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TWO-NORM CONTINUOUS FUNCTIONALS ON L_∞

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Abstract. In this paper we regard L_∞ as a two-norm space and prove a representation theorem for two-norm continuous functionals defined on L_∞.

1. INTRODUCTION

The classical Riesz representation theorems is well-known. It shows that every continuous linear functional F defined on the space $C[0,1]$ of all continuous functions on $[0,1]$ can be expressed in terms of a Riemann-Stieltjes integral. That is, if F is a continuous linear functional on $C[0,1]$, then there is a function g of bounded variation on $[0,1]$ such that

$$F(f) = \int_0^1 f(x)dg(x) \qquad for\ f \in C[0,1].$$

Let $BV[0,1]$ denote the space of all functions of bounded variation on $[0,1]$. In the language of functional analysis, the Riesz theorem says that the Banach dual of $C[0,1]$ is $BV[0,1]$. However, the Banach dual of $BV[0,1]$ is not $C[0,1]$ if we endorse $BV[0,1]$ with the usual norm, namely, $|f(0)| + V(f;[0,1])$ where $V(f;[0,1]$ denotes the total variation of f on $[0,1]$. The main difficulty is that $BV[0,1]$ is non-separable. Hence the usual technique of proving such representation theorem no longer applies. More precisely, the proof often contains the following two steps. First, we prove the representation for some elementary functions, for example, step functions. Secondly, we approximate a general function by a sequence of elementary functions. Thus the representation for general functions follows from a convergence

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theorem for the integral. If the space is non-separable, the second step does not work.

Hildebrandt [1] proved a representation theorem for $BV[0,1]$ by regarding $BV[0,1]$ as a two-norm space [6]. Here we state the theorem without proof in the form as given by Khaing [2].

A functional F defined on $BV[0,1]$ is said to be **two-norm continuous** if $F(f_n) \to F(f)$ as $n \to \infty$ whenever $V(f_n;[0,1]) \leq M$ for all n and $\|f_n - f\| \to 0$ as $n \to \infty$, where $\|f\| = \sup_{0\leq x\leq 1} |f(x)|$.

Theorem 1 *If F is a two-norm continuous linear functional on $BV[0,1]$ then there exist bounded functions f_1 and f_2 such that the following Henstock-Stieltjes integral and infinite series exist and*

$$F(g) = \int_a^b f_1 dg^* + \sum_{i=1}^{\infty} [g(t_i) - g^*(t_i)] f_2(t_i)$$

for every $g \in BV[0,1]$, where $t_i, i = 1, 2, \ldots,$ are the discontinuity points of g, and g^* is the normalized function of g.

In fact, the converse of Theorem 1 holds [2].

In this paper, following the same idea as above we regard L_∞ as a two-norm space and prove a representation theorem for two-norm continuous linear functionals on L_∞. Here L_∞ denotes the space of all essentially bounded functions on $[0,1]$.

2. LINEAR FUNCTIONALS ON L_∞

A function F is **essentially bounded** if it is bounded almost everywhere.

Let L_∞ be the space of all essentially bounded functions on $[0,1]$. The two norms defined on L_∞, as suggested by Orlicz [6], are the essential bound $\|f\|_\infty$ and $\int_0^1 |f(x)| dx$.

A sequence $\{f_n\}$ of functions is said to be **two-norm convergent** to f in L_∞ if $\|f_n\|_\infty \leq M$ for all n and

$$\int_0^1 |f_n(x) - f(x)| dx \;\to\; 0, \text{ as } n \to \infty.$$

A functional F defined on L_∞ is said to be **two-norm continuous** if

$$F(f_n) \to F(f) \text{ as } n \to \infty$$

whenever $\{f_n\}$ is two-norm convergent to f in L_∞.

We state without proof the big Sandwich Lemma [4]. We need it in proving a convergence theorem for the Lebesgue integral.

Lemma 2 *If* $0 \leq a_n \leq b_{kn}$ *for all* n, k *and*

$$\lim_{k\to\infty} \lim_{n\to\infty} b_{kn} = 0$$

then $\lim_{n\to\infty} a_n = 0$.

In what follows, when we say **absolutely integrable** we mean **Lebesgue integrable**.

Lemma 3 *If* f *is integrable and there is a positive constant* K *such that*

$$\int_0^1 |fg| \leq K \int_0^1 |g|,$$

for every bounded measurable functions g *on* $[0,1]$*, then* $f \in L_\infty$ *and* $\|f\|_\infty \leq K$.

Lemma 4 *If* $\{f_n\}$ *is two-norm convergent to* f *in* L_∞*, then* $f \in L_\infty$.

Proof. Let $A_n = \int_0^1 f_n$, for every n.

$$\begin{aligned} |A_n - A_m| &\leq |\int_0^1 f_n - \int_0^1 f_m| \leq \int_0^1 |f_n - f_m| \\ &\leq \int_0^1 |f_n - f| + \int -0^1 |f - f_m| \to 0, \quad \text{as } n, m \to \infty. \end{aligned}$$

That means $\{A_n\}$ is Cauchy in real system, there exists a real number A such that $A_n \to A$ as $n \to \infty$. Let $\epsilon > 0$ be given. There is a positive integer n_o such that for every positive integer n, $n \geq n_o$,

$$|A_n - A| < \epsilon.$$

Therefore,

$$|A - \int_0^1 f| \leq |A - A_{n_o}| + |A_{n_o} - \int_0^1 f_{n_o}| + |\int_0^1 f_{n_o} - \int_0^1 f| < 3\epsilon.$$

That is, f is integrable. If g is bounded measurable function on $[0,1]$, then g is integrable on $[0,1]$, then

$$|\int_0^1 fg| \leq \int_0^1 |(f - f_{n_o})g| + \int_0^1 |f_{n_o} g| \leq \int_0^1 |(f - f_{n_o})| \|g\|_1 + \|f_{n_o}\|_\infty \|g\|_1.$$

By, Lemma 3, $f \in L_\infty$. □

Theorem 5 *Let* g *be absolutely integrable on* $[0,1]$*. If* $\{f_n\}$ *is two-norm convergent to* f *in* L_∞ *then*

$$\lim_{n\to\infty} \int_0^1 f_n(x) g(x) dx = \int_0^1 f(x) g(x) dx.$$

Proof. Since g is absolutely integrable on $[0,1]$, there is a sequence $\{g_k\}$ of essentially bounded functions on $[0,1]$ such that

$$\int_0^1 |g_k(x) - g(x)|dx \to 0 \text{ as } k \to \infty.$$

Since $\{f_n\}$ is two-norm convergent, we have $\|f_n\|_\infty \leq M$ for all n. Hence the convergence of the integrals follows from Lemma 2 and the inequality

$$\begin{aligned} |\int_0^1 f_n(x)g(x)dx - \int_0^1 f(x)g(x)dx| &\leq 2M\int_0^1 |g_k(x) - g(x)|dx \\ &\quad + \|g_k\|_\infty \int_0^1 |f_n(x) - f(x)|dx. \quad \square \end{aligned}$$

Corollary 6 *If g is absolutely integrable on $[0,1]$ and*

$$F(f) = \int_0^1 f(x)g(x)dx \text{ for } f \in L_\infty,$$

then F defines a two-norm continuous linear functional on L_∞.

We define

$$\gamma_t(x) = \begin{cases} 1 & \text{untuk } 0 \leq x < t \\ 0 & \text{untuk } t \leq x \leq 1. \end{cases}$$

A function G defined on $[0,1]$ is said to be **absolutely continuous** if for every $\epsilon > 0$ there is $\delta > 0$ such that

$$|(\mathcal{D})\sum\{G(v) - G(u)\}| < \epsilon$$

whenever $(\mathcal{D})\sum|v-u| < \delta$, where $\mathcal{D} = \{[u,v]\}$ denotes a partial division of $[0,1]$ in which $[u,v]$ stands for a typical interval in the partial division. We are using the notation of Henstock integral ([5], [7]).

Lemma 7 *Let F be a two-norm continuous linear functional on L_∞. If $G(t) = F(\gamma_t)$ for $t \in [0,1]$ then G is absolutely continuous.*

Proof. Suppose G is not absolutely continuous on $[0,1]$. Then there is $\epsilon > 0$ such that for every δ there exists a partial division $\mathcal{D} = \{[u,v]\}$ satisfying

$$(\mathcal{D})\sum|v-u| < \delta \text{ and } |(\mathcal{D})\sum\{G(v) - G(u)\}| \geq \epsilon.$$

For each n, take $\delta = \frac{1}{n}$ and $\mathcal{D}_n = \mathcal{D}$. For every $x \in [0,1]$, there is $[u,v] \in \mathcal{D}$, put $f_n(x) = (\mathcal{D}_n)\sum|\gamma_v - \gamma_u|$. Then $\|f_n\|_\infty \leq 1$ for all n and

$$\int_0^1 |f_n(x)|dx = (\mathcal{D}_n)\sum|v-u| \downarrow 0 \text{ as } n \to \infty.$$

That is, $\{f_n\}$ is two-norm convergent to 0 in L_∞. Yet we have

$$F(f_n) = |(\mathcal{D}_n) \sum \{G(v) - G(u)\}| \geq \epsilon \quad \text{for all } n.$$

It contradicts the fact that F is two-norm continuous. Hence G is absolutely continuous on $[0, 1]$. □

Theorem 8 *If F is a two-norm continuous linear functional on L_∞ then there is an absolutely integrable function g such that*

$$F(f) = \int_0^1 f(x)g(x)dx \quad \text{for} \quad f \in L_\infty.$$

PROOF. In view of Lemma 7 and using notation there, we obtain

$$F(\gamma_t) = G(t) = \int_0^1 \gamma_t(x)dG(x)$$

where the integral is the Riemann-Stieltjes integral and G is absolutely continuous on $[0, 1]$. Note that $G(0) = 0$. Since F is linear,

$$F(f) = \int_0^1 f(x)dG(x)$$

for any step function f.
Next, write $g(x) = G'(x)$ almost everywhere in $[0, 1]$. In view of integration by substitution [4] p.74 Exercise 2.20, we have

$$F(f) = \int_0^1 f(x)g(x)dx$$

for any step function f. Take $f \in L_\infty$. Then there is a sequence $\{f_n\}$ of step functions two-norm convergent to f. Hence the general case of the theorem follows from Theorem 5.

3. CONCLUDING REMARKS

In conclusion, we have characterized completely two-norm continuous linear functionals on L_∞.

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