J. Indones. Math. Soc. (MIHMI)
Vol. 15, No. 1 (2009), pp. 61–67.

# THE RAMSEY NUMBERS OF LINEAR FOREST VERSUS $3K_3 \cup 2K_4$

I W. Sudarsana, E. T. Baskoro, H. Assiyatun, and S. Uttunggadewa

**Abstract.** For given two graphs $G$ and $H$, a graph $F$ is called a $(G, H)$-*good graph* if $F$ contains no $G$ and $\overline{F}$ contains no $H$. Furthermore, any $(G, H)$-good graph on $n$ vertices will be denoted by $(G, H, n)$-good graph. The *Ramsey number* $R(G, H)$ is defined as the smallest natural number $n$ such that no $(G, H, n)$-good graph exists. In this paper, we determine the Ramsey numbers $R(G, H)$ for disconnected graphs $G$ and $H$. In particular, $G = \bigcup_{i=1}^{k} P_{n_i}$ and $H = 3K_3 \cup 2K_4$.

## 1. INTRODUCTION

We consider finite undirected graphs without loops and multiple edges. Let $G(V, E)$ be a graph, the notation $V(G)$ and $E(G)$ (in short $V$ and $E$) stand for the vertex set and the edge set of the graph $G$, respectively. A graph $H(V', E')$ is a *subgraph* of $G$ if $V' \subseteq V$ and $E' \subseteq E$. For $A \subseteq V$, $G[A]$ represents the *subgraph induced* by $A$ in $G$.

For given two graphs $G$ and $H$, a graph $F$ is called a $(G, H)$-*good graph* if $F$ contains no $G$ and $\overline{F}$ contains no $H$. Furthermore, any $(G, H)$-good graph on $n$ vertices will be denoted by $(G, H, n)$-good graph. The *Ramsey number* $R(G, H)$ is defined as the smallest natural number $n$ such that no $(G, H, n)$-good graph exists. The Ramsey numbers $R(G, H)$ for connected graphs $G$ and $H$ have been intensively studied since Chvátal and Harary [4] established the general lower bound $R(G, H) \geq (c(G) - 1)(h - 1) + 1$, where $h$ is the chromatic number of $H$ and $c(G)$ is the number of vertices of the largest component of $G$. A connected graph $G$ is

Received 20-05-2009, Accepted 05-08-2009.
*2000 Mathematics Subject Classification*: 05C55
*Key words and Phrases*: Ramsey number, union graph, good graph, forest, linear forest

called $H$-*good* if $R(G,H) = (|G|-1)(h-1)+s$, where $s$ is the chromatic surplus of $H$. The chromatic surplus of $H$ is the minimum cardinality of a color class taken over all proper $h$ colorings of $H$.

Let $G_i$ be a graph with the vertex set $V_i$ and the edge set $E_i$, $i = 1, 2, ..., k$. The *union* $G = \bigcup_{i=1}^{k} G_i$ has the vertex set $V = \bigcup_{i=1}^{k} V_i$ and the edge set $E = \bigcup_{i=1}^{k} E_i$. We denote the union by $kF$ when $G_i = F$ for every $i$. If $G_i$ is isomorphic to a tree for every $i$ then the union is called a *forest*. A forest is called linear forest, if all the components are a path.

Some recent results on the Ramsey number for a combination of disconnected (union) and connected graphs can be found in ([1], [2], [6], [7], [9], [10], [11]). Other results concerning graph Ramsey numbers can be seen in [8]. In this note, we determine the Ramsey numbers $R(\bigcup_{i=1}^{k} P_{n_i}, 3K_3 \cup 2K_4)$.

## 2. MAIN RESULTS

Let us note firstly the previous theorems and lemmas used in the proof of our results.

**Theorem 1** *(Chvátal [5]). Let $T_n$ and $K_m$ be a tree of order $n \geq 1$ and a clique of order $m \geq 1$, respectively. Then, $R(T_n, K_m) = (n-1)(m-1)+1$.*

**Theorem 2** *(Sudarsana et al. [11]). Let $k \geq 1$ and $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 6$ be integers. If $G = \bigcup_{i=1}^{k} P_{n_i}$ then*

$$R(G, 2K_4) = \max_{1 \leq i \leq k} \left\{ 2n_i + \sum_{j=i}^{k} n_j \right\} - 1. \tag{1}$$

**Lemma 1** *(Sudarsana et al. [11]). Let $k, t \geq 1$ be integers. Let $H$ be a connected graph with the chromatic number $h \geq 2$ and the chromatic surplus $s \geq 1$. Let $G_i$ be connected graphs and satisfies $|G_k| \geq |G_{k-1}| \geq ... \geq |G_1| \geq \left(\frac{|H|-s}{h-1}\right)st + 1$. If $G = \bigcup_{i=1}^{k} G_i$ and $G_i$ is a $H$-good for every $i = 1, 2, ..., k$, then*

$$R(G, tH) \geq \max_{1 \leq i \leq k} \left\{ (|G_i| - 1)(h-2) + \sum_{j=i}^{k} |G_j| \right\} + st - 1. \tag{2}$$

**Lemma 2** *(Sudarsana et al. [11]). If $P_n$ is a path of order $n \geq 4$ then $R(P_n, 2K_3) = 2n$.*

Now, we are ready to prove our main results in the following.

**Lemma 3** *If $P_n$ is a path of order $n \geq 7$ then $R(P_n, 3K_3) = 2n + 1$.*

**Proof.** The inequality $R(P_n, 3K_3) \geq 2n+1$ is derived from Lemma 1. We will show the reverse inequality $R(P_n, 3K_3) \leq 2n+1$ by the following reason. Take an arbitrary graph $G$ on $2n+1$ vertices and contains no $P_n$. We will show that $\overline{G}$ contains $3K_3$. Since $|G| > 2n$ then by Lemma 2 we obtain $\overline{G} \supseteq 2K_3$. Let $B = \{a_1, b_1, c_1\} \cup \{a_2, b_2, c_2\}$ be the vertex set of $2K_3$ in $\overline{G}$. We let $D = V(G) \backslash B$ and $T = G[D]$. Clearly, $|T| = 2(n-2)-1$. Theorem 1 gives that $T \supseteq P_{n-2}$ or $\overline{T} \supseteq K_3$. If $\overline{T} \supseteq K_3$ then we finish the proof. Now, consider $T$ contains $P_{n-2}$ and call $P_{n-2} = (p_1 p_2 ... p_{n-3} p_{n-2})$. Let $K = V(T) \backslash V(P_{n-2})$ and hence $|K| = n-3$. Now, we consider the connection of the end vertices of $P_{n-2}$ and the vertex set $K$. Since $G$ contains no $P_n$ then we obtain the following facts.

**Fact 1**. *The vertex $p_1$ or $p_{n-2}$ is adjacent to at least two vertices in $K$.*

We let $p_1$ adjacent to $y_1$ and $y_2$ in $K$. Since $G$ does not contain $P_n$ then $\{p_{n-2}, y_1, y_2\}$ is an independent set in $G$. Therefore, the vertex set $\{p_{n-2}, y_1, y_2\}$ forms a $K_3$ in $\overline{T}$ and together with $B$ we have $\overline{G} \supseteq 3K_3$.

**Fact 2**. *The vertex $p_1$ or $p_{n-2}$ is adjacent to exactly one vertex in $K$.*

We let $p_1$ adjacent to $x$ in $K$. Since $G$ contains no $P_n$ then $p_{n-2}$ must not adjacent to any vertex in $K$ except the vertex $x$. If $K \backslash x$ contains two independent vertices, call $x_1$ and $x_2$, in $T$ then the vertex $p_{n-2}$ together with $x_1$ and $x_2$ induce a $K_3$ in $\overline{T}$ and hence $\overline{G} \supseteq 3K_3$. Therefore, the vertex set $K \backslash x$ forms a $K_{n-4}$ in $T$. Now, if there exists one vertex, say $y$, in $K \backslash x$ that is not adjacent to one vertex in $B$ then the vertex set $\{p_{n-2}, x, y\} \cup B$ induces a $3K_3$ in $\overline{G}$. Since otherwise we will get that every vertex in $K \backslash x$ is adjacent to every vertex in $B$, which is impossible since $G$ does not contain $P_n$ with $n \geq 7$. Thus, $\overline{G}$ contains $3K_3$.

**Fact 3**. *The vertex $p_1$ and $p_{n-2}$ do not adjacent to any vertex in $K$.*

If $K$ contains two independent vertices, call $x_1$ and $x_2$, in $T$ then the vertex set $\{p_{n-2}, x_1, x_2\}$ induce a $K_3$ in $\overline{T}$. Therefore, we finish the proof since we have $\overline{G} \supseteq 3K_3$. Now, consider $K$ shapes a $K_{n-3}$ in $T$. Thus without loss of generality, one of the following conditions holds:

**(i)**. *The vertex $p_1$ or $p_{n-2}$ is adjacent to every vertices in $B$.*

We let $p_1$ adjacent to every vertex in $B$. Since $G$ does not contain $P_n$ then $\{p_{n-2}\} \cup B$ is an independent set in $T$ which each element does not adjacent to any vertex in $K$. Therefore, it can be verified that the set $\{k_1, k_2, p_{n-2}\} \cup B$ induce a $3K_3$ in $\overline{T}$, for any $k_1, k_2$ in $K$. So, $\overline{G} \supseteq 3K_3$.

**(ii)**. *The vertex $p_1$ or $p_{n-2}$ is adjacent to five vertices in $B$.*

We let $p_1$ adjacent to every vertex in $B\backslash a_2$. Since $G$ contains no $P_n$ then it is not difficult to verify that the sets $\{p_{n-2}, a_2, c_1\}$, $\{a_1, c_2, k_2\}$ and $\{b_1, b_2, k_1\}$ form a $3K_3$ in $\overline{G}$, for any $k_1, k_2$ in $K$.

**(iii)**. *The vertex $p_1$ or $p_{n-2}$ is adjacent to four vertices in $B$.*

We let $p_1$ adjacent to every vertex in $B\backslash\{a_2, c_2\}$. Again, since $G$ contains no $P_n$ then it can be verified that the sets $\{p_{n-2}, a_2, c_2\}$, $\{a_1, c_1, k_2\}$ and $\{b_1, b_2, k_1\}$ form a $3K_3$ in $\overline{G}$, for any $k_1, k_2$ in $K$.

**(iv)**. *The vertex $p_1$ or $p_{n-2}$ is adjacent to three vertices in $B$.*

Without loss of generality, we distinguish the following two cases.

*Case 1. The vertex $p_1$ is adjacent to $a_1$, $b_1$ and $c_1$ in $B$*

Thus the set $\{p_1, a_2, b_2, c_2\}$ is an independent set in $T$. Since $G$ contain no $P_n$ then it is easy verify that the sets $\{p_{n-2}, a_1, c_2\}$, $\{a_2, b_2, p_1\}$ and $\{b_1, c_1, k\}$ form a $3K_3$ in $\overline{G}$, for every $k \in K$.

*Case 2. The vertex $p_1$ is adjacent to $a_1$, $b_1$ and $c_2$ in $B$*

Therefore, the set $\{p_1, a_2, b_2\}$ is an independent set in $T$. Since $G$ contain no $P_n$ then it is easy verify that the sets $\{p_1, a_2, b_2\}$, $\{a_1, c_1, p_{n-2}\}$ and $\{b_1, c_2, k\}$ form a $3K_3$ in $\overline{G}$, for every $k \in K$.

**(v)**. *The vertex $p_1$ or $p_{n-2}$ is adjacent to two vertices in $B$.*

Without loss of generality, we distinguish the following two cases.

*Case 1. The vertex $p_1$ is adjacent to $a_1$ and $b_1$ in $B$*

Thus the set $\{p_1, a_2, b_2, c_2\}$ is an independent set in $T$. Now, if there exists one vertex, call $y$, in $K$ that is not adjacent to one vertex, say $c_2$, in $B\backslash\{a_1, b_1\}$ then the vertex sets $\{p_1, a_2, b_2\}$, $\{b_1, c_2, y\}$ and $\{a_1, c_1, p_{n-2}\}$ form a $3K_3$ in $\overline{G}$. Since otherwise we get that vertex set $K \cup \{a_2, b_2, c_2, c_1\}$ induces a graph $K_{n-3} + (\overline{K}_3 \cup K_1)$ in $T$, which is impossible since $G$ does not contain $P_n$ with $n \geq 7$. Thus, $\overline{G}$ contains $3K_3$.

*Case 2. The vertex $p_1$ is adjacent to $a_1$ and $a_2$ in $B$*

Since $G$ contains no $P_n$ then it is easy verify that the sets $\{p_{n-2}, b_1, c_1\}$, $\{p_1, b_2, c_2\}$ and $\{a_1, a_2, k\}$ form a $3K_3$ in $\overline{G}$, for every $k \in K$.

**(vi)**. *The vertex $p_1$ or $p_{n-2}$ is adjacent to one vertex in $B$.*

We let $p_1$ adjacent to $a_1$ in $B$. Now, consider the vertex set $\{b_1, c_1, a_2, b_2, c_2\}$. If there exists a vertex, say $k_1$, in $K$ that is not adjacent to one vertex in $\{b_1, c_1, a_2, b_2, c_2\}$ then the vertex set $\{p_1, p_{n-2}, k_1\} \cup B$ induces a $3K_3$ in $\overline{G}$. Since otherwise we obtain that the vertex set $K \cup \{b_1, c_1, a_2, b_2, c_2\}$ induces a graph $K_{n-3} + (\overline{K}_3 \cup \overline{K}_2)$ in $T$, which is impossible since $G$ does not contain $P_n$ with $n \geq 7$. Thus, $\overline{G}$ contains $3K_3$.

**(vii)**. *The vertex $p_1$ or $p_{n-2}$ does not adjacent to any vertices in $B$.*

If there exists a vertex, say $k$, in $K$ that is not adjacent to one vertex in $B$ then the vertex set $\{p_1, p_{n-2}, k\} \cup B$ induces a $3K_3$ in $\overline{G}$. Since otherwise we derive that the vertex set $K \cup \{b_1, c_1, a_2, b_2, c_2\}$ induces a graph $K_{n-3} + (2\overline{K}_3)$ in $T$, which is impossible since $G$ does not contain $P_n$ with $n \geq 7$. Thus $\overline{G} \supseteq 3K_3$. This completes the proof. □

**Theorem 3** *Let $k \geq 1$ and $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 7$ be integers. If $G = \bigcup_{i=1}^{k} P_{n_i}$ then*

$$R(G, 3K_3) = \max_{1 \leq i \leq k} \left\{ n_i + \sum_{j=i}^{k} n_j \right\} + 1. \tag{3}$$

**Proof.** For $1 \leq i \leq k$, let $G = \bigcup_{i=1}^{k} P_{n_i}$ and $G_i = \bigcup_{j=i}^{k} P_{n_j}$. Obviously, $G = G_1$. Suppose that the maximum of the right side of the equation (3) is achieved for $i_0$. Write $t_0 = \sum_{j=i_0}^{k} n_j$ and $t = n_{i_0} + t_0$. The lower bound $R(G, 3K_3) \geq t + 1$ can be obtained by using Lemma 1. We will prove $R(G, 3K_3) \leq t + 1$.

Let $F$ be a graph of order $t + 1$ and suppose that $\overline{F}$ contains no $3K_3$. We shall show that $F$ contains $G$. We prove this by induction on $i$. For $i = k$, we get $G = P_{n_k}$. Since $t + 1 \geq 2n_k + 1$ and $\overline{F} \not\supseteq 3K_3$ then the theorem holds by Lemma 3. Let us state the inductive hypothesis: $F$ contains $G_{i+1}$ for some $1 \leq i \leq k$. We will show that $F$ contains $G_i$ for any $i \geq 1$. By induction hypothesis, we have $F \supseteq G_{i+1}$. Clearly, $|G_{i+1}| = \sum_{j=i+1}^{k} n_j$. Let $A = V(F) \backslash V(G_{i+1})$ and $W = F[A]$, then $|W| = (t + 1) - \sum_{j=i+1}^{k} n_j$. By definition of $t$, we get $t \geq n_i + \sum_{j=i}^{k} n_j$ for every $i = 1, 2, ..., k$. Therefore, $|W| \geq 2n_i + 1$. Since $\overline{W} \not\supseteq 3K_3$ then Lemma 3 guarantees that $W$ contains $P_{n_i}$. Therefore, $F \supseteq G_i$ for any $i \geq 1$. Thus $F \supseteq G_1$. □

**Theorem 4** *Let $k \geq 1$ and $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 9$ be integers. Let $G = \bigcup_{i=1}^{k} P_{n_i}$ and $H = 3K_3 \cup 2K_4$. If $R(G, 2K_4) - R(G, 3K_3) \geq 9$ then*

$$R(G, H) = R(G, 2K_4). \tag{4}$$

**Proof.** By Theorem 2, we let $R(G, 2K_4) = l - 1$. Since $2K_4 \subset H$ then $R(G, H) \geq l - 1$. Now, we will show that $R(G, H) \leq l - 1$. Let $U$ be a graph of order $l - 1$ and contains no $G$. We shall show that $\overline{U}$ contains $H$. Theorem 2 provides

$\overline{U} \supseteq 2K_4$. Let $L = V(U)\backslash V(2K_4)$ and $Q = U[L]$. Clearly, $|Q| = l - 9$. By Theorem 3, we let $R(G, 3K_3) = l'$. Thus, $|Q| = l - 9 = l' + (l - l') - 9 \geq l'$ when $l - l' \geq 9$. Since $Q \not\supseteq G$ then $\overline{Q} \supseteq 3K_3$. This concludes that $\overline{U}$ contains $H$. $\square$

**Remark.** If $n_i = n$ for every $i = 1, 2, ..., k$, then the union $G$ is isomorphic to $kP_n$. Therefore, by Theorem 3 we obtain $R(kP_n, 3K_3) = (k+1)n + 1$ when $n \geq 7$. Meanwhile, Theorem 2 gives $R(kP_n, 2K_4) = (k+2)n - 1$ when $n \geq 6$ and Theorem 4 also provides $R(kP_n, 3K_3 \cup 2K_4) = (k+2)n - 1$ when $n \geq 9$. Furthermore, if $G = \bigcup_{i=1}^{k} l_i P_{n_i}$ and $l_i$ is the number of the paths of order $n_i$ in $G$ then the following corollaries hold.

**Corollary 1** *Let $k \geq 1$ and $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 7$ be integers. If $G = \bigcup_{i=1}^{k} l_i P_{n_i}$ then*

$$R(G, 3K_3) = \max_{1 \leq i \leq k} \left\{ n_i + \sum_{j=i}^{k} l_j n_j \right\} + 1. \tag{5}$$

**Corollary 2** *Let $k \geq 1$ and $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 9$ be integers. Let $G = \bigcup_{i=1}^{k} l_i P_{n_i}$ and $H = 3K_3 \cup 2K_4$. If $R(G, 2K_4) - R(G, 3K_3) \geq 9$ then*

$$R(G, H) = \max_{1 \leq i \leq k} \left\{ 2n_i + \sum_{j=i}^{k} l_j n_j \right\} - 1. \tag{6}$$

## REFERENCES


1. E.T Baskoro, Hasmawati, and H. Assiyatun, "The Ramsey number for disjoint unions of trees", *Discrete Math.* **306** (2006), 3297–3301.
2. H. Bielak, "Ramsey numbers for a disjoint of some graphs", *Appl. Math. Lett.* **22** (2009), 475–477.
3. S. A. Burr, "Ramsey numbers involving graphs with long suspended paths", *J. London Math. Soc. (2)* **24** (1981), 405–413.
4. V. Chvátal and F. Harary, "Generalized Ramsey theory for graphs, III: small off-diagonal numbers", *Pac. J. Math.* **41** (1972), 335–345.
5. V. Chvátal, "Tree complete graphs Ramsey number", *J. Graph Theory* **1** (1977), 93.
6. Hasmawati, E.T Baskoro, and H. Assiyatun, "The Ramsey number for disjoint unions of graphs", *Discrete Math.* **308** (2008), 2046-2049.
7. Hasmawati, H. Assiyatun, E.T. Baskoro, and A.N.M. Salman, "Ramsey numbers on a union of identical stars versus a small cycle", *LNCS* **4535** (2008), 85-89.
8. S. P. Radziszowski, "Small Ramsey numbers", *Electron. J. Combin.* August 2006 DS1.9. ⟨ http://www.combinatorics.org/ ⟩.

9. I W. Sudarsana, E. T. Baskoro, H. Assiyatun, and S. Uttunggadewa, "On the Ramsey Numbers $R(S_{2,m}, K_{2,q})$ and $R(sK_2, K_s + C_n)$", *Ars Combin.* to appear
10. I W. Sudarsana, E. T. Baskoro, H. Assiyatun, and S. Uttunggadewa, "On the Ramsey numbers of certain forest respect to small wheels", *J. Combin. Math. Combin. Comput.* **71** (2009), 257–264.
11. I W. Sudarsana, E. T. Baskoro, H. Assiyatun, and S. Uttunggadewa, "The Ramsey numbers for the union graph with $H$-good components", *Far East J. Math. Sci. (FJMS)* **39:1** (2010), 29-40.

I W. Sudarsana: Combinatorial Mathematics Research Division, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung (ITB), Jalan Ganesha No. 10 Bandung 40132, Indonesia dan Jurusan Matematika, FMIPA, Universitas Tadulako (UNTAD), Jl. Soekarno-Hatta Km. 8 Palu 94118, Indonesia. E-mail: sudarsanaiwayan@yahoo.co.id

E. T. Baskoro: Combinatorial Mathematics Research Division, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung (ITB), Jalan Ganesha No. 10 Bandung 40132, Indonesia.
E-mail: ebaskoro@math.itb.ac.id

H. Assiyatun: Combinatorial Mathematics Research Division, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung (ITB), Jalan Ganesha No. 10 Bandung 40132, Indonesia.
E-mail: hilda@math.itb.ac.id

S. Uttunggadewa: Combinatorial Mathematics Research Division, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung (ITB), Jalan Ganesha No. 10 Bandung 40132, Indonesia.
E-mail: s_uttunggadewa }@math.itb.ac.id