STAR PROJECTIVE AND STAR INJECTIVE $H_v$-MODULES

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Abstract. In this paper meantime the check and the defining concepts of product and direct sum, star projective and star injective in $H_v$-modules, we introduce a generalization extra of some notions in homological algebra to prove the five lemma and star projective and star injective Theorems in $H_v$-modules. We determine the conditions equivalent to split sequences in $H_v$-modules and also some interesting results on these concepts are given.

Key words and Phrases: $H_v$-module, direct product and direct sum, star projective, star injective, split sequence.

1. Introduction

A couple $(H, \circ)$ of a non-empty set $H$ and a mapping on $H \times H$ into the family of non-empty subsets of $H$ is called a hyperstructure (or hypergroupoid). A hypergroup is a hyperstructure $(H, \circ)$ with associative law: $(x \circ y) \circ z = x \circ (y \circ z)$ for every $x, y, z \in H$ and the reproduction axiom is valid: $x \circ H = H \circ x = H$ for every $x \in H$, i.e., for every $x, y \in H$ there exist $u, v \in H$ such that $y \in x \circ u$.
and \( y \in v \circ x \). This concept is introduced by Marty in 1934 [11]. If \( A \) and \( B \) are non-empty subsets of \( H \), then \( A \circ B \) is given by
\[
A \circ B = \bigcup_{a \in A} a \circ b.
\]

Also, \( x \circ A \) is used for \( \{x\} \circ A \) and \( A \circ x \) for \( A \circ \{x\} \). Hypermorphs, hypermodules and other hyperstructures are defined and several books have been written till now [1, 2, 8, 14]. The concept of \( H_v \)-structures as a larger class than the well known hyperstructures was introduced by Vougiouklis at fourth congress of AHA (Algebraic Hyperstructures and Applications) [15], where the axioms are replaced by the weak ones, that is instead of the equality on sets one has non-empty intersections. The basic definitions and results of \( H_v \)-structures can be found in [4, 5, 6, 7, 9, 10, 12, 13, 14]. The fundamental relations, weak equality, weak commutative, weak monic, weak epic, weak isomorphism, star homomorphism, star isomorph, direct product and direct sum, isomorph sequences and star projective and split sequences in \( H_v \)-modules are defined and is proved some results in [3, 14, 16, 17]. Also, some famous lemmas such as five short lemma, Snake lemma, Shanuels lemma are derived in the context of \( H_v \)-modules. The notions of \( M[-] \) and \(-[M]\) functors are introduced in [17] and the authors investigated the exactness of them and other problems.

The notion of exact sequences is a fundamental concept and it has been widely used in many areas such as ring and module theory. Our aim in this paper meantime the defining concept star injective, introduce a generalization extra of some notions in homological algebra to prove the five lemma and theorems star meantime the defining concept star injective, introduce a generalization extra of widely used in many areas such as ring and module theory. Our aim in this paper

2. Basic concepts

The hyperstructure \((H, \circ)\) is called an \( H_v \)-group if “\( \circ \)” is weak associative: \( x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset \) and the reproduction axiom is hold: \( x \circ H = H \circ x = H \) for every \( x \in H \). The \( H_v \)-group \( H \) is weak commutative if for every \( x, y \in H \), \( x \circ y \cap y \circ x \neq \emptyset \).

A multivalued system \((R, +, \cdot)\) is an \( H_v \)-ring if \((R, +)\) is a weak commutative \( H_v \)-group, \((R, \cdot)\) is a weak associative hyperstructure where “\( \cdot \)” hyperoperation is weak distributive with respect to “\( + \)” i.e. for every \( x, y, z \in R \) we have \( x \cdot (y + z) \cap (x \cdot y + x \cdot z) \neq \emptyset \) and \( (x + y) \cdot z \cap (x \cdot z + y \cdot z) \neq \emptyset \). A non-empty set \( M \) is a (left) \( H_v \)-module over an \( H_v \)-ring \( R \) if \((M, +)\) is a weak commutative \( H_v \)-group and there exists a map \( \cdot : R \times M \rightarrow P^\ast(M) \) denoted by \((r, m) \mapsto rm\) such that for every \( r_1, r_2 \in R \) and every \( m_1, m_2 \in M \) we have \( r_1(m_1 + m_2) \cap (r_1m_1 + r_1m_2) \neq \emptyset \), \((r_1 + r_2)m_1 \cap (r_1m_1 + r_2m_1) \neq \emptyset \) and \((r_1r_2)m_1 \cap r_1(r_2m_1) \neq \emptyset \).

A mapping \( f : M_1 \rightarrow M_2 \) of \( H_v \)-modules \( M_1 \) and \( M_2 \) over an \( H_v \)-ring \( R \) is a strong homomorphism (homomorphism) if for every \( x, y \in M_1 \) and every \( r \in R \) we have \( f(x + y) = f(x) + f(y) \) and \( f(rx) = rf(x) \).
By using a certain type of equivalence relations we can connect hyperstructures to ordinary structures.

Let \((R, +, \cdot)\) be an \(H_v\)-ring. Vougiouklis in [14] defined the relation \(\gamma^*\) as the smallest equivalence relation on \(R\) such that the quotient set \(R/\gamma^* = \{\gamma^*(r) \mid r \in R\}\) is a ring. The \(\gamma^*\) is called the fundamental equivalence relation on \(R\) and \(R/\gamma^*\) is called the fundamental ring. Let us denote the set of all finite polynomials of elements of \(R\), over \(\mathbb{N}\), by \(\mathcal{U}\). We define the relation \(\gamma\) as follows:

\[
x \gamma y \iff \{x, y\} \subseteq u, \text{ for some } u \in \mathcal{U}.
\]

**Theorem 2.1.** [14] The relation \(\gamma^*\) is the transitive closure of the relation \(\gamma\), and the addition and multiplication operations on \(R/\gamma^*\) are defined as follows:

\[
\gamma^*(a) + \gamma^*(b) = \gamma^*(c), \text{ for all } c \in \gamma^*(a) + \gamma^*(b),
\]

\[
\gamma^*(a) \cdot \gamma^*(b) = \gamma^*(c), \text{ for all } c \in \gamma^*(a) \cdot \gamma^*(b).
\]

Now, suppose that \(M\) is an \(H_v\)-module over an \(H_v\)-ring \(R\). Vougiouklis in [13] defined the relation \(\varepsilon^*\) as the smallest equivalence relation on \(M\) such that the quotient set \(M/\varepsilon^*(x) \mid x \in M\) is a module over the ring \(R/\gamma^*\). The relation \(\varepsilon^*\) is called the fundamental equivalence relation on \(M\) and \(M/\varepsilon^*\) is called the fundamental module. Let us denote \(\vartheta\) the set of all expressions consisting of finite hyperoperations either on \(R\) and \(M\) or the external hyperoperation applied on finite sets of elements of \(R\) and \(M\) [13]. We consider the relation \(\varepsilon\) on \(M\) as follows:

\[
x \varepsilon y \iff \{x, y\} \subseteq v, \text{ for some } v \in \vartheta.
\]

**Theorem 2.2.** [13] The relation \(\varepsilon^*\) is the transitive closure of the relation \(\varepsilon\), and the addition and external product on \(M/\varepsilon^*\) are defined as follows:

\[
\varepsilon^*(x) + \varepsilon^*(y) = \varepsilon^*(z), \text{ for all } z \in \varepsilon^*(x) + \varepsilon^*(y),
\]

\[
\varepsilon^*(r) \cdot \varepsilon^*(x) = \varepsilon^*(t), \text{ for all } t \in \varepsilon^*(r) \cdot \varepsilon^*(x).
\]

The heart of an \(H_v\)-module \(M\) over an \(H_v\)-ring \(R\) is denoted by \(\omega_M\) and is defined by \(\omega_M = \{x \in M \mid \varepsilon_M^*(x) = 0\}\), where 0 is the unit element of the group \((M/\varepsilon^*, \oplus)\). One can prove that the unit element of the group \((M/\varepsilon^*, \oplus)\) is equal to \(\omega_M\). By the definition of \(\omega_M\) we have \(\omega_{M_H} = \text{Ker}(\phi : \omega_M \longrightarrow \omega_M/\varepsilon_M^* = 0) = \omega_M\).

Let \(M_1\) and \(M_2\) be two \(H_v\)-modules over an \(H_v\)-ring \(R\) and \(\varepsilon_M^1, \varepsilon_M^2\) and \(\varepsilon^*\) be the fundamental relations on \(M_1, M_2\) and \(M_1 \times M_2\) respectively, then \((x_1, x_2) \varepsilon^*(y_1, y_2)\) if and only if \(x_1 \varepsilon^1_M(y_1)\) and \(x_2 \varepsilon^2_M(y_2)\) for all \((x_1, x_2), (y_1, y_2) \in M_1 \times M_2\) [13, 14].

Weak equality (monic, epic), exact sequences and relative results in \(H_v\)-modules are defined as follows [3]:

Let \(M\) be an \(H_v\)-module. The non-empty subsets \(X\) and \(Y\) of \(M\) are weakly equal if for every \(x \in X\) there exists \(y \in Y\) such that \(\varepsilon_M^*(x) = \varepsilon_M^*(y)\) and for every \(y \in Y\) there exists \(x \in X\) such that \(\varepsilon_M^*(x) = \varepsilon_M^*(y)\) that is denoted by \(X \equiv Y\).

The sequence

\[
M_o \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n
\]
of $H_v$-modules and strong homomorphisms is exact if for every $2 \leq i \leq n$,

$$\text{Im}(f_{i-1}) \cong \text{Ker}(f_i),$$

where $\text{Ker}(f_i) = \{a \in M_{i-1} \mid f_i(a) \in \omega_{M_i}\}$ (that is an $H_v$-submodule of $M_{i-1}$).

The strong homomorphism $f : M_1 \to M_2$ is called weak monic if for every $m_1, m_1' \in M_1$ the equality $f(m_1) = f(m_1')$ implies $\varepsilon_{M_1}^*(m_1) = \varepsilon_{M_1}^*(m_1')$ and $f$ is called weak epic if for every $m_2 \in M_2$ there exists $m_1 \in M_1$ such that $\varepsilon_{M_2}^*(m_2) = \varepsilon_{M_2}^*(f(m_1))$. Finally, $f$ is called weak-isomorphism if $f$ is weak monic and weak epic.

It is easy to see that every one to one (onto) strong homomorphism is weak monic (weak epic), but the converse is not true necessarily. In fact the concept of weak monic (weak epic) is a generalization of the concept of one to one (onto).

Let $f : A \to B$ be a strong homomorphism of $H_v$-modules over an $H_v$-ring $R$. Then, we have $f(\omega_A) \subseteq \omega_B$ and so $\omega_A \subseteq \text{Ker}(f)$. Moreover, $\text{Ker}(f) = \omega_A$ if and only if $f$ is weak monic.

A mapping $f : M_1 \to M_2$ of $H_v$-modules $M_1$ and $M_2$ over an $H_v$-ring $R$ is called a star homomorphism if for every $x, y \in M_1$ and every $r \in R$: $\varepsilon_{M_2}^*(f(x+y)) = \varepsilon_{M_2}^*(f(x) + f(y))$ and $\varepsilon_{M_2}^*(f(rx)) = \varepsilon_{M_2}^*(rf(x))$; i.e. $f(x+y) \equiv f(x) + f(y)$ and $f(rx) \equiv rf(x)$.

Two mappings $f, g : M \to N$ on $H_v$-modules are called weak equal if for every $m \in M$: $\varepsilon_N^*(f(m)) = \varepsilon_N^*(g(m))$ and denote by $f \equiv g$. The following diagram of $H_v$-modules and strong homomorphisms is called star commutative if $g \circ f \equiv h$.

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
C
\end{array}$$

Also, it is called commutative if for every $a \in A$, $g \circ f(a) = h(a)$.

The sequences

$$\omega_A \to A \xrightarrow{f} B \xrightarrow{g} C \to \omega_C$$

and

$$\omega_A' \to A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to \omega_C'$$

are called isomorph (star isomorph) if there exist weak-isomorphisms (star homomorphisms) $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : C \to C'$ such that the following diagram is commutative (star commutative):

$$\begin{array}{ccc}
\omega_A & \xrightarrow{f} & B \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\omega_A' & \xrightarrow{f'} & B' \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
C & \xrightarrow{g} C' \to \omega_C' \end{array}$$
According to [3] for every strong homomorphism \( f : M \rightarrow N \) there is the \( R/\gamma^* \)-homomorphism \( F : M/\varepsilon_M^* \rightarrow N/\varepsilon_N^* \) of \( R/\gamma^* \)-modules defined by \( F(\varepsilon_M^*(m)) = \varepsilon_N^*(f(m)) \).

Let \( M \) be an \( H_e \)-module and \( H, K \) be \( H_e \)-submodules of \( M \). Then \( M \) is called the direct sum of \( H \) and \( K \) if \( H \cap K \subseteq \omega_M \) and \( \varepsilon^*(H + K) = \varepsilon^*(M) \). We denote it by \( H \oplus K = M \).

Then, \( f \) is weak epic if and only if \( F \) is onto. Moreover, \( f \) is weak monic if and only if \( F \) is one to one. Finally, \( f \) is a weak-isomorphism if and only if \( F \) is an isomorphism.

3. PRODUCT AND DIRECT SUM IN \( H_e \)-MODULES

In this section, we meantime the check concept product and direct sum, we introduce a generalization of some notions in homological algebra to prove the five lemma in \( H_e \)-modules. Also, we determine the conditions equivalent to split the exact sequences in \( H_e \)-modules and some interesting results on these concepts are given.

**Proposition 3.1.** Let \( f : M \rightarrow N \) and \( g : N \rightarrow M \) be strong homomorphisms of \( H_e \)-modules such that \( gf = 1 \). Then, \( N \) is the direct sum of \( \text{Im}(f) \) and \( \text{Ker}(f) \).

**Proof.** Let \( m \in \text{Im}(f) \cap \text{Ker}(g) \). Then \( m = f(m_1) \) for some \( m_1 \in M \) and

\[
g(m) \in \omega_N.
\]

By applying \( f \) on Eq. (1) we obtain

\[
g(m) = gf(m_1) = m_1.
\]

hence \( m_1 \in \omega_N \). Since \( f \) is a strong homomorphism, it follows that \( m = f(m_1) \in \omega_N \). So, \( \text{Im}(f) \cap \text{Ker}(f) \subseteq \omega_N \). Now, for every \( m \in M \) we have:

\[
G(F(\varepsilon^*(m))) = G(\varepsilon^*(f(m))) = \varepsilon^*(gf(m)) = \varepsilon^*(m).
\]

So \( \text{Im}(F) + \text{Ker}(G) = N/\varepsilon^*_N \), since \( G \) and \( F \) are \( R/\gamma^* \)-homomorphisms such that \( GF = 1 \). Therefore, \( \varepsilon^*(\text{Im}(f) + \text{Ker}(g)) = \varepsilon^*(N) \). \( \square \)

\[\prod_{i \in I} M_i\] is called the (external) direct product of the family of \( H_e \)-modules \( \{M_i \mid i \in I\} \) and \( \sum_{i \in I} M_i \) is its (external) direct sum. If the index set is finite, say \( I = \{1, 2, \cdots, n\} \), then the product and direct sum coincide and will be written \( M_1 \oplus M_2 \oplus \cdots \oplus M_n \). The maps \( \Pi_k \) [resp. \( \iota_k \)] are called the canonical projections [resp. injections]. Let \( \{M_i\}_{i \in I} \) be a non-empty collection of \( H_e \)-modules. The product of this collection \( \prod_{i \in I} M_i = \{(x_i) \mid x_i \in M_i; \forall i \in I\} \), with the following hyperoperations is an \( H_e \)-module: \( (x_i) + (y_i) = \{(z_i) \mid z_i \in x_i + y_i\} \) and \( r(x_i) = \{(aw_i) \mid w_i \in r x_i\} \). Similarly, we define hyperoperations on \( \sum_{i \in I} M_i \) which if \( 0 \neq \{m_i\} \in \sum_{i \in I} M_i \), then only finitely many of the \( a_i \) are nonzero, say \( a_{i_1}, a_{i_2}, \cdots, a_{i_v} \).

**Proposition 3.2.** Let \( \{M_i\} \) be a non-empty collection of \( H_e \)-modules. For every \( H_e \)-module \( X \) and every collection of strong homomorphisms \( \{\psi_i : M_i \rightarrow X\} \) there
exists an unique strong homomorphism \( \psi : \sum_{i \in I} M_i \rightarrow X \) defined by \( \psi_i = \psi_{t_i} \) such that for every \( i \in I \) the following diagram is commutative.

\[
\begin{array}{ccc}
\sum_{i \in I} M_i & \xrightarrow{t_i} & M_i \\
\downarrow{\psi_i} & \searrow{\psi} & \\
X & \rightarrow & X
\end{array}
\]

**Proof.** The proof is straightforward. \( \square \)

**Proposition 3.3.** Let \( R \) be a \( H_v \)-ring and \( A, A_1, A_2, \cdots, A_n \) \( H_v \)-modules. Then \( A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \) if and only if for each \( i = 1, 2, \cdots, n \) there are strong homomorphisms \( \Pi_i : A \rightarrow A_i \) and \( t_i : A_i \rightarrow A \) such that for every \( i \)

1. \( \Pi_i t_i = 1_A \) for \( i = 1, 2, \cdots, n \);
2. \( \Pi_j t_i = 0 \) for \( i \neq j \);
3. \( t_1 \Pi_i + t_1 \Pi_1 + \cdots + t_1 \Pi_1 = 1_A \).

**Proof.** (\( \Rightarrow \)): If \( A \) is the \( H_v \)-module \( A_1 \oplus A_2 \oplus \cdots \oplus A_n \), then the canonical injections \( t_i \) and projections \( \Pi_i \) satisfy (1)-(3) as the readers may easily verify. Likewise if \( A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \) under an weak isomorphism \( A \rightarrow A_1 \oplus A_2 \oplus \cdots \oplus A_n \) then the homomorphisms \( \Pi_i f : A \rightarrow A_i \) and \( f^{-1} t_i : A_i \rightarrow A \) satisfy (1)-(3).

(\( \Leftarrow \)): Suppose that \( \Pi_i : A \rightarrow A_i \) and \( t_i : A_i \rightarrow A \) (\( i = 1, \cdots, n \)) satisfy (1)-(3). Let \( \Pi'_i : A_1 \oplus A_2 \oplus \cdots \oplus A_n \rightarrow A_i \) and \( t'_i : A_i \rightarrow A_1 \oplus A_2 \oplus \cdots \oplus A_n \) be the canonical projections and injections. Let \( \varphi : A_1 \oplus A_2 \oplus \cdots \oplus A_n \rightarrow A \) be given by \( \varphi = t_1 \Pi'_1 + t_2 \Pi'_2 + \cdots + t_n \Pi'_n \) and \( \psi : A \rightarrow A_1 \oplus A_2 \oplus \cdots \oplus A_n \) by \( \psi = t'_1 \Pi_1 + t'_2 \Pi_2 + \cdots + t'_n \Pi_n \). Then

\[
\varphi \psi = \sum_{i=1}^{n} t_i \Pi_i \sum_{j=1}^{n} t'_j \Pi_j = \sum_{i=1}^{n} \sum_{j=1}^{n} t_i \Pi_i' t_j \Pi_j = \sum_{i=1}^{n} t_i \Pi_i', t_j \Pi_j = \sum_{i=1}^{n} t_i 1_A \Pi_i = \sum_{i=1}^{n} \Pi_i A_i = 1_A
\]

\( \square \)

**Example 1.** Note first that for any \( H_v \)-module \( A \), there are unique strong homomorphisms \( \omega_A \rightarrow A \) and \( A \rightarrow \omega_A \). If \( A \) and \( B \) are any \( H_v \)-modules then the sequences

\[
\omega_A \rightarrow A \xrightarrow{i} A \oplus B \xrightarrow{\Pi} B \rightarrow \omega_B \text{ and } \omega_B \rightarrow B \xrightarrow{i} B \oplus A \xrightarrow{\Pi} A \rightarrow \omega_A
\]

are exact, where the \( i \) and \( \Pi \) are the canonical injections and projections respectively. Similarly, if \( C \) is a submodule of \( D \), then the sequence

\[
\omega_C \rightarrow C \rightarrow D \rightarrow D/C \rightarrow \omega_{D/C}
\]

is exact, where \( i \) is the inclusion map and \( p \) is the canonical epimorphism.
Proposition 3.4. (Five Lemma in Hv-modules) Let

\[
\begin{array}{cccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\alpha_1 & \downarrow & \alpha_2 & \downarrow & \alpha_3 & \downarrow & \alpha_4 & \downarrow & \alpha_5 \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
\end{array}
\]

be a commutative diagram of Hv-modules and Hv-homomorphisms over an Hv-ring \(R\) with both rows exact. Then,

1. if \(\alpha_1\) is weak monic and \(\alpha_2, \alpha_4\) weak epic then \(\alpha_3\) is weak epic.
2. if \(\alpha_3\) is weak epic and \(\alpha_2, \alpha_4\) weak monic then \(\alpha_3\) is weak monic.
3. if \(\alpha_1\) is weak monic, \(\alpha_5\) weak epic and \(\alpha_2, \alpha_4\) weak isomorphism then \(\alpha_3\) is weak isomorphism.

Proof. (1) By Theorem 4.8 of [4] and Lemma 3.4 of [16] the following diagram of \(R/\gamma\)-modules and \(R/\gamma\)-homomorphisms is commutative with both rows exact:

\[
\begin{array}{cccccc}
A_1/\varepsilon^+ & \xrightarrow{\pi_1} & A_2/\varepsilon^+ & \xrightarrow{\pi_2} & A_3/\varepsilon^+ & \xrightarrow{\pi_3} & A_4/\varepsilon^+ & \xrightarrow{\pi_4} & A_5/\varepsilon^+ \\
\phi_1 & \downarrow & \phi_2 & \downarrow & \phi_3 & \downarrow & \phi_4 & \downarrow & \phi_5 \\
B_1/\varepsilon^+ & \xrightarrow{\gamma_1} & B_2/\varepsilon^+ & \xrightarrow{\gamma_2} & B_3/\varepsilon^+ & \xrightarrow{\gamma_3} & B_4/\varepsilon^+ & \xrightarrow{\gamma_4} & B_5/\varepsilon^+
\end{array}
\]

By Lemma 4.5 of [16], \(\pi_1\) is monic and \(\pi_2, \pi_4\) epic \(R/\gamma\)-homomorphisms. By five lemma in modules \(\pi_3\) is monic \(R/\gamma\)-homomorphism. Thus, by Lemma 4.5 of [16], \(\alpha_3\) is weak monic \(R\)-homomorphism.

(2) The proof is similar to the proof of (1).

(3) The proof follows from (1) and (2). \(\square\)

Proposition 3.5.  
1. Let \(M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3\) be an exact sequence of Hv-modules and Hv-homomorphisms such that \(\varphi\) is epic and \(\psi\) is weak monic, then \(M_2 \equiv \omega_{M_2}\).

2. Let \(M_1 \xrightarrow{\varphi} M_2 \xrightarrow{f} M_3 \xrightarrow{\psi} M_4\) be an exact sequence of Hv-modules and Hv-homomorphisms. Then \(\varphi\) is weak epic if and only if \(\psi\) is weak monic.

3. Let \(M_1 \xrightarrow{\varphi} M_2 \xrightarrow{f} M_3 \xrightarrow{g} M_4 \xrightarrow{\psi} M_5\) be an exact sequence of Hv-modules and Hv-homomorphisms such that \(\varphi\) is weak epic and \(\psi\) is weak monic, then \(M_3 \equiv \omega_{M_3}\).

Proof. (1) Let \(m_2 \in M_2\). Since \(\varphi\) is epic, there is \(m_1 \in M_1\) such that \(\varphi(m_1) = m_2\). Hence \(M_2 = \text{Im } \varphi \equiv \text{Ker } \psi\). Since \(\psi\) is weak monic, thus \(\text{Ker } \psi \equiv \omega_{M_2}\). Then \(M_2 \equiv \omega_{M_2}\).

(2) Let \(\varphi\) be weak epic and \(x \in \text{Ker } \psi\). We have \(\text{Im } \varphi \equiv \text{Ker } f\) and \(\text{Im } f \equiv \text{Ker } \psi\).

Hence, there is \(m_2 \in M_2\) such that \(f(m_2) = x\). Since \(\varphi\) is weak epic, thus there is \(m_1 \in M_1\) such that \(\varphi(m_1) = m_2\). Then \(x \equiv f(\varphi(m_1)) \equiv \omega_{M_3}\). There for
Proposition 3.6. Let

\[
\begin{array}{c}
M' \xrightarrow{f} M \xrightarrow{g} M'' \\
\downarrow \alpha \quad \downarrow \beta \quad \downarrow \gamma \\
\omega_{N'} \xrightarrow{\varphi} N' \xrightarrow{\psi} N''
\end{array}
\]

be a commutative diagram of \(H_v\)-modules and \(H_v\)-homomorphisms over an \(H_v\)-ring \(R\) with both rows exact. Then, there is the exact sequence

\[
\text{Ker } \alpha \xrightarrow{\overline{f}} \text{Ker } \beta \xrightarrow{\overline{g}} \text{Ker } \gamma \xrightarrow{d} N' \xrightarrow{\overline{d}} N' \xrightarrow{\overline{\psi}} N'' \xrightarrow{\overline{\psi}} N''/\text{Im } \gamma.
\]

Withal, if \(f\) is weak monic, then \(\overline{f}\) is weak monic and if \(\psi\) is weak epic, then \(\overline{\psi}\) is weak epic.

Proof. We define \(\overline{\varphi}(n' + \text{Im } \alpha) \equiv \varphi(n') + \text{Im } \beta\) and \(\overline{\psi}(n + \text{Im } \beta) \equiv \psi(n) + \text{Im } \gamma\). Also we define \(\overline{\varphi}\) and \(\overline{\psi}\), scowl the mapping \(f\) and \(g\) on \(\text{Ker } \alpha\) and \(\text{Ker } \beta\) respectively. It can be easily we seen that \(\overline{\varphi}, \overline{\psi}, \overline{f}\) and \(\overline{g}\) are well define also \(\text{Im } \overline{f} \equiv \text{Im } f\), \(\text{Ker } \overline{\varphi} \equiv \text{Ker } \varphi\), \(\text{Im } \overline{\varphi} \equiv \text{Im } \varphi\), \(\text{Ker } \overline{\psi} \equiv \text{Ker } \psi\). We have \(\text{Im } \overline{\psi} \equiv \text{Ker } \overline{\psi}\) and \(\text{Im } \overline{\psi} \equiv \text{Ker } \overline{\psi}\). Now, we defining the mapping \(d : \text{Ker } \gamma \xrightarrow{\mu} N'/\text{Im } \alpha\) by \(d(m') = n' + \text{Im } \alpha\). We show that \(d\) is well define. If \(m' \in \text{Ker } \gamma\), then \(\gamma(m') \equiv \omega_{N'}\). Since \(g\) is weak epic, then there is \(m \in M\) such that \(g(m) \equiv m''\). Thus \(\gamma(g(m)) \equiv \omega_{N''}\). We obtain \(\psi(\beta(m)) \equiv \omega_{N''}\). Hence \(\beta(\text{Im } \psi) \equiv \text{Im } \varphi\). Therefor, there is \(n' \in N'\) such that \(\beta(m) \equiv \varphi(n')\). Since \(\varphi\) is weak monic, then \(n'\) is unique. Thus, if \(m_1' = m_2'\) we obtain \(g(m_1') \equiv g(m_2')\). Then there is \(n_1', n_2' \in N'\) such that \(\varphi(n_1') \equiv \beta(m_1')\) and \(\varphi(n_2') \equiv \beta(m_2')\). Thus \(n_1' + \text{Im } \alpha \equiv n_2' + \text{Im } \alpha\). Then \(d\) is well define. It can be easily we seen that \(d\) is a \(H_v\)-homomorphism.

Now, we have \(\text{Im } d = \{n' : n' = \text{Im } \alpha \mid m' \in \text{Ker } \gamma\}\) and \(\text{Ker } \overline{d} = \{n' : n' = \text{Im } \alpha \mid \varphi(n') + \text{Im } \beta \equiv \omega_{N'}\}\). If \(m'' \in \text{Ker } \gamma\), then \(\gamma(m'') \equiv \omega_{N''}\). Since \(g\) is weak epic, there is \(m \in M\) such that \(g(m) \equiv m''\). Thus \(\gamma(g(m)) \equiv \omega_{N''}\). We obtain \(\psi(\beta(m)) \equiv \omega_{N''}\). Hence \(\beta(\text{Im } \psi) \equiv \text{Im } \varphi\). Therefor there is \(n' \in N'\) such that \(\beta(m) \equiv \varphi(n')\in \text{Im } g\). Then \(\text{Ker } \overline{\varphi} \equiv \text{Im } d\). Also, \(\text{Ker } \overline{d} = \{m'' : m'' \in \text{Ker } \gamma \mid m'' + \text{Im } \alpha \equiv \omega_{N''}\}\). Thus \(\text{Im } \overline{d} \equiv N'/\text{Im } \gamma\). Then we obtain the exact sequence

\[
\text{Ker } \alpha \xrightarrow{\overline{f}} \text{Ker } \beta \xrightarrow{\overline{g}} \text{Ker } \gamma \xrightarrow{d} N' \xrightarrow{\overline{d}} N' \xrightarrow{\overline{\psi}} N'' \xrightarrow{\overline{\psi}} N''/\text{Im } \gamma.
\]
Now, if $f$ is weak monic, since $\mathcal{F}$ is scowl the mapping $f$ on $\text{Ker} \, \alpha$, hence $\mathcal{F}$ is weak monic. Let $\psi$ be weak epic. Consider $n'' \in N''$. Then, $n'' + \text{Im} \, \gamma \in N'' / \text{Im} \, \gamma$. Since $\psi$ is weak epic, then there is $n \in N$ such that $\varphi(n) = n''$ and $n + \text{Im} \, \beta \in N / \text{Im} \, \beta$. We obtain $\bar{\psi}(n + \text{Im} \, \beta) \equiv n'' + \text{Im} \, \gamma$. Then $\psi$ is weak epic.

**Proposition 3.7.** Let

\[
\begin{array}{cccccc}
\omega_K' \\
\downarrow \\
K' \\
\downarrow \alpha \\
\omega_K \\
\downarrow \\
K \\
\downarrow \beta \\
M \\
\downarrow \\
C \\
\downarrow \\
\omega_C \\
\downarrow \\
C' \\
\downarrow \\
\omega_{C'}
\end{array}
\]

be a star commutative diagram of $H_v$-modules and strong homomorphisms which rows horizontal and diagonal are exact. If $\alpha$ is weak monic and $\beta$ weak epic, then $K' / \text{Im} \, \alpha \neq \omega_{K'}$ if and only if $\text{Ker} \, \beta \neq \omega_C$.

**Proof.** Let $x + \text{Im} \, \alpha \in K' / \text{Im} \, \alpha$ such that $x \in K'$ and $x \notin \text{Im} \, \alpha$. On the other hand $g_1(\alpha) = f_1$, $\beta f_2 = g_2, g_1(x) \in M$ and $f_2 g_1(x) \in C$. hence $\beta f_2 g_1(x) = g_2 g_1(x) = \omega_{C'}$. therefore $f_2 g_1(x) \in \text{Ker} \, \beta$. If $f_2 g_1(x) = \omega_{C'}$, then $g_1(x) \in \text{Ker} \, f_2 \equiv \text{Im} \, f_1$. hence there is $t \in K$ such that $g_1(x) = f_1(t)$. Thus $g_1(x) \equiv g_1 \alpha(t)$. Since $g_1$ is weak monic, then $x \equiv \alpha(t)$. therefor $x \in \text{Im} \, \alpha$ which is a contradiction. hence $K' / \text{Im} \, \alpha \neq \omega_{K'}$.

For the converse, let $\text{Ker} \, \beta \notin \omega_C$. We show that $K' / \text{Im} \, \alpha \neq \omega_{K'}$. Let $x \in \text{Ker} \, \beta$. Since $f_2$ is weak monic, it follows that there is $m \in M$ such that $f_2(m) = x$ for every $x \in C$. hence $\beta f_2(m) \equiv \omega_{C'}$, then $g_2(m) \equiv \omega_{C'}$. Thes $m \in \text{Ker} \, g_2 \equiv \text{Im} \, g_1$. therefor there is $k' \in K'$ such that $m = g_1(k')$. We show that $k' \notin \text{Im} \, \alpha$. If $k' \in \text{Im} \, \alpha$, then there is $k \in K$ such that $k' \equiv \alpha(k)$. Thus $g_1(\alpha(k)) \equiv m$. therefor $f_1(k) \equiv m$. Then $f_2(m) = f_2 f_1(k) \equiv \omega_C$. Hence $k' \in \text{Im} \, \alpha$. Then $K' / \text{Im} \, \alpha \neq \omega_{K'}$.

**Proposition 3.8.** Let $M_1, M_2$ and $M$ be three $H_v$-modules and the sequence

\[
\omega_{A_1} \longrightarrow A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \longrightarrow \omega_{A_2}
\]

is exact. Then the following conditions are equivalent.
There is a star homomorphism $h : A_2 \rightarrow B$ with $gh = 1_{A_2}$.

(2) There is a star homomorphism $k : B \rightarrow A_1$ with $kf = 1_{A_1}$.

(3) the given sequence is star isomorphic (with identity maps on $A_1$ and $A_2$) to the direct sum short exact sequence

$\omega_{A_1} \rightarrow A_1 \rightarrow A_1 \oplus A_2 \rightarrow A_2 \rightarrow \omega_{A_2};$

in particular $B \cong A_1 \oplus A_2$.

Proof. The proof is similar to the proof of Theorem 4.7 in [17]. \hfill \square

4. Star projective and star injective $H_v$-Modules

In this section we define concepts star projective and star injective in $H_v$-modules and introduce a generalization extra of some notions in homological algebra to prove Theorems star projective and star injective in $H_v$-modules.

Definition 4.1. [17] An $H_v$-module $P$ is a star projective if for every diagram of star homomorphisms and $H_v$-modules as follows

\[ P \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{\omega} \omega_N \]

that its row is exact, there exists a star homomorphism $\phi : P \rightarrow M$ such that for each $p \in P$, $\varepsilon_N(g(\phi(p))) = \varepsilon_N(f(p))$.

Proposition 4.2. Let $R$ be an $H_v$-ring. Then,

(1) If $P$ is a star projective $H_v$-module, then every short exact sequence

$\omega_{A} \rightarrow A \xrightarrow{f} B \xrightarrow{g} P \rightarrow \omega_P$

is split exact (hence $B \cong A \oplus P$);

(2) If every short exact sequence

$\omega_{A} \rightarrow A \xrightarrow{f} B \xrightarrow{g} P \rightarrow \omega_P$

is split exact, then $P$ is a star projective $H_v$-module.

Proof. (1) Consider the diagram

\[ P \xrightarrow{f} B \xrightarrow{g} P \rightarrow \omega_P \]
of star homomorphisms and with bottom row exact. Since \( P \) is star projective, it follows that there exists a star homomorphism \( h : P \longrightarrow B \) such that \( \varepsilon^*(gh(p)) = \varepsilon^*(1_P(p)) \) for all \( p \in P \). Therefore, the short exact sequence

\[
\begin{array}{c}
\xrightarrow{w_A} A \xrightarrow{f} B \xrightarrow{g} P \xrightarrow{w_P}
\end{array}
\]

is split exact by Theorem 3.8 and \( B \cong A \oplus P \).

(2) Suppose that every short exact sequence

\[
\begin{array}{c}
\xrightarrow{w_A} A \xrightarrow{f} B \xrightarrow{g} P \xrightarrow{w_P}
\end{array}
\]

is split exact. We show that \( P \) is a star projective \( H_v \)-module. So, we show that for every diagram

\[
\begin{array}{c}
P \\
\downarrow f \\
T_1 \xrightarrow{g} T_2 \xrightarrow{w} T_2
\end{array}
\]

of \( H_v \)-modules and star homomorphism such that bottom row is exact, there is a star homomorphism \( \varphi : P \longrightarrow T_1 \) such that \( \varepsilon^*(g\varphi(p)) = \varepsilon^*(f(p)) \) for all \( p \in P \). Now, we consider the mapping \( h : T_1 \longrightarrow P \) by \( h(t) \in f^{-1}g(t) \) for every \( t \in T_1 \). Since \( f^{-1} \) and \( g \) are star homomorphisms. Hence \( h \) is star homomorphism. Now, we have the exact sequence

\[
\begin{array}{c}
w_{T_1} \longrightarrow \ker h \longrightarrow T_1 \xrightarrow{h} P \longrightarrow w_{T_2}.
\end{array}
\]

The exact sequence (3) is split, hence there exists a star homomorphism \( \psi : P \longrightarrow T_1 \) such that \( h\psi \cong 1_P \). Therefore \( f(h\psi) \cong f1_P \cong f \). Then \( (fh)\psi \cong f \). Hence \( g\psi \cong f \). \( \square \)

**Proposition 4.3.** Let \( R \) be an \( H_v \)-ring. A direct sum \( \sum_{i \in I} P_i \) of \( H_v \)-modules is star projective if and only if each \( P_i \) is star projective.

**Proof.** Let \( \sum_{i \in I} P_i \) be star projective. We consider the diagram

\[
\begin{array}{c}
P_i \\
\downarrow f \\
A \xrightarrow{g} B \xrightarrow{w} w_B
\end{array}
\]
of $H_{\nu}$-modules and star homomorphisms such that bottom row is exact. Therefore, there is the diagram

\[
\begin{array}{c}
\sum_{i \in I} P_i \\
\downarrow \pi_i \\
P_i \\
\downarrow f \\
A \xrightarrow{g} B \xrightarrow{w_B} \\
\end{array}
\]

of $H_{\nu}$-modules and star homomorphisms. Since $\sum_{i \in I} P_i$ is star projective, it follows that there exists a strong homomorphism $\varphi : \sum_{i \in I} P_i \longrightarrow A$ such that $g\varphi \equiv f\Pi_i$. We define the mapping $h : \sum_{i \in I} P_i \longrightarrow A$ by $h(x) \equiv \varphi_i(x)$ for every $x \in P_i$. Now, we have

\[gh(x) \equiv g\varphi_i(x) \equiv f\Pi_i\varphi_i(x) \equiv f(x).\]

Then $gh \equiv f$. Nothing that $h$ is a star homomorphism. Therefore, $P_i$ is star projective. The converse is proved by a similar techniques and using the diagram

\[
\begin{array}{c}
P_i \\
\uparrow \iota_i \\
\sum_{i \in I} P_i \\
\downarrow f \\
A \xrightarrow{g} B \xrightarrow{w_B} \\
\end{array}
\]

If each $P_i$ is star projective, then for each $i$ there exists a star homomorphism $h_i : P_i \longrightarrow A$ such that $gh_i \equiv f\iota_i$. By Theorem 3.2 there is a unique star homomorphism $h : \sum_{i \in I} P_i \longrightarrow A$ with $h\iota_i \equiv h_i$ for every $i \in I$. Hence $h \equiv h_i\Pi_i$. Then

\[gh \equiv gh_i\Pi_i \equiv f\iota_i\Pi_i \equiv f.\]

Verify that $gh = f$. □

**Definition 4.4.** An $H_{\nu}$-module $P$ is a star injective if for every diagram of star homomorphisms and $H_{\nu}$-modules as follows

\[
\begin{array}{c}
w_N \\
\downarrow j \\
N \xrightarrow{g} M \\
\downarrow f \\
E \\
\end{array}
\]

that it’s row is exact, there exists a star homomorphism $\phi : M \longrightarrow E$ such that for each $n \in N$, $\varepsilon_N(\phi(g(n))) = \varepsilon_N(f(n))$. 
Proposition 4.5. Let $R$ be an $H_v$-ring. Then,

1. If $E$ is a star injective $H_v$-module, then every short exact sequence

$$\omega_E \longrightarrow E \longrightarrow A \longrightarrow g \longrightarrow B \longrightarrow \omega_B$$

is split (hence $A \cong B \bigoplus E$);
2. If every short exact sequence

$$w_E \longrightarrow E \longrightarrow A \longrightarrow g \longrightarrow B \longrightarrow w_B$$

is split, then $E$ is a star injective $H_v$-module.

Proof. (1$\Rightarrow$2) Consider the diagram

$$w_E \longrightarrow E \longrightarrow A \quad \xrightarrow{1_E} \quad E \quad \xrightarrow{1_E}$$

of $H_v$-modules and star homomorphisms such that upper row exact. Since $E$ is star injective, it follows that there is a star homomorphism $h : A \longrightarrow E$ such that $\varepsilon^*(hg(e)) = \varepsilon^*(1_E(e))$ for all $e \in E$. Therefore, the short exact sequence

$$w_E \longrightarrow E \longrightarrow A \longrightarrow g \longrightarrow B \longrightarrow w_B$$

is split, by Theorem 3.8 and $A \cong B \bigoplus E$.

(2$\Rightarrow$1) Suppose that every short exact sequence

$$w_E \longrightarrow E \longrightarrow A \longrightarrow g \longrightarrow B \longrightarrow w_E$$

is split. We show that for every diagram

$$w_{T_1} \longrightarrow T_1 \longrightarrow T_2 \quad \xrightarrow{f} \quad E \quad \xrightarrow{f}$$

of $H_v$-modules and star homomorphism such that upper row is exact, there is a star homomorphism $\varphi : T_2 \longrightarrow E$ such that $\varepsilon^*(\varphi g(t)) = \varepsilon^*(f(t))$ for all $t \in T_1$. Now we defined the mapping $h : E \longrightarrow T_2$ by $h(e) \in gf^{-1}(e)$ for every $e \in E$. According to what was said in case $H_v$-modules star projective, since $f^{-1}$ and $g$ are star homomorphisms. Hence $h$ is a star homomorphism. Now

$$w_{T_1} \longrightarrow E \longrightarrow h \longrightarrow T_2 \longrightarrow \text{Coker } h \longrightarrow w_{\text{Coker } h}$$

is an exact sequence. By hypothesis, the above exact sequence is split. So, there exists a star homomorphism $\varphi : T_2 \longrightarrow E$ such that $\varphi h \cong 1_E$. Therefore $\varphi h(f) \cong 1_E f \cong f$. Then $\varphi(hf) \cong \psi(g) \cong f$. hence $\varphi y \cong f$, then $E$ is star injective. \[\square\]
Proposition 4.6. Let $R$ be an $H_v$-ring. A direct product $\prod_{i \in I} E_i$ of $H_v$-modules is star injective if and only if each $E_i$ is star injective.

Proof. Let $\prod_{i \in I} E_i$ be star injective. We consider the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow{f} & & \downarrow{E} \\
E_i & & \Pi_i E_i
\end{array}
$$

of $H_v$-modules and star homomorphism such that upper row is exact. Therefore we have

$$
\begin{array}{ccc}
w_A & \xrightarrow{A} & A \\
\downarrow{f} & & \downarrow{E_i} \\
E_i & & \Pi_i E_i
\end{array}
$$

Since $\prod_{i \in I} E_i$ is star injective, it follows that there exists a star homomorphism $\varphi : B \rightarrow \prod_{i \in I} E_i$ such that $\varphi g \equiv \Pi_i f$. We define the mapping $h : B \rightarrow E_i$ by $h(x') \equiv \Pi_i \varphi(x')$ for every $x' \in B$. Now for every $x \in A$ we have

$$
hg(x) \equiv \Pi_i \varphi g(x) \equiv \Pi_i \Pi_i f(x) \equiv f(x).
$$

Then $gh \equiv f$. Therefore $E_i$ is star injective. The converse can be proved by a similar techniques and using the diagram

$$
\begin{array}{ccc}
w_A & \xrightarrow{A} & A \\
\downarrow{f} & & \downarrow{E_i} \\
\Pi_i E_i & & E_i
\end{array}
$$

If each $E_i$ is star injective, then for each $i \in I$ there exists a star homomorphism $h_i : B \rightarrow E_i$ such that $h_i g \equiv \Pi_i f$. By Theorem 4.4 of [17], there is a unique star homomorphism $h : B \rightarrow \prod_{i \in I} E_i$ with $\Pi_i h \equiv h_i$ for every $i$. Hence $h \equiv \Pi_i h_i$. Then

$$
hg \equiv \Pi_i h_i g \equiv \Pi_i \Pi_i f \equiv f.
$$
Verify that $gh = f$. □

**Corollary 4.7.** The following conditions on an $H_v$-ring $R$ are equivalent:

1. Every $H_v$-module is star projective.
2. Every short exact sequence of $H_v$-modules is split exact.
3. Every $H_v$-module is star injective.

**Proof.** (1$\Rightarrow$2) According to Proposition 4.2, the proof is obvious.

(3$\Rightarrow$2) According to Proposition 4.5, the proof is obvious.

(3$\Rightarrow$1) Suppose that $P$ is star injective. We show that $P$ is star projective.

Since $P$ is star injective, hence for every diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{g} \\
P & & P
\end{array}
\] (3)

of $H_v$-modules $A$ and $B$ and star homomorphisms, there is a star homomorphism $h' : B \rightarrow P$ such that diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{g} \\
P & & P
\end{array}
\] (4)

star commutative, i.e., $h'f \equiv g$. Now, we show that $P$ is star projective. we consider diagram

\[
\begin{array}{ccc}
P & & A \\
\downarrow{k} & & \downarrow{k} \\
& & B
\end{array}
\]

of $H_v$-modules and star homomorphism such that bottom row is exact. We show that there is a star homomorphism $h : P \rightarrow A$ such that $fh \equiv k$. Since in the diagrams (4) and (5) row horizontal is exact, hence $f$ is weak monic and weak epic. Then there is $f^{-1} : P \rightarrow A$. Now, we consider the mapping $h = f^{-1}k : P \rightarrow A$ which since $f^{-1}$ and $k$ are star homomorphisms, hence $h$ is a star homomorphism and $fh \equiv ff^{-1}k \equiv k$. Then $P$ is star projective. □

**References**


