J. Indones. Math. Soc.
Vol. 24, No. 1 (2018), pp. 79–94.

STAR PROJECTIVE AND STAR INJECTIVE H_v-MODULES

A. Mortazavi and B. Davvaz

Department of Mathematics, Yazd University, Yazd, Iran
davvaz@yazd.ac.ir

Abstract. In this paper meantime the check and the defining concepts of product and direct sum, star projective and star injective in H_v-modules, we introduce a generalization extra of some notions in homological algebra to prove the five lemma and star projective and star injective Theorems in H_v-modules. We determine the conditions equivalent to split sequences in H_v-modules and also some interesting results on these concepts are given.

Key words and Phrases: H_v-module, direct product and direct sum, star projective, star injective, split sequence.

Abstrak. Dalam makalah ini, konsep perkalian dan jumlah langsung, projektif bintang, dan injektif bintang di modul-modul-H_v diperiksa dan didefinisikan. Kita memperkenalkan suatu ekstra perumuman dari beberapa gagasan dalam aljabar homologi untuk membuktikan *five lemma* dan teorema *star projective* dan *star injective* di modul-modul-H_v. Kita menentukan kondisi-kondisi yang setara dengan barisan terpisah di modul-modul-H_v. Beberapa hasil menarik pada konsep-konsep ini juga diberikan.

Kata kunci: modul-H_v, perkalian dan jumlah langsung, *star projective*, *star injective*, barisan terpisah.

1. Introduction

A couple $(H, \circ)$ of a non-empty set H and a mapping on $H \times H$ into the family of non-empty subsets of H is called a hyperstructure (or hypergroupoid). A hypergroup is a hyperstructure $(H, \circ)$ with associative law: $(x \circ y) \circ z = x \circ (y \circ z)$ for every $x, y, z \in H$ and the reproduction axiom is valid: $x \circ H = H \circ x = H$ for every $x \in H$, i.e., for every $x, y \in H$ there exist $u, v \in H$ such that $y \in x \circ u$

2000 Mathematics Subject Classification: 16E99, 18G99,20N20.
Received: 30-04-2017, revised: 05-07-2017, accepted: 17-07-2017.

and $y \in v \circ x$. This concept is introduced by Marty in 1934 [11]. If A and B are non-empty subsets of H, then $A \circ B$ is given by

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b.$$

Also, $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. Hyperrings, hypermodules and other hyperstructures are defined and several books have been written till now [1, 2, 8, 14]. The concept of H_v-structures as a larger class than the well known hyperstructures was introduced by Vougiouklis at fourth congress of AHA (Algebraic Hyperstructures and Applications) [15], where the axioms are replaced by the weak ones, that is instead of the equality on sets one has non-empty intersections. The basic definitions and results of H_v-structures can be found in [4, 5, 6, 7, 9, 10, 12, 13, 14]. The fundamental relations, weak equality, weak commutative, weak monic, weak epic, weak isomorphism, star homomorphism, star isomorph, direct product and direct sum, isomorph sequences and star projective and split sequences in H_v-modules are defined and is proved some results in [3, 14, 16, 17]. Also, some famous lemmas such as five short lemma, Snake lemma, Shanuels lemma are derived in the context of H_v-modules. The notions of $M[-]$ and $-[M]$ functors are introduced in [17] and the authors investigated the exactness of them and other problems.

The notion of exact sequences is a fundamental concept and it has been widely used in many areas such as ring and module theory. Our aim in this paper meantime the defining concept star injective, introduce a generalization extra of some notions in homological algebra to prove the five lemma and theorems star projective and star injective in H_v-modules. Determine the conditions to split a sequence (in H_v-modules) and finally some interesting results are given.

2. Basic concepts

The hyperstructure $(H, \circ)$ is called an H_v-group if " $\circ$ " is weak associative: $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \varnothing$ and the reproduction axiom is hold: $x \circ H = H \circ x = H$ for every $x \in H$. The H_v-group H is weak commutative if for every $x, y \in H$, $x \circ y \cap y \circ x \neq \varnothing$.

A multivalued system $(R, +, \cdot)$ is an H_v-ring if $(R, +)$ is a weak commutative H_v-group, $(R, \cdot)$ is a weak associative hyperstructure where " $\cdot$ " hyperoperation is weak distributive with respect to " $+$ "; i.e. for every $x, y, z \in R$ we have $x \cdot (y + z) \cap (x \cdot y + x \cdot z) \neq \varnothing$ and $(x + y) \cdot z \cap (x \cdot z + y \cdot z) \neq \varnothing$.

A non-empty set M is a (left) H_v-module over an H_v-ring R if $(M, +)$ is a weak commutative H_v-group and there exists a map $\cdot : R \times M \longrightarrow P^*(M)$ denoted by $(r, m) \longmapsto rm$ such that for every $r_1, r_2 \in R$ and every $m_1, m_2 \in M$ we have $r_1(m_1 + m_2) \cap (r_1 m_1 + r_1 m_2) \neq \varnothing$, $(r_1 + r_2)m_1 \cap (r_1 m_1 + r_2 m_1) \neq \varnothing$ and $(r_1 r_2)m_1 \cap r_1(r_2 m_1) \neq \varnothing$.

A mapping $f : M_1 \longrightarrow M_2$ of H_v-modules M_1 and M_2 over an H_v-ring R is a strong homomorphism (homomorphism)if for every $x, y \in M_1$ and every $r \in R$ we have $f(x + y) = f(x) + f(y)$ and $f(rx) = rf(x)$.

By using a certain type of equivalence relations we can connect hyperstructures to ordinary structures.

Let $(R, +, \cdot)$ be an H_v-ring. Vougiouklis in [14] defined the relation γ^* as the smallest equivalence relation on R such that the quotient set $R/\gamma^* = \{\gamma^*(r) \mid r \in R\}$ is a ring. The γ^* is called the fundamental equivalence relation on R and R/γ^* is called the fundamental ring. Let us denote the set of all finite polynomials of elements of R, over $\mathbb{N}$, by $\mathcal{U}$. We define the relation γ as follows:

$$x\gamma y \;\Leftrightarrow\; \{x, y\} \subseteq u, \text{ for some } u \in \mathcal{U}.$$

Theorem 2.1. [14] *The relation γ^* is the transitive closure of the relation γ, and the addition and multiplication operations on R/γ^* are defined as follows:*

$$\begin{aligned} \gamma^*(a) \oplus \gamma^*(b) &= \gamma^*(c), \text{ for all } c \in \gamma^*(a) + \gamma^*(b), \\ \gamma^*(a) \odot \gamma^*(b) &= \gamma^*(c), \text{ for all } c \in \gamma^*(a) \cdot \gamma^*(b). \end{aligned}$$

Now, suppose that M is an H_v-module over an H_v-ring R. Vougiouklis in [13] defined the relation ε^* as the smallest equivalence relation on M such that the quotient $\{M/\varepsilon^*(x) \mid x \in M\}$ is a module over the ring R/γ^*. The relation ε^* is called the fundamental equivalence relation on M and M/ε^* is called the fundamental module. Let us denote ϑ the set of all expressions consisting of finite hyperoperations either on R and M or the external hyperoperation applied on finite sets of elements of R and M [13]. We consider the relation ε on M as follows:

$$x\varepsilon y \;\Leftrightarrow\; \{x, y\} \subseteq v, \text{ for some } v \in \vartheta.$$

Theorem 2.2. [13] *The relation ε^* is the transitive closure of the relation ε, and the addition and external product on M/ε^* are defined as follows:*

$$\begin{aligned} \varepsilon^*(x) \oplus \varepsilon^*(y) &= \varepsilon^*(z), \text{ for all } z \in \varepsilon^*(x) + \varepsilon^*(y), \\ \gamma^*(r) \odot \varepsilon^*(x) &= \varepsilon^*(t), \text{ for all } t \in \gamma^*(r) \cdot \varepsilon^*(x). \end{aligned}$$

The heart of an H_v-module M over an H_v-ring R is denoted by ω_M and is defined by $\omega_M = \{x \in M \mid \varepsilon^*_M(x) = 0\}$, where 0 is the unit element of the group $(M/\varepsilon^*, \oplus)$. One can prove that the unit element of the group $(M/\varepsilon^*, \oplus)$ is equal to ω_M. By the definition of ω_M we have $\omega_{\omega_M} = \mathrm{Ker}(\phi : \omega_M \longrightarrow \omega_M/\varepsilon^*_{\omega_M} = 0) = \omega_M$.

Let M_1 and M_2 be two H_v-modules over an H_v-ring R and $\varepsilon^*_1, \varepsilon^*_2$ and ε^* be the fundamental relations on M_1, M_2 and $M_1 \times M_2$ respectively, then $(x_1, x_2)\varepsilon^*(y_1, y_2)$ if and only if $x_1\varepsilon^*_1 y_1$ and $x_2\varepsilon^*_2 y_2$ for all $(x_1, x_2), (y_1, y_2) \in M_1 \times M_2$ [13, 14].

Weak equality (monic, epic), exact sequences and relative results in H_v-modules are defined as follows [3]:

Let M be an H_v-module. The non-empty subsets X and Y of M are weakly equal if for every $x \in X$ there exists $y \in Y$ such that $\varepsilon^*_M(x) = \varepsilon^*_M(y)$ and for every $y \in Y$ there exists $x \in X$ such that $\varepsilon^*_M(x) = \varepsilon^*_M(y)$ that is denoted by $X \overset{w}{=} Y$. The sequence

$$M_o \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n$$

of H_v-modules and strong homomorphisms is exact if for every $2 \leq i \leq n$,

$$\mathrm{Im}(f_{i-1}) \overset{w}{=} \mathrm{Ker}(f_i),$$

where $\mathrm{Ker}(f_i) = \{a \in M_{i-1} \mid f_i(a) \in \omega_{M_i}\}$ (that is an H_v-submodule of M_{i-1}).

The strong homomorphism $f : M_1 \longrightarrow M_2$ is called *weak monic* if for every $m_1, m_1^{'} \in M_1$ the equality $f(m_1) = f(m_1^{'})$ implies $\varepsilon^*_{M_1}(m_1) = \varepsilon^*_{M_1}(m_1^{'})$ and f is called *weak epic* if for every $m_2 \in M_2$ there exists $m_1 \in M_1$ such that $\varepsilon^*_{M_2}(m_2) = \varepsilon^*_{M_2}(f(m_1))$. Finally, f is called *weak-isomorphism* if f is weak monic and weak epic.

It is easy to see that every one to one (onto) strong homomorphism is weak monic (weak epic), but the converse is not true necessarily. In fact the concept of weak monic (weak epic) is a generalization of the concept of one to one (onto).

Let $f : A \longrightarrow B$ be a strong homomorphism of H_v-modules over an H_v-ring R. Then, we have $f(\omega_A) \subseteq \omega_B$ and so $\omega_A \subseteq \mathrm{Ker}(f)$. Moreover, $\mathrm{Ker}(f) = \omega_A$ if and only if f is weak monic.

A mapping $f : M_1 \longrightarrow M_2$ of H_v-modules M_1 and M_2 over an H_v-ring R is called a star homomorphism if for every $x, y \in M_1$ and every $r \in R : \varepsilon^*_{M_2}(f(x+y)) = \varepsilon^*_{M_2}(f(x) + f(y))$ and $\varepsilon^*_{M_2}(f(rx)) = \varepsilon^*_{M_2}(rf(x))$; i.e. $f(x+y) \overset{w}{=} f(x) + f(y)$ and $f(rx) \overset{w}{=} rf(x)$.

Two mappings $f, g : M \longrightarrow N$ on H_v-modules are called *weak equal* if for every $m \in M$; $\varepsilon^*_N(f(m)) = \varepsilon^*_N(g(m))$ and denote by $f \overset{w}{=} g$. The following diagram of H_v-modules and strong homomorphisms is called *star commutative* if $g \circ f \overset{w}{=} h$.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, "h"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

Also, it is called *commutative* if for every $a \in A$, $g \circ f(a) = h(a)$.

The sequences

$$\omega_A \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \omega_C$$

and

$$\omega_{A'} \longrightarrow A^{'} \xrightarrow{f^{'}} B^{'} \xrightarrow{g^{'}} C^{'} \longrightarrow \omega_{C'}$$

are called *isomorph* (*star isomorph*) if there exist weak-isomorphisms (star homomorphisms) $\alpha : A \longrightarrow A^{'}$, $\beta : B \longrightarrow B^{'}$ and $\gamma : C \longrightarrow C^{'}$ such that the following diagram is commutative (star commutative):

$$\begin{tikzcd} \omega_A \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"'] & \omega_C \\ \omega_{A'} \arrow[r] & A^{'} \arrow[r, "f^{'}"'] & B^{'} \arrow[r, "g^{'}"'] & C^{'} \arrow[r] & \omega_{C'} \end{tikzcd}$$

According to [3] for every strong homomorphism $f : M \longrightarrow N$ there is the R/γ^*-homomorphism $F : M/\varepsilon^*_M \longrightarrow N/\varepsilon^*_N$ of R/γ^*-modules defined by $F(\varepsilon^*_M(m)) = \varepsilon^*_N(f(m))$.

Let M be an H_v-module and H, K be H_v-submodules of M. Then M is called the *direct sum* of H and K if $H \cap K \subseteq \omega_M$ and $\varepsilon^*(H + K) = \varepsilon^*(M)$. We denote it by $H \oplus K = M$.

Then, f is weak epic if and only if F is onto. Moreover, f is weak monic if and only if F is one to one. Finally, f is a weak-isomorphism if and only if F is an isomorphism

3. Product and direct sum in H_v-Modules

In this section, we meantime the check concept product and direct sum, we introduce a generalization of some notions in homological algebra to prove the five lemma in H_v-modules. Also, we determine the conditions equivalent to split the exact sequences in H_v-modules and some interesting results on these concepts are given.

Proposition 3.1. *Let $f : M \longrightarrow N$ and $g : N \longrightarrow M$ be strong homomorphisms of H_v-modules such that $gf = 1$. Then, N is the direct sum of* $\mathrm{Im}(f)$ *and* $\mathrm{Ker}(f)$.

Proof. Let $m \in \mathrm{Im}(f) \cap \mathrm{Ker}(g)$. Then

$$m = f(m_1) \text{ for some } m_1 \in M$$

and

$$g(m) \in \omega_N. \tag{1}$$

By applying f on Eq. (1) we obtain $g(m) = gf(m_1) = m_1$. hence $m_1 \in \omega_N$. Since f is a strong homomorphism, it follows that $m = f(m_1) \in \omega_N$. So, $\mathrm{Im}(f) \cap \mathrm{Ker}(f) \subseteq \omega_N$. Now, for every $m \in M$ we have:

$$G(F(\varepsilon^*(m))) = G(\varepsilon^*(f(m))) = \varepsilon^*(gf(m)) = \varepsilon^*(m).$$

So $\mathrm{Im}(F) + Ker(G) = N/\varepsilon^*_N$, since G and F are R/γ^*-homomorphisms such that $GF = 1$. Therefore, $\varepsilon^*(\mathrm{Im}(f) + Ker(g)) = \varepsilon^*(N)$. □

$\prod_{i\in I} M_i$ is called the (external) *direct product* of the family of H_v-modules $\{M_i \mid i \in I\}$ and $\sum_{i\in I} M_i$ is its (external) *direct sum*. If the index set is finite, say $I = \{1, 2, \cdots, n\}$, then the product and direct sum coincide and will be written $M_1 \oplus M_2 \oplus \cdots \oplus M_n$. The maps Π_k [resp. ι_k] are called the *canonical projections* [resp. *injections*]. Let $\{M_i\}_{i\in I}$ be a non-empty collection of H_v-modules. The product of this collection $\prod_{i\in I} M_i = \{(x_i) \mid x_i \in M_i; \forall i \in I\}$, with the following hyperoperations is an H_v-module: $(x_i) + (y_i) = \{(z_i) \mid z_i \in x_i + y_i\}$ and $r(x_i) = \{(w_i)|w_i \in rx_i\}$. Similarly, we define hyperoperations on $\sum_{i\in I} M_i$ which if $0 \neq \{m_i\} \in \sum_{i\in I} M_i$, then only finitely many of the a_i are nonzero, say $a_{i_1}, a_{i_2}, \cdots, a_{i_r}$.

Proposition 3.2. *Let $\{M_i\}$ be a non-empty collection of H_v-modules. For every H_v-module X and every collection of strong homomorphisms $\{\psi_i : M_i \longrightarrow X\}$ there* *exists an unique strong homomorphism* $\psi : \sum_{i\in I} M_i \longrightarrow X$ *defined by* $\psi_i = \psi\iota_i$ *such that for every* $i \in I$ *the following diagram is commutative.*

$$\begin{tikzcd} & \sum\limits_{i\in I} M_i \arrow[d, "\psi"] \\ M_i \arrow[ur, "\iota_i"] \arrow[r, "\psi_i"'] & X \end{tikzcd}$$

Proof. The proof is straightforward. □

Proposition 3.3. *Let* R *be a* H_v*-ring and* $A, A_1, A_2, \cdots, A_n$ H_v*-modules. Then* $A \overset{w}{=} A_1 \oplus A_2 \oplus \cdots \oplus A_n$ *if and only if for each* $i = 1, 2, \cdots, n$ *there are strong homomorphisms* $\Pi_i : A \longrightarrow A_i$ *and* $\iota_i : A_i \longrightarrow A$ *such that*

(1) $\Pi_i\iota_i \overset{w}{=} 1_i$ for $i = 1, 2, \cdots, n$;
(2) $\Pi_j\iota_i \overset{w}{=} 0$ for $i \neq j$;
(3) $\iota_1\Pi_1 + \iota_1\Pi_1 + \cdots + \iota_1\Pi_1 \overset{w}{=} 1_A$.

Proof. ($\Rightarrow$): If A is the H_v-module $A_1 \oplus A_2 \oplus \cdots \oplus A_n$, then the canonical injections ι_i and projections Π_i satisfy (1)-(3) as the readers may easily verify. Likewise if $A \overset{w}{=} A_1 \oplus A_2 \oplus \cdots \oplus A_n$ under an weak isornorphism $A \longrightarrow A_1 \oplus A_2 \oplus \cdots \oplus A_n$ then the homomorphisms $\Pi_i f : A \longrightarrow A_i$ and $f^{-1}\iota_i : A_i \longrightarrow A$ satisfy (1)-(3).

($\Leftarrow$): Suppose that $\Pi_i : A \longrightarrow A_i$ and $\iota_i : A_i \longrightarrow A$ ($i = 1, \cdots, n$) satisfy (1)-(3). Let $\Pi_i^{'} : A_1 \oplus A_2 \oplus \cdots \oplus A_n \longrightarrow A_i$ and $\iota_i^{'} : A_i \longrightarrow A_1 \oplus A_2 \oplus \cdots \oplus A_n$ be the canonical projections and injections. Let $\varphi : A_1 \oplus A_2 \oplus \cdots \oplus A_n \longrightarrow A$ be given by $\varphi \overset{w}{=} \iota_1\Pi_1^{'} + \iota_2\Pi_2^{'} + \cdots + \iota_n\Pi_n^{'}$ and $\psi : A \longrightarrow A_1 \oplus A_2 \oplus \cdots \oplus A_n$ by $\psi \overset{w}{=} \iota_1^{'}\Pi_1 + \iota_2^{'}\Pi_2 + \cdots + \iota_n^{'}\Pi_n$. Then

$$\begin{aligned} \varphi\psi &\overset{w}{=} (\sum_{i=1}^{n} \iota_i\Pi_i^{'})(\sum_{j=1}^{n} \iota_j^{'}\Pi_j) \overset{w}{=} \sum_{i=1}^{n}\sum_{j=1}^{n} \iota_i\Pi_i^{'}\iota_j^{'}\Pi_j \overset{w}{=} \sum_{i=1}^{n} \iota_i\Pi_i^{'}\iota_i^{'}\Pi_i \\ &\overset{w}{=} \sum_{i=1}^{n} \iota_i 1_{A_i}\Pi_i \overset{w}{=} \sum_{i=1}^{n} \iota_i\Pi_{A_i} \overset{w}{=} 1_A \end{aligned}$$

□

EXAMPLE 1. Note first that for any H_v-module A, there are unique strong hornomorphisrns $\omega_A \longrightarrow A$ and $A \longrightarrow \omega_A$. If A and B are any H_v-modules then the sequences

$$\omega_A \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{\Pi} B \longrightarrow \omega_B \text{ and } \omega_B \longrightarrow B \xrightarrow{i} B \oplus A \xrightarrow{\Pi} A \longrightarrow \omega_A$$

are exact, where the i and Π are the canonical injections and projections respectively. Similarly, if C is a submodule of D, then the sequence

$$\omega_C \longrightarrow C \longrightarrow D \longrightarrow D/C \longrightarrow \omega_{D/C}$$

is exact, where i is the inclusion map and p is the canonical epimorphism.

Proposition 3.4. (**Five Lemma in Hv-modules**) *Let*

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"'] & A_3 \arrow[r, "f_3"] \arrow[d, "\alpha_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "\alpha_4"'] & A_5 \arrow[d, "\alpha_5"'] \\ B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4 \arrow[r, "g_4"'] & B_5 \end{tikzcd}$$

be a commutative diagram of H_v-modules and H_v-homomorphisms over an H_v-ring R with both rows exact. Then,

(1) *if α_1 is weak monic and α_2, α_4 weak epic then α_3 is weak epic.*
(2) *if α_5 is weak epic and α_2, α_4 weak monic then α_3 is weak monic.*
(3) *if α_1 is weak monic, α_5 weak epic and α_2, α_4 weak isomorphism then α_3 is weak isomorphism.*

Proof. (1) By Theorem 4.8 of [4] and Lemma 3.4 of [16] the following diagram of R/γ^*-modules and R/γ^*-homomorphisms is commutative with both rows exact

$$\begin{tikzcd} A_1/\varepsilon^* \arrow[r, "F_1"] \arrow[d, "\overline{\alpha}_1"'] & A_2/\varepsilon^* \arrow[r, "F_2"] \arrow[d, "\overline{\alpha}_2"'] & A_3/\varepsilon^* \arrow[r, "F_3"] \arrow[d, "\overline{\alpha}_3"] & A_4/\varepsilon^* \arrow[r, "F_4"] \arrow[d, "\overline{\alpha}_4"'] & A_5/\varepsilon^* \arrow[d, "\overline{\alpha}_5"'] \\ B_1/\varepsilon^* \arrow[r, "G_1"'] & B_2/\varepsilon^* \arrow[r, "G_2"'] & B_3/\varepsilon^* \arrow[r, "G_3"'] & B_4/\varepsilon^* \arrow[r, "G_4"'] & B_5/\varepsilon^* \end{tikzcd}$$

By Lemma 4.5 of [16], $\overline{\alpha}_1$ is monic and $\overline{\alpha}_2$, $\overline{\alpha}_4$ epic R/γ^*-homomorphisms. By five lemma in modules $\overline{\alpha}_3$ is monic R/γ^*-homomorphism. Thus, by Lemma 4.5 of [16], α_3 is weak monic R-homomorphism.

(2) The proof is similar to the proof of (1).

(3) The proof follows from (1) and (2). □

Proposition 3.5. (1) *Let $M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3$ be an exact sequence of H_v-modules and H_v-homomorphisms such that φ is epic and ψ is weak monic, then $M_2 \overset{w}{=} \omega_{M_2}$.*

(2) *Let $M_1 \xrightarrow{\varphi} M_2 \xrightarrow{f} M_3 \xrightarrow{\psi} M_4$ be an exact sequence of H_v-modules and H_v-homomorphisms. Then φ is weak epic if and only if ψ is weak monic.*

(3) *Let $M_1 \xrightarrow{\varphi} M_2 \xrightarrow{f} M_3 \xrightarrow{g} M_4 \xrightarrow{\psi} M_5$ be an exact sequence of H_v-modules and H_v-homomorphisms such that φ is weak epic and ψ is weak monic, then $M_3 \overset{w}{=} \omega_{M_3}$.*

Proof. (1) Let $m_2 \in M_2$. Since φ is epic, there is $m_1 \in M_1$ such that $\varphi(m_1) = m_2$. Hence $M_2 = \operatorname{Im}\varphi \overset{w}{=} \operatorname{Ker}\psi$. Since ψ is weak monic, thus $\operatorname{Ker}\psi \overset{w}{=} \omega_{M_2}$. Then $M_2 \overset{w}{=} \omega_{M_2}$.

(2) Let φ be weak epic and $x \in \operatorname{Ker}\psi$. We have $\operatorname{Im}\varphi \overset{w}{=} \operatorname{Ker} f$ and $\operatorname{Im} f \overset{w}{=} \operatorname{Ker}\psi$. Hence, there is $m_2 \in M_2$ such that $f(m_2) \overset{w}{=} x$. Since φ is weak epic, thus there is $m_1 \in M_1$ such that $\varphi(m_1) \overset{w}{=} m_2$. Then $x \overset{w}{=} f(\varphi(m_1)) \overset{w}{=} \omega_{M_3}$. There for

$\operatorname{Ker}\psi \overset{w}{=} \omega_{M_3}$. Then ψ is weak monic.

(3) Let $m_3 \in M_3$, hence $g(m_3) \in \operatorname{Im}(g)$. We have $\operatorname{Im} g \overset{w}{=} \operatorname{Ker}\psi$ and ψ is weak monic. Thus $g(m_3) \overset{w}{=} \omega_{M_4}$. Then $a \in \operatorname{Ker} g$ and $\operatorname{Ker} g \overset{w}{=} \operatorname{Im} f$. Therefor $m_2 \in M_2$ such that $f(m_2) \overset{w}{=} m_3$. Since φ is weak epic, then there is $m_1 \in M_1$ such that $\varphi(m_1) \overset{w}{=} m_2$. Hence $m_3 \overset{w}{=} f(\varphi(m_1)) \overset{w}{=} \omega_{M_3}$. Then $M_3 \overset{w}{=} \omega_{M_3}$. $\square$

Proposition 3.6. *Let*

$$\begin{tikzcd} & M' \arrow[r, "f"] \arrow[d, "\alpha"'] & M \arrow[r, "g"] \arrow[d, "\beta"'] & M'' \arrow[r] \arrow[d, "\gamma"] & \omega_{M''} \\ \omega_{N'} \arrow[r] & N' \arrow[r, "\varphi"'] & N \arrow[r, "\psi"'] & N'' & \end{tikzcd}$$

be a commutative diagram of H_v-modules and H_v-homomorphisms over an H_v-ring R with both rows exact. Then, there is the exact sequence

$$\operatorname{Ker}\alpha \xrightarrow{\overline{f}} \operatorname{Ker}\beta \xrightarrow{\overline{g}} \operatorname{Ker}\gamma \xrightarrow{d} \frac{N'}{\operatorname{Im}\alpha} \xrightarrow{\overline{\varphi}} \frac{N}{\operatorname{Im}\beta} \xrightarrow{\overline{\psi}} \frac{N''}{\operatorname{Im}\gamma}.$$

Withal, if f is weak monic, then $\overline{f}$ is weak monic and if ψ is weak epic, then $\overline{\psi}$ is weak epic.

Proof. We define $\overline{\varphi}(n' + \operatorname{Im}\alpha) \overset{w}{=} \varphi(n') + \operatorname{Im}\beta$ and $\overline{\psi}(n + \operatorname{Im}\beta) \overset{w}{=} \psi(n) + \operatorname{Im}\gamma$. Also we define $\overline{f}$ and $\overline{g}$, scowl the mapping f and g on $\operatorname{Ker}\alpha$ and $\operatorname{Ker}\beta$ respectively. It can be easily we seen that $\overline{\varphi}, \overline{\psi}, \overline{f}$ and $\overline{g}$ are well define also $\operatorname{Im}\overline{f} \overset{w}{=} \operatorname{Im} f$, $\operatorname{Ker}\overline{g} \overset{w}{=} \operatorname{Ker} g$, $\operatorname{Im}\overline{\varphi} \overset{w}{=} \operatorname{Im}\varphi$, $\operatorname{Ker}\overline{\psi} \overset{w}{=} \operatorname{Ker}\psi$. We have $\operatorname{Im}\overline{f} \overset{w}{=} \operatorname{Ker}\overline{g}$ and $\operatorname{Im}\overline{\varphi} \overset{w}{=} \operatorname{Ker}\overline{\psi}$. Now, we defining the mapping $d : \operatorname{Ker}\gamma \longrightarrow N'/\operatorname{Im}\alpha$ by $d(m'') = n' + \operatorname{Im}\alpha$. We show that d is well define. If $m" \in \operatorname{Ker}\gamma$, then $\gamma(m") \overset{w}{=} \omega_{N"}$. Since g is weak epic, then there is $m \in M$ such that $g(m) \overset{w}{=} m"$. Thus $\gamma g(m) \overset{w}{=} \omega_{N"}$. We obtain $\psi\beta(m) \overset{w}{=} \omega_{N"}$. Hence $\beta(m) \in \operatorname{Ker}\psi \overset{w}{=} \operatorname{Im}\varphi$. Therefor, there is $n' \in N'$ such that $\beta(m) \overset{w}{=} \varphi(n')$. Since φ is weak monic, then n' is unique. Thus, if $m''_1 = m''_2$ we obtain $g(m''_1) \overset{w}{=} g(m''_2)$. Then there is $n'_1, n'_2 \in N'$ such that $\varphi(n'_1) \overset{w}{=} \beta(m''_1)$ and $\varphi(n'_2) \overset{w}{=} \beta(m''_2)$. Therefor $n'_1 + \operatorname{Im}\alpha \overset{w}{=} n'_2 + \operatorname{Im}\alpha$. Then d is well define. It can be easily we seen that d is a H_v-homomorphism.

Now, we have $\operatorname{Im} d = \{n' = \operatorname{Im}\alpha \mid m" \operatorname{Ker}\gamma\}$ and $\operatorname{Ker}\overline{\varphi} = \{n' = \operatorname{Im}\alpha \mid \varphi(n') + \operatorname{Im}\beta \overset{w}{=} \omega_N\}$. If $m'' \in \operatorname{Ker}\gamma$, then $\gamma(m'') \overset{w}{=} \omega_{N''}$. Since g is weak epic, there is $m \in M$ such that $g(m) \overset{w}{=} m''$. Thus $\gamma g(m) \overset{w}{=} \omega_{N''}$. we abtain $\psi\beta(m) \overset{w}{=} \omega_{N''}$. Hence $\beta(m) \in \operatorname{Ker}\psi \overset{w}{=} \operatorname{Im}\varphi$. Then there is $n' \in N'$ such that $\beta(m) \overset{w}{=} \varphi(n') \in Img\beta$. Then $\operatorname{Ker}\overline{\varphi} \overset{w}{=} \operatorname{Im} d$. Also, $\operatorname{Ker} d = \{m'' \in \operatorname{Ker}\gamma \mid n' + \operatorname{Im}\alpha \overset{w}{=} \omega_{N'}\} \overset{w}{=} \{g(m) \mid m \in M, n' \in \operatorname{Im}\alpha\} = \operatorname{Im}\overline{g}$. Then we obtain the exact sequence

$$\operatorname{Ker}\alpha \xrightarrow{\overline{f}} \operatorname{Ker}\beta \xrightarrow{\overline{g}} \operatorname{Ker}\gamma \xrightarrow{d} \frac{N'}{\operatorname{Im}\alpha} \xrightarrow{\overline{\varphi}} \frac{N}{\operatorname{Im}\beta} \xrightarrow{\overline{\psi}} \frac{N''}{\operatorname{Im}\gamma}.$$

Now, if f is weak monic, since $\overline{f}$ is scowl the mapping f on $\operatorname{Ker}\alpha$, hence $\overline{f}$ is weak monic. Let ψ be weak epic. Consider $n^{''} \in N^{''}$. Then, $n^{''} + \operatorname{Im}\gamma \in N^{''}/\operatorname{Im}\gamma$. Since ψ is weak epic, then there is $n \in N$ such that $\varphi(n) \overset{w}{=} n^{''}$ and $n + \operatorname{Im}\beta \in N/\operatorname{Im}\beta$. We obtain $\overline{\psi}(n + \operatorname{Im}\beta) \overset{w}{=} n^{''} + \operatorname{Im}\gamma$. Then $\overline{\psi}$ is weak epic. $\square$

Proposition 3.7. *Let*

$$\begin{tikzcd}
\omega_{K'} \arrow[dr] & & & & \\
& K' \arrow[dr] & & & \\
\omega_K \arrow[r] & K \arrow[r] \arrow[u, "\alpha"] & M \arrow[r] \arrow[dr] & C \arrow[r] \arrow[d, "\beta"] & \omega_C \\
& & & C' \arrow[dr] & \\
& & & & \omega_{C'}
\end{tikzcd}$$

be a star commutative diagram of H_v-modules and strong homomorphisms which rows horizontal and diagonal are exact. If α is weak monic and β weak epic, then $K^{'}/\operatorname{Im}\alpha \neq \omega_{K'}$ if and only if $\operatorname{Ker}\beta \neq \omega_C$.

Proof. Let $x + \operatorname{Im}\alpha \in K^{'}/\operatorname{Im}\alpha$ such that $x \in K^{'}$ and $x \notin \operatorname{Im}\alpha$. On the other hand $g_1\alpha \overset{w}{=} f_1$, $\beta f_2 \overset{w}{=} g_2$, $g_1(x) \in M$ and $f_2 g_1(x) \in C$. hence $\beta f_2 g_1(x) \overset{w}{=} g_2 g_1(x) \overset{w}{=} \omega_{C'}$. therefore $f_2 g_1(x) \in \operatorname{Ker}\beta$. If $f_2 g_1(x) \overset{w}{=} \omega_{C'}$ then $g_1(x) \in \operatorname{Ker} f_2 \overset{w}{=} \operatorname{Im} f_1$. hence there is $t \in K$ such that $g_1(x) \overset{w}{=} f_1(t)$. Thus $g_1(x) \overset{w}{=} g_1\alpha(t)$. Since g_1 is weak monic, then $x \overset{w}{=} \alpha(t)$. therefor $x \in \operatorname{Im}\alpha$ which is a contradiction. hence $\operatorname{Ker}\beta \overset{w}{\not\models} \omega_C$.

For the converse, let $\operatorname{Ker}\beta \overset{w}{\not\models} \omega_C$. We show that $K^{'}/\operatorname{Im}\alpha \overset{w}{\not\models} \omega_{K'}$. Let $x \in \operatorname{Ker}\beta$. Since f_2 is weak monic, it follows that there is $m \in M$ such that $f_2(m) \overset{w}{=} x$ for every $x \in C$. hence $\beta f_2(m) \overset{w}{=} \omega_{C'}$, then $g_2(m) \overset{w}{=} \omega_{C'}$. Thes $m \in \operatorname{Ker} g_2 \overset{w}{=} \operatorname{Im} g_1$.. therefor there is $k^{'} \in K^{'}$ such that $m \overset{w}{=} g_1(k^{'})$. We show that $k^{'} \notin \operatorname{Im}\alpha$. If $k^{'} \in \operatorname{Im}\alpha$, then there is $k \in K$ such that $k^{'} \overset{w}{=} \alpha(k)$. Thus $g_1\alpha(k) \overset{w}{=} m$. therefor $f_1(k) \overset{w}{=} m$. Then $f_2(m) \overset{w}{=}$
$f_2 f_1(k) \overset{w}{=} \omega_C$. Hence $k^{'} \in \operatorname{Im}\alpha$. Then $K^{'}/\operatorname{Im}\alpha \overset{w}{\not\models} \omega_{K'}$. $\square$

Proposition 3.8. *Let M_1, M_2 and M be three H_v-modules and the sequence*

$$\omega_{A_1} \longrightarrow A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \longrightarrow \omega_{A_2}$$

is exact. Then the following conditions are equivalent.

(1) *There is a star homomorphism* $h : A_2 \longrightarrow B$ *with* $gh = 1_{A_2}$*;*
(2) *There is a star homomorphism* $k : B \longrightarrow A_1$ *with* $kf = 1_{A_1}$*;*
(3) *the given sequence is star isomorphic (with identity maps on* A_1 *and* A_2*) to the direct sum short exact sequence*

$$\omega_{A_1} \longrightarrow A_1 \longrightarrow A_1 \bigoplus A_2 \longrightarrow A_2 \longrightarrow \omega_{A_2} \ ;$$

in particular $B \overset{w}{=} A_1 \bigoplus A_2$.

Proof. The proof is similar to the proof of Theorem 4.7 in [17]. □

4. Star projective and star injective H_v-Modules

In this section we define concepts star projective and star injective in H_v-modules and introduce a generalization extra of some notions in homological algebra to prove Theorems star projective and star injective in H_v-modules.

Definition 4.1. [17] An H_v-module P is a star projective if for every diagram of star homomorphisms and H_v-modules as follows

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ M \arrow[r, "g"] & N \arrow[r] & \omega_N \end{tikzcd}$$

that it's row is exact, there exists a star homomorphism $\phi : P \longrightarrow M$ such that for each $p \in P$, $\varepsilon^*_N(g(\phi(p))) = \varepsilon^*_N(f(p))$.

Proposition 4.2. *Let* R *be an* H_v*-ring. Then,*

(1) *If* P *is a star projective* H_v*-module, then every short exact sequence*

$$\omega_A \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow \omega_P$$

is split exact (hence $B \overset{w}{=} A \bigoplus P$*);*

(2) *If every short exact sequence*

$$\omega_A \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow \omega_P$$

is split exact, then P *is a star projective* H_v*-module.*

Proof. (1) Consider the diagram

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ B \arrow[r, "g"] & P \arrow[r] & \omega_P \end{tikzcd}$$

of star homomorphisms and with bottom row exact. Since P is star projective, it follows that there exists a star homomorphism $h : P \longrightarrow B$ such that $\varepsilon^*(gh(p)) = \varepsilon^*(1_P(p))$ for all $p \in P$. Therefore, the short exact sequence

$$w_A \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow w_P$$

is split exact by Theorem 3.8 and $B \overset{w}{=} A \bigoplus P$.

(2) Suppose that every short exact sequence

$$w_A \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow w_P$$

is split exact. We show that P is a star projective H_v-module. So, we show that for every diagram

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ T_1 \arrow[r, "g"] & T_2 \arrow[r] & w_{T_2} \end{tikzcd}$$

of H_v-modules and star homomorphism such that bottom row is exact, there is a star homomorphism $\varphi : P \longrightarrow T_1$ such that $\varepsilon^*(g\varphi(p)) = \varepsilon^*(f(p))$ for all $p \in P$. Now, we consider the mapping $h : T_1 \longrightarrow P$ by $h(t) \in f^{-1}g(t)$ for every $t \in T_1$. since f^{-1} and g are star homomorphisms. Hence h is star homomorphism. Now, we have the exact sequence

$$w_{T_1} \longrightarrow \ker h \longrightarrow T_1 \xrightarrow{h} P \longrightarrow w_{T_2} \ . \tag{2}$$

The exact sequence (3) is split, hence there exists a star homomorphism $\psi : P \longrightarrow T_1$ such that $h\psi \overset{w}{=} 1_p$. Therefore $f(h\psi) \overset{w}{=} f1_p \overset{w}{=} f$. Then $(fh)\psi \overset{w}{=} f$. Hence $g\psi \overset{w}{=} f$. $\square$

Proposition 4.3. *Let R be an H_v-ring. A direct sum $\sum\limits_{i\in I} P_i$ of H_v-modules is star projective if and only if each P_i is star projective.*

Proof. Let $\sum\limits_{i\in I} P_i$ be star projective. We consider the diagram

$$\begin{tikzcd} & P_i \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & w_B \end{tikzcd}$$

of H_v-modules and star homomorphisms such that bottom row is exact. Therefore, there is the diagram

$$\begin{tikzcd} & \sum\limits_{i\in I} P_i \arrow[d, shift left, "\Pi"'] & \\ & P_i \arrow[u, shift left, "\iota_i"'] \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & w_B \end{tikzcd}$$

of H_v-modoules and star homomorphisms. Since $\sum\limits_{i\in I} P_i$ is star projective, it follows that there exists a strong homomorphism $\varphi : \sum\limits_{i\in I} P_i \longrightarrow A$ such that $g\varphi \overset{w}{=} f\Pi_i$. we define the mapping $h : P_i \longrightarrow A$ by $h(x) \overset{w}{=} \varphi\iota_i(x)$ for every $x \in P_i$. Now, we have

$$gh(x) \overset{w}{=} g\varphi\iota_i(x) \overset{w}{=} f\Pi_i\iota_i(x) \overset{w}{=} f(x).$$

Then $gh \overset{w}{=} f$. Nothing that h is a star homomorphism. Therefore, P_i is star projective. The converse is proved by a similar techniques and using the diagram

$$\begin{tikzcd} & P_i \arrow[d, shift right, "\iota_i"'] & \\ & \sum\limits_{i\in I} P_i \arrow[u, shift right, "\pi_i"'] \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & w_B \end{tikzcd}$$

If each P_i is star projective, then for each i there exists a star homomorphism $h_i : P_i \longrightarrow A$ such that $gh_i \overset{w}{=} f\iota_i$. By Theorem 3.2 there is a unique star homomorphism $h : \sum\limits_{i\in I} P_i \longrightarrow A$ with $h\iota_i \overset{w}{=} h_i$ for every $i \in I$. Hence $h \overset{w}{=} h_i\Pi_i$. Then

$$gh \overset{w}{=} gh_i\Pi_i \overset{w}{=} f\iota_i\Pi_i \overset{w}{=} f.$$

Verify that $gh = f$. □

Definition 4.4. An H_v-module P is a *star injective* if for every diagram of star homomorphisms and H_v-modules as follows

$$\begin{tikzcd} w_N \arrow[r] & N \arrow[r, "g"] \arrow[d, "f"] & M \\ & E & \end{tikzcd}$$

that it's row is exact, there exists a star homomorphism $\phi : M \longrightarrow E$ such that for each $n \in N$, $\varepsilon^*_N(\phi(g(n))) = \varepsilon^*_N(f(n))$.

Proposition 4.5. *Let R be an H_v-ring. Then,*

(1) *If E is a star injective H_v-module, then every short exact sequence*

$$\omega_E \longrightarrow E \xrightarrow{f} A \xrightarrow{g} B \longrightarrow \omega_B$$

is split (hence $A \overset{w}{=} B \bigoplus E$);

(2) *If every short exact sequence*

$$w_E \longrightarrow E \xrightarrow{f} A \xrightarrow{g} B \longrightarrow w_B$$

is split, then E is a star injective H_v-module.

Proof. (1⇒2) Consider the diagram

$$\begin{tikzcd} w_E \arrow[r] & E \arrow[r, "g"] \arrow[d, "1_E"] & A \\ & E & \end{tikzcd}$$

of H_v-modules and star homomorphisms such that upper row exact. Since E is star injective, it follows that there is a star homomorphism $h : A \longrightarrow E$ such that $\varepsilon^*(hg(e)) = \varepsilon^*(1_E(e)))$ for all $e \in E$. Therefore, the short exact sequence

$$w_E \longrightarrow E \xrightarrow{f} A \xrightarrow{g} B \longrightarrow w_E$$

is split, by Theorem 3.8 and $A \overset{w}{=} B \bigoplus E$.

(2⇒1) Suppose that every short exact sequence

$$w_E \longrightarrow E \xrightarrow{f} A \xrightarrow{g} B \longrightarrow w_E$$

is split. We show that for every diagram

$$\begin{tikzcd} w_{T_1} \arrow[r] & T_1 \arrow[r, "g"] \arrow[d, "f"] & T_2 \\ & E & \end{tikzcd}$$

of H_v-modules and star homomorphism such that upper row is exact, there is a star homomorphism $\varphi : T_2 \longrightarrow E$ such that $\varepsilon^*(\varphi g(t_1)) = \varepsilon^*(f(t_1))$ for all $t_1 \in T_1$. Now we defined the maping $h : E \longrightarrow T_2$ by $h(e) \in gf^{-1}(e)$ for every $e \in E$. According to what was said in case H_v-modules star projective, since f^{-1} and g are star homomorphisms. Hence h is a star homomorphism. Now

$$w_{T_1} \longrightarrow E \xrightarrow{h} T_2 \longrightarrow \operatorname{Coker} h \longrightarrow w_{\operatorname{Coker} h}\ .$$

is an exact sequence. By hypothesis, the above exact sequence is split. So, there exists a star homomorphism $\varphi : T_2 \longrightarrow E$ such that $\varphi h \overset{w}{=} 1_E$. Therefore $\varphi h(f) \overset{w}{=} 1_E f \overset{w}{=} f$. Then $\varphi(hf) \overset{w}{=} \psi(g) \overset{w}{=} f$. hence $\varphi g \overset{w}{=} f$. then E is star injective. □

Proposition 4.6. *Let R be an H_v-ring. A direct product $\prod_{i\in I} E_i$ of H_v-modules is star injective if and only if each E_i is star injective.*

Proof. Let $\prod_{i\in I} E_i$ be star injective. We consider the diagram

$$\begin{tikzcd} w_A \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"] & B \\ & E & \end{tikzcd}$$

of H_v-modules and star homomorphism such that upper row is exact. Therefore we have

$$\begin{tikzcd} w_A \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"] & B \\ & E_i \arrow[d, shift left, "\iota_i"] & \\ & \prod_{i\in I} E_i \arrow[u, shift left, "\Pi_i"] & \end{tikzcd}$$

Since $\prod_{i\in I} E_i$ is star injective, it follows that there exists a star homomorphism $\varphi : B \longrightarrow \prod_{i\in I} E_i$ such that $\varphi g \overset{w}{=} \iota_i f$. We define the mapping $h : B \longrightarrow E_i$ by $h(x^{'}) \overset{w}{=} \Pi_i\varphi(x^{'})$ for every $x^{'} \in B$. Now for every $x \in A$ we have

$$hg(x) \overset{w}{=} \Pi_i\varphi g(x) \overset{w}{=} \Pi_i\iota_i f(x) \overset{w}{=} f(x).$$

Then $gh \overset{w}{=} f$. Therefore E_i is star injective. The converse can be proved by a similar techniques and using the diagram

$$\begin{tikzcd} w_A \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"] & B \\ & \prod_{i\in I} E_i \arrow[d, shift left, "\Pi_i"] & \\ & E_i \arrow[u, shift left, "\iota_i"] & \end{tikzcd}$$

If each E_i is star injective, then for each $i \in I$ there exists a star homomorphism $h_i : B \longrightarrow E_i$ such that $h_i g \overset{w}{=} \Pi_i f$. By Theorem 4.4 of [17], there is a unique star homomorphism $h : B \longrightarrow \prod_{i\in I} E_i$ with $\Pi_i h \overset{w}{=} h_i$ for every i. Hence $h \overset{w}{=} \iota_i h_i$. Then

$$hg \overset{w}{=} \iota_i h_i g \overset{w}{=} \iota_i \Pi_i f \overset{w}{=} f.$$

Verify that $gh = f$. □

Corollary 4.7. The following conditions on an H_v-ring R are equivalent:

(1) Every H_v-module is star projective.
(2) Every short exact sequence of H_v-modules is split exact.
(3) Every H_v-module is star injective.

Proof. (1⇒2) According to Proposition 4.2, the proof is obvious.

(3⇒2) According to Proposition 4.5, the proof is obvious.

(3⇒1) Suppose that P is star injective. We show that P is star projective. Since P is star injective, hence for every diagram

$$\begin{tikzcd} w_A \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"] & B \\ & P & \end{tikzcd} \tag{3}$$

of H_v-modules A and B and star homomorphisms, there is a star homomorphism $h^{'} : B \longrightarrow P$ such that diagram

$$\begin{tikzcd} w_A \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"] & B \\ & P & \end{tikzcd} \tag{4}$$

star commutative, i.e., $h^{'}f \overset{w}{=} g$. Now, we show that P is star projective. we consider diagram

$$\begin{tikzcd} & P \arrow[d, "k"] & \\ A \arrow[r, "f"] & B \arrow[r] & w_B \end{tikzcd}$$

of H_v-modules and star homomorphism such that bottom row is exact. We show that there is a star homomorphism $h : P \longrightarrow A$ such that $fh \overset{w}{=} k$. Since in the diagrams (4) and (5) row horizontal is exact, hence f is weak monic and weak epic. Then there is $f^{-1} : P \longrightarrow A$. Now, we consider the mapping $h = f^{-1}k : P \longrightarrow A$ which since f^{-1} and k are star homomorphisms, hence h is a star homomorphism and $fh \overset{w}{=} ff^{-1}k \overset{w}{=} k$. Then P is star projective. □

References

[1] Corsini, P., *Prolegomena of Hypergroup Theory*. Udine, Tricesimo, Italy: Second edition, Aviani editor, 1993.

[2] Corsini, P., and Leoreanu, V., *Applications of Hyperstructure Theory*, The Netherlands, Dordrecht: Kluwer Academic Publishers (Advances in Mathematics), 2003.

[3] Davvaz, B., and Ghadiri, M., *Weak equality and exact sequences in H_v-modules.* Southeast Asian Bull. Math., 25 (2001), 403-411.

[4] Davvaz, B., "Remarks on weak hypermodules", *Bull. Korean Math. Soc.*, **36**(1999), 599-608.

[5] B. Davvaz, "H_v-module of fractions", *Proc 8^{th} Algebra Seminar of Iranian Math Soc; 17-18 December; University of Tehran*, 1996, 37-46.

[6] Davvaz, B., "A brief survey of the theory of H_v-structures", *Proc 8^{th} International Congress on Algebraic Hyperstructures and Applications*, 1-9 September 2002; Samothraki, Greece, Spanidis: Press, 2003, 39-70.

[7] Davvaz, B., "Approximations in H_v-modules", *Taiwanese J. Math.*, **6**(2002), 499-505.

[8] Davvaz, B., and Leoreanu-Fotea, V., *Hyperring Theory and Applications*, USA: International Academic Press, 2007.

[9] Davvaz, B., and Vougiouklis, T., "n-ary H_v-modules with external n-ary P-hyperoperation", *Politehn Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, **76**(2014), 141-150.

[10] Ghadiri, M., and Davvaz, B., "Direct system and direct limit of H_v-modules", *Iran. J. Sci. Technol. Trans. A Sci.*, **28**(2004), 267-275.

[11] Marty, F., *Sur une generalization de la notion de groupe*, In Proceedings of the 8^{th} Congress des Mathematiciens; Scandinavia, Stockholm, Sweden: 1934, 45-49.

[12] Taghavi, A., Vougiouklis, T., and Hosseinzadeh, R., "A note on operators on normed finite dimensional weak hypervector spaces", *Politehn Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, **74**(2012), 103-108.

[13] Vougiouklis, T., *H_v-vector spaces*, Proc 5 th International Congress on Algebraic Hyperstructures and Applications; 4-10 July 1993; Iasi Romania: Hadronic Press, Inc Palm Harbor, 1994, pp. 181-190.

[14] Vougiouklis, T., *Hyperstructures and Their Representations*, Florida, USA: Hadronic Press Inc, Palm Harbor, 1994.

[15] Vougiouklis, T., *The fundamental relation in hyperrings*, The general hyperfield. Algebraic hyperstructures and applications (Xanthi, 1990), Teaneck, NJ: World Sci Publishing, 1991, pp. 203-211.

[16] Vaziri, Y., Ghadiri, M., and Davvaz, B., "The M[$-$] and [$-$]M functor and five short lemma in H_v-modules", *Turkish J. Math.*, **40**(2016), 397-410.

[17] Vaziri, Y., and Ghadiri, M., "Shanuels lemma, Snake lemma, product and direct sum in H_v-modules", *Turkish J. Math.*, to appear.