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ON ASYMPTOTICALLY f-STATISTICAL EQUIVALENT SEQUENCES

Şükran Konca[1] and Mehmet Küçükaslan[2]

[1]**Department of Mathematics, Bitlis Eren University, 13000, Turkey, skonca@beu.edu.tr**

[2]**Department of Mathematics, Mersin University, 33343, Turkey, mkucukaslan@mersin.edu.tr**

Abstract. In this work, via modulus functions, we have obtained a generalization of statistical convergence of asymptotically equivalent sequences, a new non-matrix convergence method, which is intermediate between the ordinary convergence and the statistical convergence. We also have examined some inclusion relations related to this concept. Addition to all these results, in the last part of the paper, we obtain very nice results for nonnegative real numbers with respect to the partial order on the set of real numbers.

Key words and Phrases: Statistical convergence; strong Cesaro summability; sequence space; modulus function; asymptotically equivalent sequences

Abstrak. Dalam penelitian ini, dengan menggunakan fungsi modulus, diperoleh perumuman kekonvergenan statistik dari barisan yang ekivalen secara asimtotis, suatu metode kekonvergenan non-matriks baru, yang merupakan pertengahan antara kekonvergenan biasa dan kekonvergenan statistik. Selain itu, diperoleh juga beberapa relasi inklusi yang terkait dengan konsep-konsep di atas. Sebagai tambahan hasil, pada bagian akhir artikel, diperoleh hasil yang sangat menarik terkait dengan bilangan riil tak-negatif terhadap urutan parsial pada himpunan bilangan riil.

Kata kunci: Kekonvergenan statistik; Cesaro *summability* kuat; ruang barisan; fungsi modulus; barisan ekivalen secara asimtotik.

1. INTRODUCTION

In 1993, Marouf [8] presented definitions for asymptotically equivalent and asymptotic regular matrices. In 2003, Patterson [11] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and

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natural regularity for nonnegative summability matrices. In 2006, Patterson and Savaş extended the definitions presented in [12] to lacunary sequences.

The concept of statistical convergence was defined by Steinhaus [15] and Fast [6] and later reintroduced by Schoenberg [14] independently. Although statistical convergence was introduced over nearly the last sixty years, it has become an active area of research in recent years.

The notion of a modulus function was introduced by Nakano [9]. Ruckle [13] and Maddox [7] introduced and discussed some properties of sequence spaces defined by using a modulus function. In 2014, Aizpuru et al. [1] defined a new concept of density with the help of an unbounded modulus function and, as a consequence, they obtained a new concept of nonmatrix convergence, namely, f-statistical convergence, which is intermediate between the ordinary convergence and the statistical convergence and agrees with the statistical convergence when the modulus function is the identity mapping. Quite recently, Bhardwaj and Dhawan [2], and Bhardwaj et al. [3], have introduced and studied the concepts of f-statistical convergence of order α and f-statistical boundedness, respectively, by using the approach of Aizpuru et al. [1] (see also [4] and [5]).

By using modulus functions, we have defined a generalization of statistical convergence of asymptotically equivalent sequences and obtained some inclusion relations related to this concept.

2. DEFINITIONS AND PRELIMINARIES

In this section, we present some definitions and notations needed throughout the paper. By $\mathbb{N}$ and $\mathbb{R}$, we mean the set of all natural and real numbers, respectively. For brevity, we also mean $\lim_{k\to\infty} x_k$ by the notation $\lim_k x_k$.

Definition 2.1. [10] A number sequence $x = (x_k)$ is said to be statistically convergent to the number ℓ if for each $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}$ has natural density zero, where the natural density of a subset $K \subset \mathbb{N}$ is defined by $d(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|$, where $|\{k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n. Obviously, we have $d(K) = 0$ provided that K is a finite set of positive integers. If a sequence is statistically convergent to ℓ, then we write it as S-$\lim_k x_k = \ell$ or $x_k \to \ell(S)$. The set of all statistically convergent sequences is denoted by S.

Definition 2.2. [8] Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if $\lim_k \frac{x_k}{y_k} = 1$ (denoted by $x \sim y$). If the limit is ℓ, then it will be denoted by $x \overset{\ell}{\sim} y$.

Definition 2.3. [11] Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple ℓ provided that for every $\varepsilon > 0$ $\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - \ell \right| \geq \varepsilon \right\} \right| = 0$ (denoted by $x \overset{S_\ell}{\sim} y$).

Recall that a modulus function f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(1) $f(x)=0 \Leftrightarrow x=0$,
(2) $f(x+y) \leq f(x)+f(y)$ for $x \geq 0$, $y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0.

Definition 2.4. [1] Let f be an unbounded modulus function. The f-density of a set $K \subset \mathbb{N}$ is defined by

$$d^f(K)=\lim_{n\to\infty}\frac{f(|\{k\leq n: k\in A\}|)}{f(n)}$$

in case this limit exists. Clearly, finite sets have zero f-density but in difference of the natural density, $d^f(\mathbb{N}-K)=1-d^f(K)$ does not hold, in general. But if $d^f(K)=0$, then $d^f(\mathbb{N}-K)=1$.

For example, if we take $f(x)=log(x+1)$ and $K=\{2n: n\in\mathbb{N}\}$, then $d^f(K)=d^f(\mathbb{N}-K)=1$. For any unbounded modulus f and $K\subset\mathbb{N}$, $d^f(K)=0$ implies that $d(K)=0$. But converse need not be true in the sense that a set having zero natural density may have non-zero f-density with respect to some unbounded modulus f. For example, if we take $f(x)=log(x+1)$ and $K=\{1,4,9,...\}$, then $d(K)=0$ but $d^f(K)=1/2$. However, $d(K)=0$ implies $d^f(K)=0$ is always true in case of any finite set $K\subset\mathbb{N}$, irrespective of the choice of unbounded modulus f (see, [2]).

Definition 2.5. [1] Let f be an unbounded modulus function. A number sequence $x=(x_k)$ is said to be f-statistically convergent to ℓ or S^f-convergent to ℓ, if for each $\varepsilon>0$

$$d^f(\{k\in\mathbb{N}: |x_k-\ell|\geq\varepsilon\})=0,$$

that is,

$$\lim_{n\to\infty}\frac{f(|\{k\leq n: |x_k-\ell|\geq\varepsilon\}|)}{f(n)}=0,$$

and one writes it as $S^f-\lim_k x_k=l$ or $x_k\to l\left(S^f\right)$. The set of all f-statistically convergent sequences is denoted by S^f.

Lemma 2.6. [7] Let $f:[0,\infty)\to[0,\infty)$ be a modulus. Then there is a finite $\lim_{t\to\infty}\frac{f(t)}{t}$ and equality

$$\lim_{t\to\infty}\frac{f(t)}{t}=\inf\left\{\frac{f(t)}{t}: t\in(0,\infty)\right\}$$

holds.

The well-known space $w(f)$ of strongly Cesaro summable sequences is defined as [7]

$$w(f):=\left\{x=(x_k): \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(|x_k-\ell|)=0,\ \text{for some } \ell\in\mathbb{R}\right\}.$$

3. MAIN RESULTS

Definition 3.1. Let f be an unbounded modulus function. Two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically f-statistical equivalent of multiple ℓ provided that for every $\varepsilon>0$

$$d^f\left(\left\{k\in\mathbb{N}:\left|\frac{x_k}{y_k}-\ell\right|\geq\varepsilon\right\}\right)=0,$$

that is,

$$\lim_{n\to\infty}\frac{1}{f(n)}f\left(\left|\left\{k\leq n:\left|\frac{x_k}{y_k}-\ell\right|\geq\varepsilon\right\}\right|\right)=0$$

(denoted by $x\overset{S_\ell^f}{\sim}y$) and simply asymptotically f-statistical equivalent if $\ell=1$. Furthermore, let S_ℓ^f denote the set of x and y such that $x\overset{S_\ell^f}{\sim}y$.

Definition 3.2. Two number sequences $x=(x_k)$ and $y=(y_k)$ are said to be strong Cesaro asymptotically equivalent of multiple ℓ with respect to a modulus function f provided that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{x_k}{y_k}-\ell\right|\right)=0$$

(denoted by $x\overset{w^\ell(f)}{\sim}y$) and simply strong Cesaro asymptotically equivalent if $\ell=1$. In addition, let $w^\ell(f)$ denote the set of x and y such that $x\overset{w^\ell(f)}{\sim}y$.

Theorem 3.3. Let f be any unbounded modulus function for which $\lim_{t\to\infty}\frac{f(t)}{t}>0$ and c be a positive constant such that $f(xy)\geq cf(x)f(y)$ for all $x\geq0$, $y\geq0$. If $x\overset{w^\ell(f)}{\sim}y$, then $x\overset{S_\ell^f}{\sim}y$.

Proof. Let f be any unbounded modulus function for which $\lim_{t\to\infty}\frac{f(t)}{t}>0$ and c be a positive constant such that $f(xy)\geq cf(x)f(y)$ for all $x\geq0$, $y\geq0$. For $x\overset{w^\ell(f)}{\sim}y$ and $\varepsilon\in(0,\infty)$, by the definition of a modulus function (2) and (3) we have

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{x_k}{y_k}-\ell\right|\right) &\geq\frac{1}{n}f\left(\sum_{k=1}^{n}\left|\frac{x_k}{y_k}-\ell\right|\right)\\
&\geq\frac{1}{n}f\left(\sum_{\substack{k=1\\ |x_k/y_k-\ell|\geq\varepsilon}}^{n}\left|\frac{x_k}{y_k}-\ell\right|\right)\\
&\geq\frac{1}{n}f\left(\left|\left\{k\leq n:\left|\frac{x_k}{y_k}-\ell\right|\geq\varepsilon\right\}\right|.\varepsilon\right)\\
&\geq\frac{c}{n}f\left(\left|\left\{k\leq n:\left|\frac{x_k}{y_k}-\ell\right|\geq\varepsilon\right\}\right|\right).f(\varepsilon)\\
&=\frac{1}{f(n)}f\left(\left|\left\{k\leq n:\left|\frac{x_k}{y_k}-\ell\right|\geq\varepsilon\right\}\right|\right).\frac{f(n)}{n}.c.f(\varepsilon)
\end{aligned}$$

from where it follows that $x \overset{S_\ell^f}{\sim} y$.

Theorem 3.4. If $x \overset{S_\ell^{(f)}}{\sim} y$, then $x \overset{S_\ell}{\sim} y$.

PROOF. Suppose that $x \overset{S_\ell^{(f)}}{\sim} y$. Then by the definition of the limit and the fact that f being modulus is subadditive, for every $p \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$\begin{aligned} f\left(\left|\left\{k \leq n : \left|\frac{x_k}{y_k} - \ell\right| \geq \varepsilon\right\}\right|\right) &\leq \frac{1}{p} f(n) \\ &\leq \frac{1}{p}.p.f\left(\frac{n}{p}\right) \\ &= f\left(\frac{n}{p}\right) \end{aligned}$$

and since f is increasing, we have

$$\frac{1}{n}\left|\left\{k \leq n : \left|\frac{x_k}{y_k} - \ell\right| \geq \varepsilon\right\}\right| \leq \frac{1}{p}.$$

Hence $x \overset{S_\ell}{\sim} y$.

Now, we give a corollary as a result of Theorem 3.3 and Theorem 3.4.

Corollary 3.5. Let f be an unbounded modulus function such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and c be a positive constant such that $f(xy) \geq cf(x)f(y)$ for all $x \geq 0$, $y \geq 0$. If $x \overset{w^\ell(f)}{\sim} y$, then $x \overset{S_\ell}{\sim} y$.

Theorem 3.6. If $x \in l_\infty$ (the space of all bounded real-valued sequences) and $x \overset{S_\ell^{(f)}}{\sim} y$, then $x \overset{w^\ell(f)}{\sim} y$ for any unbounded modulus f.

PROOF. Suppose that $x = (x_k) \in l_\infty$ and $x \overset{S_\ell^{(f)}}{\sim} y$. Then we can assume that there exists a $M > 0$ such that

$$\left|\frac{x_k}{y_k} - \ell\right| \leq M$$

for all k. Given $\varepsilon > 0$

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k} - \ell\right|\right) &= \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{x_k}{y_k} - \ell\right| \geq \varepsilon}}^{n} f\left(\left|\frac{x_k}{y_k} - \ell\right|\right) + \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{x_k}{y_k} - \ell\right| < \varepsilon}}^{n} f\left(\left|\frac{x_k}{y_k} - \ell\right|\right) \\ &\leq \frac{1}{n}\left|\left\{k \leq n : \left|\frac{x_k}{y_k} - \ell\right| \geq \varepsilon\right\}\right|.f(M) + \frac{1}{n}.n.f(\varepsilon). \end{aligned}$$

Taking limit on both sides as $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k} - \ell\right|\right) = 0$$

in view of Theorem 3.4 and the fact that f is increasing.

Theorem 3.7. Let f be a modulus function such that $\lim_{t\to\infty}\frac{f(t)}{t} > 0$. If $x \overset{w^{\ell}(f)}{\sim} y$, then $x \overset{w^{\ell}}{\sim} y$.

Proof. Following the proof of Proposition 1 of Maddox [7], we have

$$\beta = \lim_{t\to\infty}\frac{f(t)}{t} = \inf\left\{\frac{f(t)}{t} : t > 0\right\}.$$

By the definition of β, we have $\frac{f(t)}{t} \geq \beta$ for all $t \geq 0$. Since $\beta > 0$, we have $\frac{t}{f(t)} \leq \beta^{-1}$. Hence

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k}-\ell\right| &= \frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k}-\ell\right|\frac{1}{f\left(\left|\frac{x_k}{y_k}-\ell\right|\right)}f\left(\left|\frac{x_k}{y_k}-\ell\right|\right)\\ &\leq \beta^{-1}\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{x_k}{y_k}-\ell\right|\right)\end{aligned}$$

from where it follows that $x \overset{w^{\ell}}{\sim} y$.

Theorem 3.8. For any modulus f, if $x \overset{w^{\ell}}{\sim} y$, then $x \overset{w^{\ell}(f)}{\sim} y$.

Proof. The proof can be done in a similar manner as in (Theorem 3.4, [2]).

The following corollary is a result of Theorem 3.7 and Theorem 3.8.

Corollary 3.9. Let f be any modulus such that $\lim_{t\to\infty}\frac{f(t)}{t} > 0$. Then $x \overset{w^{\ell}(f)}{\sim} y$ $\Leftrightarrow x \overset{w^{\ell}}{\sim} y$.

3.1. **S_{ℓ}^{f}-Equivalence of sequences.** Let $x = (x_n)$ and $y = (y_n)$ be sequences of nonnegative real numbers. We use the notation "$x \prec y$" if $x_n \leq y_n$ holds for all $n \in \mathbb{N}$. In this part, we present some nice results for nonnegative real numbers with respect to the partial order on the set of real numbers. So it will be assumed that the sequences given in this part are nonnegative real numbers unless otherwise stated.

Theorem 3.10. Let f be an unbounded modulus. If $z \prec x$ and $x - z \overset{S_{\ell'}^{f}}{\sim} y$ then $x \overset{S_{\ell}^{f}}{\sim} y$ implies $z \overset{S_{(\ell-\ell')}^{f}}{\sim} y$.

Proof. Suppose that $x - z \overset{S_{\ell'}{}^{f}}{\sim} y$. We need $z \prec x$ to guarantee the sequence $x - z = x_k - z_k$ to be a sequence of nonnegative real numbers. Then

$$\left|\frac{z_k}{y_k} - (\ell - \ell')\right| \leq \left|\frac{x_k}{y_k} - \ell\right| + \left|\frac{x_k - z_k}{y_k} - \ell'\right| \tag{1}$$

holds for all $k \in \mathbb{N}$. Then for a given $\varepsilon > 0$ the following inequality

$$\left|\left\{k \leq n : \left|\frac{z_k}{y_k} - (\ell - \ell')\right| \geq \varepsilon\right\}\right| \leq \left|\left\{k \leq n : \left|\frac{x_k}{y_k} - \ell\right| \geq \frac{\varepsilon}{2}\right\}\right| + \left|\left\{k \leq n : \left|\frac{x_k - z_k}{y_k} - \ell'\right| \geq \frac{\varepsilon}{2}\right\}\right|$$

is satisfied. Since f is an unbounded increasing modulus, then we obtain the following

$$\frac{f\left(\left|\left\{k \le n : \left|\frac{z_k}{y_k} - (\ell - \ell')\right| \ge \varepsilon\right\}\right|\right)}{f(n)} \le \frac{f\left(\left|\left\{k \le n : \left|\frac{x_k}{y_k} - \ell\right| \ge \frac{\varepsilon}{2}\right\}\right|\right)}{f(n)} + \frac{f\left(\left|\left\{k \le n : \left|\frac{x_k - z_k}{y_k} - \ell'\right| \ge \frac{\varepsilon}{2}\right\}\right|\right)}{f(n)}.$$

Hence, the desired result is obtained while taking the limit for $n \to \infty$.

Corollary 3.11. Let f be an unbounded modulus. If $y \prec z$ and $x \overset{S^f_{\ell'}}{\sim} z - y$ then $x \overset{S^f_{\ell}}{\sim} y$ implies $x \overset{S^f_{1/\ell''}}{\sim} z$ where $\ell'' := 1/\ell + 1/\ell'$.

4. CONCLUDING REMARKS

In this work, we have obtained a generalization of statistical convergence of asymptotically equivalent sequences, a new non-matrix convergence method, which is intermediate between the ordinary convergence and the statistical convergence with respect to modulus functions. Theorem 3.10 and Corollary 3.11 remain true if "$z \prec x$" in Theorem 3.10 and "$y \prec z$" in Corollary 3.11 when both of them are satisfied for all $n \in \mathbb{N}$ except a set which has zero f-density.

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