we have*

$$|W^G_{ab}| - |W^G_{ba}| = d_G(b) - d_G(a).$$

Similarly to Lemma 2.6, we obtain the following result related to graphs without odd cycles.

Lemma 2.8. *If G is a connected bipartite graph and $u, v \in \Gamma_2(G)$, then*

$$|W^G_{u\underline{2}v}| - |W^G_{v\underline{2}u}| = \frac{1}{2}\Big(d_G(v) - d_G(u)\Big). \tag{10}$$

Proof. With the same reasoning of the proof of Lemma 2.6 and applying the following facts we easily find the conclusion.

$$d_G(u) = \sum_{x \in W^G_{u\underline{2}v}} d(x,u) + \sum_{x \in \underset{u\,\underline{2}\,v}{W^G}} d(x,u) + \sum_{x \in W^G_{v\underline{2}u}} d(x,u),$$

$$d_G(v) = \sum_{x \in W^G_{u\underline{2}v}} d(x,v) + \sum_{x \in \underset{u\,\underline{2}\,v}{W^G}} d(x,v) + \sum_{x \in W^G_{v\underline{2}u}} d(x,v).$$

$$\begin{aligned} d(x,v) &= d(x,u) + 2 \quad \text{if } x \in W^G_{u\underline{2}v}, \\ d(x,v) &= d(x,u) - 2 \quad \text{if } x \in W^G_{v\underline{2}u}, \\ d(x,v) &= d(x,u) \qquad\quad \text{if } x \in \underset{u\,\underline{2}\,v}{W^G}. \end{aligned}$$

Suppose that w_{uv} is the middle vertex of the path connecting $u, v \in \Gamma_2(G)$. Clearly, for any graph with no even cycle, this kind of vertex is unique for any pair of $u, v \in \Gamma_2(G)$.

Theorem 2.9. *Let G be a connected graph with k vertices.*

(i): *If G is a bipartite graph with no even cycle, that is, G is a tree, then*

$$MSz_2(G) = \frac{1}{4}\Bigg(\sum_{u,v\in\Gamma_2(G)} (k-|\Lambda_w|)^2 - \frac{1}{4}\sum_{u,v\in\Gamma_2(G)} (d_G(v)-d_G(u))^2 \Bigg);$$

(ii): *If G is a graph without even circles and satisfies (A), then*

$$MSz_2(G) = \frac{1}{4}\Bigg(\sum_{u,v\in\Gamma_2(G)} (k-|\Lambda_w|)^2 - \sum_{u,v\in\Gamma_2(G)} (d_G(v)-d_G(u))^2 \Bigg).$$

In cases (i) and (ii), Λ_w denotes the set of vertices including the vertex w_{uv} and the vertices connecting to u or v by a shortest path passing through w_{uv}.

PROOF. To prove (i), using Lemma 2.8 we get

$$d_G(v) - d_G(u) = 2(|W^G_{u\underline{2}v}| - |W^G_{v\underline{2}u}|). \tag{11}$$

Now, from the fact that $\underset{u\ \underline{2}\ v}{W^G} = \Lambda_w$, we have

$$|W^G_{u\underline{2}v}| + |W^G_{v\underline{2}u}| = k - |\Lambda_w|.$$

This together with the equality (11) implies that

$$|W^G_{u\underline{2}v}|^2 + |W^G_{v\underline{2}u}|^2 = \frac{1}{2}\Bigg((k-|\Lambda_w|)^2 + \frac{1}{4}(d_G(v)-d_G(u))^2 \Bigg).$$

Now,

$$|W^G_{u\underline{2}v}| \cdot |W^G_{v\underline{2}u}| = \frac{1}{4}\Bigg((k-|\Lambda_w|)^2 - \frac{1}{4}(d_G(v)-d_G(u))^2 \Bigg).$$

This shows that

$$\begin{aligned} MSz_2(G) &= \sum_{u,v\in\Gamma_2(G)} |W^G_{u\underline{2}v}| \cdot |W^G_{v\underline{2}u}| \\ &= \frac{1}{4}\Bigg(\sum_{u,v\in\Gamma_2(G)} (k-|\Lambda_w|)^2 - \frac{1}{4}\sum_{u,v\in\Gamma_2(G)} (d_G(v)-d_G(u))^2 \Bigg), \end{aligned}$$

which completes the proof of (i). To prove the second case we use Lemma 2.6:

$$d_G(v) - d_G(u) = |W^G_{u\underline{2}v}| - |W^G_{v\underline{2}u}|. \tag{12}$$

Now, from the fact that $\underset{u\ \underline{2}\ v}{W^G} = \Lambda_w$ we have

$$|W^G_{u\underline{2}v}| + |W^G_{v\underline{2}u}| = k - |\Lambda_w|.$$

This together with the equality (12) implies that

$$|W^G_{u\underline{2}v}|^2 + |W^G_{v\underline{2}u}|^2 = \frac{1}{2}\Bigg((k-|\Lambda_w|)^2 + (d_G(v)-d_G(u))^2 \Bigg).$$

Now,

$$|W^G_{u\underline{2}v}| \cdot |W^G_{v\underline{2}u}| = \frac{1}{4}\Bigg((k-|\Lambda_w|)^2 - (d_G(v)-d_G(u))^2 \Bigg).$$

This yields

$$\begin{aligned} MSz_2(G) &= \sum_{u,v\in\Gamma_2(G)} |W^G_{u\underline{2}v}| \cdot |W^G_{v\underline{2}u}| \\ &= \frac{1}{4}\Bigg(\sum_{u,v\in\Gamma_2(G)} (k-|\Lambda_w|)^2 - \sum_{u,v\in\Gamma_2(G)} (d_G(v)-d_G(u))^2 \Bigg), \end{aligned}$$

which completes the proof of (ii).

Here for illustration, we give an immediate consequence of case (i) of Theorem 2.9 for star graphs.

Example 2.10. *Using the notations as before, in the star graph S_k, the central vertex is w_{uv} for any pendant vertices u, v that are contained in $\Gamma_2(S_k)$. Moreover, $|\Lambda_w| = \deg(w_{uv})-1 = k-2$. Since $d_{S_k}(v) = d_{S_k}(u)$ for any pair of pendant vertices u, v, we get*

$$MSz_2(S_k) = \frac{1}{4} \sum_{u,v\in\Gamma_2(S_k)} (k-|\Lambda_w|)^2 = |\Gamma_2(S_k)| = \binom{k-1}{2}.$$

Example 2.11. *Now, let us compute the formula obtained from case (ii) of Theorem 2.9 for the graph depicted in Figure 2.*

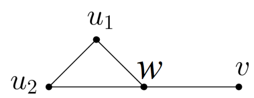

FIGURE 2. A graph G satisfying case (ii) of Theorem 2.9.

The graph shown in Figure 2 is non-cyclic and non-tree and also satisfies Remark 2.5. For $u_i, v \in \Gamma_2(G)$, $i = 1, 2$, the two shortest paths connecting any vertex to u_i and v are only overlapping to each other and not strictly contained in each other, that is, G satisfies Remark 2.5. Hence, for the pairs $(u_1, v), (u_2, v) \in \Gamma_2(G)$, we obtain

$$\begin{aligned} MSz_2(G) &= \frac{1}{4}\Bigg(\sum_{u,v\in\Gamma_2(G)} (k-|\Lambda_w|)^2 - \sum_{u,v\in\Gamma_2(G)} (d_G(v)-d_G(u))^2 \Bigg) \\ &= \frac{1}{4}\Big([(4-1)^2 + (4-1)^2] - [(5-4)^2 + (5-4)^2] \Big) = 4. \end{aligned}$$

On the other hand,

$$MSz_2(G) = \sum_{u,v\in\Gamma_2(G)} |W^G_{u\underline{2}v}| \cdot |W^G_{v\underline{2}u}| = (1\times 2) + (1\times 2) = 4.$$

Following Lemma 2.8 we see that for any bipartite 2-DB graph G, we have $|\{d_G(x) : x \in V(G)\}| = 2$. This enables us to formulate a relation for computing the Wiener index of 2-DB trees.

Proposition 2.12. *If T is a 2-DB tree with k vertices, then*

$$W(G) = \frac{k-1}{4}\Big(k + d_T(u) + d_T(v)\Big),$$

where u, v are two arbitrary adjacent vertices in G.

Proof. Following the formula of Dobrynin and Gutman [3] for the Wiener index of trees, we have

$$W(T) = \frac{1}{4}\Big[k(k-1) + \sum_{v\in V(T)} d_T(v)\deg(v)\Big].$$

Consider the disjoint sets E_1, E_2 of vertices of T. Since T is bipartite and 2-DB, we obtain

$$W(T) = \frac{1}{4}\Big[k(k-1) + \sum_{v\in V(E_1)} d_T(v)\deg(v) + \sum_{v\in V(E_2)} d_T(v)\deg(v)\Big].$$

Since $|\{d_G(x) : x \in V(G)\}| = 2$, all vertices in either E_1 or E_2 are labeled by the same total distance, that is, $d_T(u) = d_T(v)$ for $u, v \in V(E_i), i = 1, 2$, and so we have

$$\begin{aligned} W(T) &= \frac{1}{4}\Big[k(k-1) + d_T(u_0)\sum_{u\in V(E_1)} \deg(u) + d_T(v_0)\sum_{v\in V(E_2)} \deg(v)\Big] \\ &= \frac{1}{4}\Big[k(k-1) + d_T(u_0)|E(T)| + d_T(v_0)|E(T)|\Big] \\ &= \frac{k-1}{4}\Big(k + d_T(u_0) + d_T(v_0)\Big), \end{aligned}$$

where u_0 and v_0 can be chosen as two arbitrary adjacent vertices in T.

To end this section, we would like to present a conjecture with respect to the following graph invariants, the modified Szeged index with $n = 2$ and the Wiener index of trees. The upper and lower bounds in (13) are attainable for some trees, such as graphs S_3 and P_3 for upper and lower bounds, respectively; however, it seems that the lower and upper bounds can not be improved and replaced by a sharp inequality for the graphs with large order.

Conjecture 2.13. *Let T be an arbitrary tree, then we have*

$$MSz_2(T) \le W(T) \le 3MSz_2(T). \tag{13}$$

3. NnDB PROPERTY UNDER WELL-KNOWN PRODUCTS

In this section, we study the nicely n-distance-balanced graphs generated by the lexicographic, Cartesian and strong products. Note that such graph products, constructed from two graphs G and H, have the vertex set $V(G) \times V(H)$. Let (a, u) and (b, v) be two distinct vertices in $V(G) \times V(H)$. We recall that (a, u) and (b, v) are adjacent in the lexicographic product $G[H]$, if $ab \in E(G)$ or if $a = b$

and $uv \in E(H)$. They are also adjacent in the Cartesian product $G \square H$ if they coincide in one of the two coordinates and are adjacent in the other coordinate. In the strong product $G \boxtimes H$, (a, u) and (b, v) are adjacent if and only if $a = b$ and u, v are adjacent in H, or $u = v$ and a, b are adjacent in G, or a, b are adjacent in G and u, v are adjacent in H.

In the next results we use an equivalent definition for n-DB graphs of "distance partition":

$$D^n_{i,j}(u, v) = \{x \in V(G) \mid d(x, u) = i, \;\; d(x, v) = j\}, \quad \forall u, v \in \Gamma_n(G).$$

According to the definition of n-DB graph, we have

$$W^G_{u\underline{n}v} = \bigcup_{i=0}^{d-1} \bigcup_{j=1}^{n} D^n_{i,i+j}(u, v), \quad \forall u, v \in \Gamma_n(G),$$

where d is the diameter of G. Then G is n-DB if and only if

$$\sum_{i=0}^{d-1} \sum_{j=1}^{n} |D^n_{i,i+j}(u, v)| = \sum_{i=0}^{d-1} \sum_{j=1}^{n} |D^n_{i+j,i}(u, v)|, \quad \forall u, v \in \Gamma_n(G).$$

Furthermore, G is NnDB if and only if

$$\sum_{i=0}^{d-1} \sum_{j=1}^{n} |D^n_{i,i+j}(u, v)| = \sum_{i=0}^{d-1} \sum_{j=1}^{n} |D^n_{i+j,i}(u, v)| = \gamma_G, \quad \forall u, v \in \Gamma_n(G)$$

for some integer γ_G.

In order to prove the next result, we define G as a *non-adjacent* (k, l)*-regular graph* if any non-adjacent vertices $x, y \in V(G)$ have the same degree of k and $|D^r_{1,1}(x, y)| = l$, where $r = d(x, y)$. Here, we give the following results related to NnDB graphs deduced by using the lexicographic product for $n = 2, 3$.

Theorem 3.1. *Suppose that the graphs G and H are connected and G is a non-complete graph. Then $G[H]$ is N2DB if and only if G is N2DB and H is either the empty graph or the complete graph.*

Proof. Let us choose a pair of vertices from $V(G) \times V(H)$. Assume first the case where the vertices are $(a, x), (a, y)$ such that $d_{G[H]}((a, x), (a, y)) = 2$, so $d(x, y) \geq 2$. Recall that

$$d_{G[H]}((g, h), (\acute{g}, \acute{h})) = \begin{cases} d_G(g, \acute{g}) & \text{if } g \neq \acute{g}, \\ 1 & \text{if } g = \acute{g} \text{ and } h\acute{h} \in E(H), \\ 2 & \text{if } g = \acute{g} \text{ and } h\acute{h} \notin E(H). \end{cases}$$

Let (u, v) be a vertex from $V(G) \times V(H)$. We consider two cases on the pair (a, u): if $d(a, u) \geq 2$, then inspired by the definition of distance in the lexicographic product, we have $d_{G[H]}((a, x), (u, v)) = d_{G[H]}((a, y), (u, v)) = d(a, u)$, which implies

$(u,v) \in D^2_{i,i}((a,x),(a,y))$, where $i = d(a,u)$. Furthermore, if $d(a,u) \le 1$, then $d_{G[H]}((a,x),(u,v)) \le 2$ and $d_{G[H]}((a,y),(u,v)) \le 2$. Therefore, among the sets

$$D^2_{i,i+1}((a,x),(a,y)), D^2_{i+1,i}((a,x),(a,y)), D^2_{i,i+2}((a,x),(a,y)), D^2_{i+2,i}((a,x),(a,y)),$$

for $i \ge 0$ only the following sets may be nonempty:

$$\begin{aligned}
&D^2_{1,2}((a,x),(a,y)) = \{(a,v) \mid v \in N(x) \setminus N(y)\} \\
&\quad \Longrightarrow \quad |D^2_{1,2}((a,x),(a,y))| = \deg(x) + 1 - |D^r_{1,1}(x,y)|, \\
&D^2_{2,1}((a,x),(a,y)) = \{(a,v) \mid v \in N(y) \setminus N(x)\} \\
&\quad \Longrightarrow \quad |D^2_{2,1}((a,x),(a,y))| = \deg(y) + 1 - |D^r_{1,1}(x,y)|,
\end{aligned} \tag{14}$$

where $N(x)$ denotes all the neighbors of vertex x and $r = d(x,y) \ge 2$. On the other hand,

$$|D^2_{1,2}((a,x),(a,y))| = |D^2_{2,1}((a,x),(a,y))| = k - l + 1$$

if and only if H is a non-adjacent (k,l)-regular graph for some integers k,l. Moreover,

$$\begin{aligned}
\sum_{i=0}^{\operatorname{diam}(G[H])-1} \sum_{j=1}^{2} |D^2_{i,i+j}((a,x),(a,y))| &= \sum_{i=0}^{\operatorname{diam}(G[H])-1} \sum_{j=1}^{2} |D^2_{i+j,i}((a,x),(a,y))| \\
&= k - l + 1,
\end{aligned} \tag{15}$$

where $(a,x),(a,y) \in \Gamma_2(G[H])$. Now suppose that $(a,x),(b,y) \in V(G) \times V(H)$, where $a \ne b$ and $d((a,x),(b,y)) = 2$. Clearly, we have $d(a,b) = 2$. For this case, we have

$$\begin{aligned}
D^2_{2,1}((a,x),(b,y)) &= (D^2_{2,1}(a,b) \times V(H)) \cup \{(b,v) \mid v \in N(y)\} \\
&= [D^2_{2,1}(a,b) \times V(H)] \cup [\{b\} \times N(y)], \\
D^2_{1,2}((a,x),(b,y)) &= (D^2_{1,2}(a,b) \times V(H)) \cup \{(a,v) \mid v \in N(x)\} \\
&= [D^2_{1,2}(a,b) \times V(H)] \cup [\{a\} \times N(x)], \\
D^2_{2,3}((a,x),(b,y)) &= \{(u,v) \mid u \in S_2(a) \cap S_3(b)\} \\
&= [S_2(a) \cap S_3(b)] \times V(H) = D^2_{2,3}(a,b) \times V(H), \\
D^2_{3,2}((a,x),(b,y)) &= \{(u,v) \mid u \in S_3(a) \cap S_2(b)\} \\
&= [S_3(a) \cap S_2(b)] \times V(H) = D^2_{3,2}(a,b) \times V(H), \\
D^2_{3,1}((a,x),(b,y)) &= \{(u,v) \mid u \in S_1(b) \cap S_3(a)\} \\
&= [S_1(b) \cap S_3(a)] \times V(H) = D^2_{3,1}(a,b) \times V(H), \\
D^2_{1,3}((a,x),(b,y)) &= \{(u,v) \mid u \in S_3(b) \cap S_1(a)\} \\
&= [S_3(b) \cap S_1(a)] \times V(H) = D^2_{1,3}(a,b) \times V(H),
\end{aligned}$$

where $S_i(u)$ is the set of all vertices v with $d(u,v)=i$. In general, we observe that for $i \geq 2$

$$D^2_{i+1,i}((a,x),(b,y)) = D^2_{i+1,i}(a,b) \times V(H), \quad D^2_{i,i+1}((a,x),(b,y)) = D^2_{i,i+1}(a,b) \times V(H),$$
$$D^2_{i+2,i}((a,x),(b,y)) = D^2_{i+2,i}(a,b) \times V(H), \quad D^2_{i+2,i}((a,x),(b,y)) = D^2_{i+2,i}(a,b) \times V(H). \tag{16}$$

Therefore, using all relations in (16), we get

$$\begin{aligned} \sum_{i=0}^{\mathrm{diam}(G[H])-1} \sum_{j=1}^{2} |D^2_{i,i+j}((a,x),(b,y))| &= \sum_{i=0}^{\mathrm{diam}(G[H])-1} \sum_{j=1}^{2} |D^2_{i+j,i}((a,x),(b,y))| \\ \Leftrightarrow \quad \sum_{i=0}^{\mathrm{diam}(G)-1} \sum_{j=1}^{2} |D^2_{i,i+j}((a,b)| &= \sum_{i=0}^{\mathrm{diam}(G)-1} \sum_{j=1}^{2} |D^2_{i+j,i}((a,b)| = \gamma_G |V(H)| \end{aligned}$$

and it holds if and only if G is N2DB. Besides, the above equalities together with (15) shows that $G[H]$ is N2DB if and only if G is N2DB and H is either the empty graph or the complete graph. We remark that the previous case, that is, $(a,x),(a,y)$, is removed if we have $d(x,y) \leq 1$, and it means H should be as one of forms as above. More precisely, $G[H]$ is N2DB if and only if we have

$$\gamma_G |V(H)| = k - |D^r_{1,1}(x,y)| + 1,$$

but

$$\gamma_G |V(H)| + |D^r_{1,1}(x,y)| \geq |V(H)| \geq k+1,$$

and the equality happens if $k = |V(H)|-1$, that is, when H is the complete graph. Also, notice that both of the terms $\{a\} \times N(x)$ and $\{b\} \times N(y)$ have been ignored in the recent sums since they are included in $D^2_{0,2}(a,b) \times V(H)$ and $D^2_{2,0}(a,b) \times V(H)$, respectively, which is impossible.

By ignoring the second case in the previous proof, we easily conclude an immediate consequence as follows.

Theorem 3.2. *Suppose that G is a complete graph and H is an arbitrary connected graph. Then $G[H]$ is N2DB with $\gamma_{G[H]} = k-l+1$ if and only if H is a non-adjacent (k,l)-regular graph for some integers k,l.*

Proof. By considering $(a,x),(b,y)$ such that $d_{G[H]}((a,x),(b,y)) = 2$, if $a \neq b$, then $d_G(a,b) = 2$, which contradicts the fact that G is complete. Thus, we take the pair $(a,x),(a,y)$ such that $d_{G[H]}((a,x),(a,y)) = 2$. So, $xy \notin E(H)$. If we choose $(u,v) \in W_{(a,x),(a,y)}$, then $u=a$ and $v \in N(y) \setminus N(x)$. Therefore,

$$\begin{aligned} |W_{(a,x),(a,y)}| = |W_{(a,y),(a,x)}| &\iff |N(y) \setminus N(x)| = |N(x) \setminus N(y)| \\ &\iff \deg(x) - D^r_{1,1}(x,y) + |\{(a,x)\}| \\ &\qquad = \deg(y) - D^r_{1,1}(x,y) + |\{(a,b)\}|, \end{aligned}$$

where $r = d(x, y)$. This means that $G[H]$ is N2DB if and only if H is a non-adjacent (k, l)-regular graph. Moreover, $\gamma_{G[H]} = k - l + 1$ and the result follows.

Remark 3.3. *As observed in the recent result, the N2DB property is not invariant under the lexicographic product. Figure 3 shows $K_3[P_3]$ is NDB and N2DB. We notice that P_3 is a non-adjacent $(1, 1)$-regular graph and $\gamma_{K_3[P_3]} = 1 - 1 + 1 = 1$ (see also Theorem 3.2). Also in this figure, $P_3[K_3]$ is N2DB with $\gamma_{P_3[K_3]} = \gamma_{P_3}|V(K_3)| = 3$, whereas it is not NDB. This also illustrates that the lexicographic product is not commutative (see also Hammack, Imrich, and Klavžar* [7]*).*

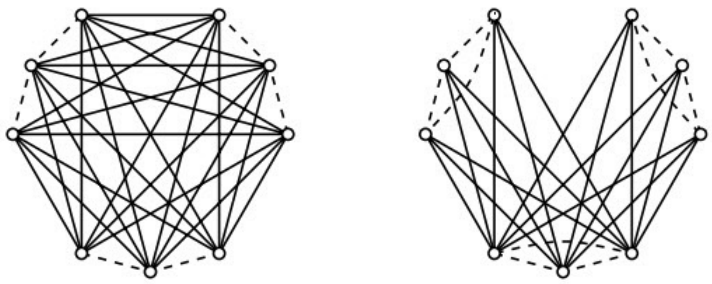

FIGURE 3. Lexicographic products $K_3[P_3]$ and $P_3[K_3]$.

Theorem 3.4. *Suppose G and H are connected graphs. Then $G[H]$ is N3DB if and only if G is N3DB and H is a k-regular graph.*

PROOF. Assume that G is N3DB and H is a regular graph. Following the proof of Theorem 3.1, if we choose $(a, x), (a, y) \in V(G) \times V(H)$ so that $d_{G[H]}((a, x), (a, y)) = 3$, then it contradicts the definition of distance in lexicographic product. Suppose then that $(a, x), (b, y) \in V(G) \times V(H)$, where $a \neq b$ and $d((a, x), (b, y)) = 3$. Clearly, we have $d(a, b) = 3$. For this case we have

$$\begin{aligned}
&D^3_{2,1}((a,x),(b,y)) = D^3_{2,1}(a,b) \times V(H), \quad D^3_{1,2}((a,x),(b,y)) = D^3_{1,2}(a,b) \times V(H),\\
&D^3_{2,3}((a,x),(b,y)) = [D^3_{2,3}(a,b) \times V(H)] \cup [\{a\} \times (V(H) - N(x)) \setminus (a,x)],\\
&D^3_{3,2}((a,x),(b,y)) = [D^3_{3,2}(a,b) \times V(H)] \cup [\{b\} \times (V(H) - N(y)) \setminus (b,y)],\\
&D^3_{3,1}((a,x),(b,y)) = [D^3_{3,1}(a,b) \times V(H)] \cup [\{b\} \times N(y)],\\
&D^3_{1,3}((a,x),(b,y)) = [D^3_{1,3}(a,b) \times V(H)] \cup [\{a\} \times N(x)],\\
&D^3_{4,1}((a,x),(b,y)) = D^3_{4,1}(a,b) \times V(H), \quad D^3_{1,4}((a,x),(b,y)) = D^3_{1,4}(a,b) \times V(H),\\
&D^3_{2,5}((a,x),(b,y)) = [D^3_{2,5}(a,b) \times V(H)] \cup [\{a\} \times (V(H) - N(x))],\\
&D^3_{5,2}((a,x),(b,y)) = [D^3_{5,2}(a,b) \times V(H)] \cup [\{b\} \times (V(H) - N(y))].
\end{aligned} \tag{17}$$

About the other such sets, we observe that for $i \geq 4$

$$
\begin{aligned}
D^3_{i-1,i}((a,x),(b,y)) &= D^3_{i-1,i}(a,b) \times V(H),\\
D^3_{i,i-1}((a,x),(b,y)) &= D^3_{i,i-1}(a,b) \times V(H),\\
D^3_{i-2,i}((a,x),(b,y)) &= D^3_{i-2,i}(a,b) \times V(H),\\
D^3_{i-2,i}((a,x),(b,y)) &= D^3_{i-2,i}(a,b) \times V(H),\\
D^3_{i-1,i+2}((a,x),(b,y)) &= D^3_{i-1,i+2}(a,b) \times V(H),\\
D^3_{i+2,i-1}((a,x),(b,y)) &= D^3_{i+2,i-1}(a,b) \times V(H).
\end{aligned} \tag{18}
$$

Consequently, we get the equality

$$
\sum_{i=0}^{\mathrm{diam}(G[H])-1} \sum_{j=1}^{3} |D^3_{i,i+j}((a,x),(b,y))| = \sum_{i=0}^{\mathrm{diam}(G[H])-1} \sum_{j=1}^{3} |D^3_{i+j,i}((a,x),(b,y))|, \tag{19}
$$

which is equal to

$$
\sum_{i=0}^{\mathrm{diam}(G)-1} \sum_{j=1}^{3} |D^3_{i,i+j}(a,b)| = \sum_{i=0}^{\mathrm{diam}(G)-1} \sum_{j=1}^{3} |D^3_{i+j,i}(a,b)| = \gamma_G |V(H)|, \tag{20}
$$

and it happens when the hypothesis holds. Therefore, $G[H]$ is N3DB with $\gamma_{G[H]} = \gamma_G |V(H)|$. To give more details about not considering the second terms in some unions of (17), we notice that these sets are repeated and they are included in $D^3_{i,j}(a,b) \times V(H)$ for some appropriate i, j. To prove the converse, using (19) and (20), one can easily show that the above equalities hold if N3DB and k-regularity properties are achieved by G and H, respectively.

Example 3.5. *In Figure 4, the top resulting graph, describes the N3DB graph $P_4[C_5]$ with constant $\gamma_{P_4[C_5]} = \gamma_G |V(H)| = 2 \cdot 5 = 10$. However, this graph is neither DB nor 2-DB. The bottom resulting graph shows the lexicographic product graph $C_5[P_4]$, which is not n-DB graph for any integer n.*

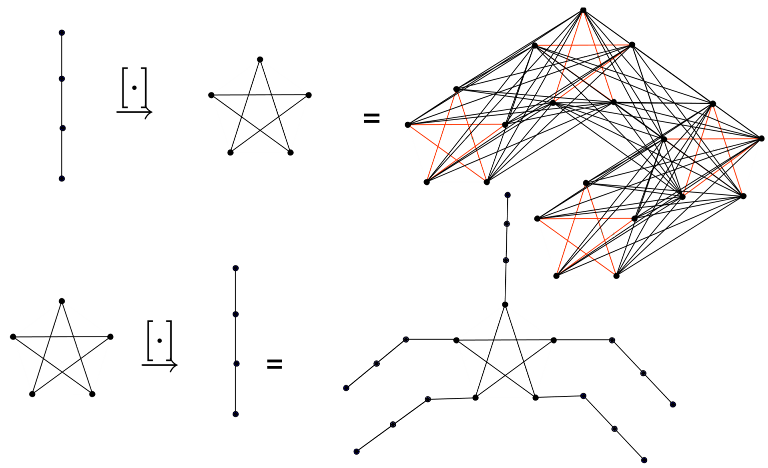

FIGURE 4. Graphs $P_4[C_5]$, $C_5[P_4]$ derived by the lexicographic product $[\cdot]$.

To investigate the nicely 2-distance-balanced property of the Cartesian product of graphs, we observe that using this kind of graph product with N2DB graphs cannot be generated a non-trivial N2DB graph. To illustrate this fact, consider the cycle C_4, which is a regular graph with the N2DB property. The Cartesian product $C_4 \times C_4$ is not even a 2DB graph. Indeed, in Figure 5, for the vertices a, b with $d(a, b) = 2$, we have

$$|W_{a\underline{2}b}^{C_4\times C_4}| = 5, \qquad |W_{b\underline{2}a}^{C_4\times C_4}| = 4$$

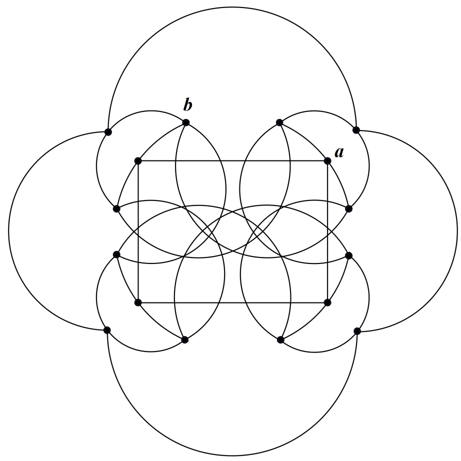

FIGURE 5. Graph $C_4 \times C_4$.

Similarly, we can construct a non-NnDB graph generated from the Cartesian product of two NnDB graph for all $n \geq 3$. The graph $P_{n+1} \times C_{2n}$ is not n-DB since it has at least two vertices c, d with $d(c,d) = n$ satisfying

$$|W_{c\underline{n}d}^{P_{n+1}\times C_{2n}}| < |W_{d\underline{n}c}^{P_{n+1}\times C_{2n}}|.$$

To be clarified, see Figure 6.

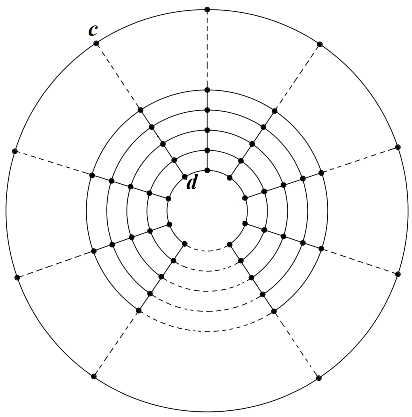

FIGURE 6. Graph $P_{n+1} \times C_{2n}$.

To discuss about NnDB graphs generated by the strong product, we have the same fact. For instance, considering $G = P_{n+1} \boxtimes C_{2n}$, we observe that this graph is not NnDB, whereas both P_{n+1} and C_{2n} are NnDB. Moreover, the graph G does not have the n-DB property for $n > 2$. See the following graphs as examples.

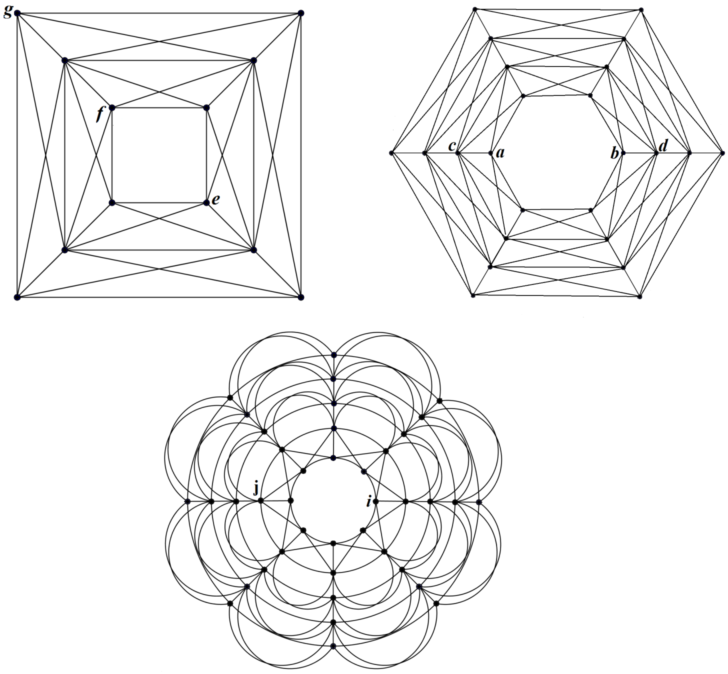

FIGURE 7. The non-N2DB Graph $P_3 \boxtimes C_4$, non-N3DB Graph $P_4 \boxtimes C_6$ and the non-N4DB Graph $P_5 \boxtimes C_8$.

In Figure 7, we have

$$|W_{e\underline{2}f}^{P_3\boxtimes C_4}| = |W_{f\underline{2}e}^{P_3\boxtimes C_4}| = 2, \quad |W_{g\underline{2}f}^{P_3\boxtimes C_4}| = |W_{f\underline{2}g}^{P_3\boxtimes C_4}| = 3, \quad |W_{g\underline{2}e}^{P_3\boxtimes C_4}| = |W_{e\underline{2}g}^{P_3\boxtimes C_4}| = 4,$$

and

$$|W_{a\underline{3}b}^{P_4\boxtimes C_6}| = |W_{b\underline{3}a}^{P_4\boxtimes C_6}| = 7, \quad |W_{c\underline{3}d}^{P_4\boxtimes C_6}| = |W_{d\underline{3}c}^{P_4\boxtimes C_6}| = 10,$$
$$\text{but } |W_{c\underline{3}b}^{P_4\boxtimes C_6}| = 14, \quad |W_{b\underline{3}c}^{P_4\boxtimes C_6}| = 7,$$

and

$$|W_{i\underline{4}j}^{P_5\boxtimes C_8}| = 10, \quad |W_{j\underline{4}i}^{P_5\boxtimes C_8}| = 21.$$

Based on the discussion as above the following problem arises naturally.

Problem 3.6. *Find a class of non-trivial NnDB graphs generated by either Cartesian or strong products. Find necessary (or sufficient) conditions for this class of graphs.*

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