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FIRST EIGENVALUES OF GEOMETRIC OPERATOR UNDER THE RICCI-BOURGUIGNON FLOW

Shahroud Azami

Department of Mathematics, Faculty of Sciences
Imam Khomeini International University, Qazvin, Iran
azami@sci.ikiu.ac.ir

Abstract. Let $(M, g(t))$ be a compact Riemannian manifold and the metric $g(t)$ evolve by the Ricci-Bourguignon flow. We find the formula variation of the eigenvalues of geometric operator $-\Delta_\phi + cR$ under the Ricci-Bourguignon flow, where Δ_ϕ is the Witten-Laplacian operator and R is the scalar curvature. In the final section, we show that some quantities dependent to the eigenvalues of the geometric operator are nondecreasing along the Ricci-Bourguignon flow on closed manifolds with nonnegative curvature.

Key words and Phrases: Laplace, Ricci-Bourguignon flow.

Abstrak. Misalkan $(M, g(t))$ adalah *manifold* Riemann kompak dan metrik $g(t)$ berevolusi mengikuti aliran *Ricci-Bourguignon.* Kita mencari variasi formula nilai-nilai eigen dari operator geometrik $-\Delta_\phi + cR$ di bawah aliran *Ricci-Bourguignon*, dengan Δ_ϕ menyatakan operator *Laplace-Witten* dan R adalah kurvatur skalar. Di bagian akhir, kita menunjukkan bahwa beberapa besaran yang bergantung pada nilai-nilai eigen dari operator geometrik bersifat tak turun sepanjang aliran *Ricci-Bourguignon* pada *manifold* tutup dengan kurvatur taknegatif.

Kata kunci: *Laplace*, aliran *Ricci-Bourguignon*

1. Introduction

Let $(M, g(t))$ be a closed Riemannian manifold. Studying the eigenvalues of geometric operators is a very powerful tool for the understanding Riemannian manifolds. Recently, there has been a lot of work on the eigenvalue problem under

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geometric flow. In [9], Perelman shows that the functional

$$F=\int_M (R+|\nabla f|^2)e^{-f}\,d\nu$$

is nondecreasing along the Ricci flow coupled to a backward heat-type equation, where R is the scalar curvature with respect to the metric $g(t)$ and $d\nu$ denote the volume form of the metric $g=g(t)$. The nondecreasing of the functional F implies that the lowest eigenvalue of the operator $-4\Delta+R$ is nondecreasing along the Ricci flow. As an application, Perelman shown that there are no nontrivial steady or expanding breathers on compact manifolds. Cao [2] extended the geometric operator $-4\Delta+R$ to the operator $\Delta+\frac{R}{2}$ on closed Riemannian manifolds, and showed that the eigenvalues of the operator $\Delta+\frac{R}{2}$ are nondecreasing along the Ricci flow with nonnegative curvature operator. Then, Li [8] and Cao [3] considered the operator $-\Delta+cR$ and both them proved that the first eigenvalue of the operator $-\Delta+cR$ for $c\geq\frac{1}{4}$ is nondecreasing along the Ricci flow. Zeng and et'al [12] studied the monotonicity of eigenvalues of the operator $-\Delta+cR$ along the Ricci-bourguignon flow. Later Fang and Yang [7] studied the evolution for the first eigenvalue of geometric operator $-\Delta_\phi+\frac{R}{2}$ under the Yamabe flow, where $-\Delta_\phi$ is the Witten-Laplacian operator, $\phi\in C^2(M)$, and constructed some monotonic quantities under the Yamabe flow. Also, Wen and et'al [10] investigated the evolution and monotonicity for eigenvalues of geometric operator $-\Delta_\phi+\frac{R}{2}$ under the Ricci flow. For the other recent research in this direction, see [5, 6, 11].

We consider an n-dimensional closed Riemannain manifold M with a time dependent Riemannian metric $g(t)$, where $g=g(t)$ is evolving according to the Ricci-Bourguignon flow equation

$$\frac{\partial}{\partial t}g=-2Ric+2\rho Rg=-2(Ric-\rho Rg),\quad g(0)=g_0 \tag{1}$$

where Ric is the Ricci tensor of the manifold, R is scalar curvature and ρ is a real constant. This family of geometric flows contains, as a special case, when $\rho=0$, this flow is the Ricci flow. At the first time the Ricci-Bourguignon introduced by Bourguignon in [1] and then Catino and et 'al in [4] shown that if $\rho<\frac{1}{2(n-1)}$, then the evolution equation (1) has a unique solution for a position time interval on any smooth n-dimensional closed Riemannian M with any initial metric g_0 and shown that some conditions on the curvature are preserved by the Ricci-Bourguignon flow. Motivated by the above works, in this paper we will study the first eigenvalue of the geometric operator whose metric satisfying the Ricci-Bourguignon flow (1).

2. Preliminaries

In this section, we will first the definitions for the first eigenvalue of the geometric operator

$$-\Delta_\phi+cR \tag{2}$$

then we will find the formula for the evolution of the first eigenvalue of the geometric operator (2) under the Ricci-Bourguignon flow on a closed manifold. Let $(M,g(t))$ be a compact Riemannian manifold, and $(M,g(t))$ be a smooth solution to the

Ricci-Bourguignon flow (1) for $t \in [0, T)$. Let ∇ be the Levi-Civita connection on $(M, g(t))$ and $f : M \to \mathbb{R}$ be a smooth function on M or f belong to the Sobolev space $W^{1,2}(M)$. The Laplacian of f defined as

$$\Delta f = div(\nabla f) = g^{ij}(\partial_i \partial_j f - \Gamma_{ij}^k \partial_k f). \tag{3}$$

Suppose that $d\nu$ the Riemannian volume measure, and $d\mu$ the weight volume measure on $(M, g(t))$; i.e.

$$d\mu = e^{-\phi(x)} d\nu \tag{4}$$

where $\phi \in C^2(M)$. The Witten-Laplacian is defined by

$$\Delta_\phi = \Delta - \nabla\phi . \nabla \tag{5}$$

which is a symmetric operator on $L^2(M, \mu)$ and satisfies the following integration by part formula:

$$\int_M < \nabla u, \nabla v > d\mu = -\int_M v \Delta_\phi u \, d\mu = -\int_M u \Delta_\phi v \, d\mu \qquad \forall u, v \in C^\infty(M),$$

The Witten-Laplacian is generalize of Laplacian operator, for example, when ϕ is a constant function, the Witten-Laplacian operator is just the Laplace-Belterami operator. In this paper we consider a generalize of the Witten-Laplacian operator as $-\Delta_\phi + cR$ where R is the scalar curvature. We say that $\lambda(t)$ is an eigenvalue of the operator $-\Delta_\phi + cR$ at time $t \in [0, T)$, and $f(x, t)$ the corresponding eigenfunction, whenever

$$-\Delta_\phi f(x, t) + cRf(x, t) = \lambda(t) f(x, t). \tag{6}$$

Normalized eigenfuctions are defined as follow:

$$\int_M f^2 \, d\mu = 1, \tag{7}$$

and assume that $f(x, t)$ is a C^1-family of smooth function on M.
Multiplying with f on both sides (6) and then by integration we get

$$\lambda(t) = \int_M (-f\Delta_\phi f + cRf^2) d\mu \tag{8}$$

where defines the evolution of the first eigenvalue of the geometric operator (2) under the variation of $g(t)$ where the eigenfunction associated to $\lambda(t)$ is normalized. In [10] Wen and et 'al shown that the following lemma.

Lemma 2.1. *([10]) Suppose that $\lambda(t)$ is an eigenvalue of the operator $-\Delta_\phi + cR$, f is the eigenfunction of $\lambda(t)$ at the time t, and the metric $g(t)$ evolves by $\frac{\partial}{\partial t} g_{ij} = v_{ij}$, where v_{ij} is a symmetric two-tensor. Then we have*

$$\frac{d}{dt}\lambda(t) = \int_M (v_{ij} f_{ij} - v_{ij}\phi_i f_j + c\frac{\partial R}{\partial t} f) d\mu + \int_M (v_{ij,i} - \frac{1}{2} V_i) f_j f d\mu \tag{9}$$

where $V = tr(v)$, $f_{ij} = \nabla_i \nabla_j f$, $f_i = \nabla_i f$, $\phi_i = \nabla_i \phi$.

Now, we find the evolution formula of eigenvalue $\lambda(t)$ under the Ricci-Bourguignon flow (1).

Theorem 2.2. *Let $g(t)$, $t\in[0,T)$, be a solution of the Ricci-Bourguignon flow (1) on an n-dimensional closed manifold M. Assume that $\lambda(t)$ is the lowest eigenvalue of $-\Delta_\phi + cR$ and $f = f(x,t)$ satisfies in (6) with (7). Then under the Ricci-Bourguignon flow, we have*

$$\begin{aligned}\frac{d}{dt}\lambda(t) &= 2A\int_M R|\frac{1}{2}\nabla f - f\nabla\phi|^2 d\mu - A\int_M Rf^2\Delta\phi d\mu \\ &+[\frac{3}{2}A - 1 + (n-2)\rho]\int_M R|\nabla f|^2 d\mu - \lambda(2A-1+n\rho)\int_M Rf^2 d\mu \qquad (10)\\ &+2\int_M R_{ij}f_if_j d\mu + 2c\int_M |Ric|^2 f^2 d\mu + c(2A-1+(n-2)\rho)\int_M R^2f^2 d\mu\end{aligned}$$

where $A = c(1-2(n-1)\rho)$.

Proof. In [4], G. Catino and et'al shown that the evolution of scalar curvature under the Ricci-Bourguignon flow is

$$\frac{\partial R}{\partial t} = (1-2(n-1)\rho)\Delta R + 2|Ric|^2 - 2\rho R^2. \qquad (11)$$

Substiuting $v_{ij} = -2R_{ij} + 2\rho R g_{ij}$ and (11) into the equality (9) we get

$$\begin{aligned}\frac{d}{dt}\lambda(t) &= \int_M [-2R_{ij}f_{ij} + 2\rho R\,\Delta f + 2R_{ij}\phi_i f_j - 2\rho R\,\nabla\phi.\nabla f]\, f d\mu \\ &-(n-2)\rho\int_M \nabla_i R\, f_i f d\mu \qquad (12)\\ &+c\int_M \left[(1-2(n-1)\rho)\Delta R\, f^2 + 2|Ric|^2 f^2 - 2\rho R^2 f^2\right] d\mu.\end{aligned}$$

Integration by parts results that

$$\begin{aligned}\int_M f^2\Delta R d\mu &= 2\int_M R|\nabla f|^2 d\mu + 2\int_M Rf\Delta_\phi f d\mu - 2\int_M Rf\nabla\phi.\nabla f d\mu \\ &- \int_M Rf^2\Delta\phi\, d\mu + \int_M Rf^2|\nabla\phi|^2 d\mu, \qquad (13)\end{aligned}$$

and

$$\int_M \nabla_i R\, f_i f d\mu = -\int_M Rf\Delta_\phi f d\mu - \int_M R|\nabla f|^2 d\mu. \qquad (14)$$

Also, using integration by parts and $\frac{1}{2}\nabla R = div\, Ric$ we have

$$\begin{aligned}
-\int_M R_{ij} f_{ij} f d\mu &= \int_M (R_{ij} f e^{-\phi})_j f_i d\nu \\
&= \int_M div(Ric) f e^{-\phi} f_i d\nu + \int_M R_{ij} f_j e^{-\phi} f_i d\nu - \int_M R_{ij} f \phi_j f_i e^{-\phi} d\nu \\
&= \frac{1}{2}\int_M \nabla_i R f f_i e^{-\phi} d\nu + \int_M R_{ij} f_j f_i d\mu - \int_M R_{ij} f \phi_j f_i d\mu \\
&= -\frac{1}{2}\int_M R(f f_i e^{-\phi})_i d\nu + \int_M R_{ij} f_j f_i d\mu - \int_M R_{ij} f \phi_j f_i d\mu \\
&= -\frac{1}{2}\int_M R|\nabla f|^2 d\mu - \frac{1}{2}\int_M R f \Delta f d\mu + \frac{1}{2}\int_M R f f_i \phi_i d\mu \qquad (15) \\
&\quad + \int_M R_{ij} f_j f_i d\mu - \int_M R_{ij} f \phi_j f_i d\mu \\
&= -\frac{1}{2}\int_M R|\nabla f|^2 d\mu - \frac{1}{2}\int_M R f \Delta_\phi f d\mu + \int_M R_{ij} f_j f_i d\mu - \int_M R_{ij} f \phi_j f_i d\mu.
\end{aligned}$$

Inserting (13), (14) and (15) in (12), yields

$$\begin{aligned}
\frac{d}{dt}\lambda(t) &= -\int_M R|\nabla f|^2 d\mu - \int_M R f \Delta_\phi f d\mu + 2\int_M R_{ij} f_i f_j d\mu - 2\int_M R_{ij} f_i \phi_j f d\mu \\
&\quad +2\rho\int_M R f \Delta f d\mu + 2\int_M R_{ij} f_j \phi_i f d\mu - 2\rho\int_M R f \nabla\phi.\nabla f d\mu \\
&\quad +(n-2)\rho\int_M R f \Delta_\phi f d\mu + (n-2)\rho\int_M R|\nabla f|^2 d\mu \\
&\quad +2c(1-2(n-1)\rho)\int_M R|\nabla f|^2 d\mu + 2c(1-2(n-1)\rho)\int_M R f \Delta_\phi f d\mu \\
&\quad -2c(1-2(n-1)\rho)\int_M R f \nabla\phi.\nabla f d\mu - c(1-2(n-1)\rho)\int_M R f^2 \Delta\phi d\mu \\
&\quad +2c(1-2(n-1)\rho)\int_M R f^2 |\nabla\phi|^2 d\mu + 2c\int_M |Ric|^2 f^2 d\mu - 2c\rho\int_M R^2 f^2 d\mu,
\end{aligned}$$

therefore

$$\begin{aligned}
\frac{d}{dt}\lambda(t) &= [2A-1+(n-2)\rho]\int_M R|\nabla f|^2 d\mu + [2A-1+n\rho]\int_M Rf\Delta_\phi f d\mu \\
&\quad +2\int_M R_{ij}f_i f_j d\mu + 2c\int_M |Ric|^2 f^2 d\mu - 2c\rho\int_M R^2 f^2 d\mu \qquad (16)\\
&\quad -2A\int_M Rf\nabla\phi.\nabla f d\mu - A\int_M Rf^2\Delta\phi d\mu + 2A\int_M Rf^2|\nabla\phi|^2 d\mu \\
&= 2A\int_M R|\frac{1}{2}\nabla f - f\nabla\phi|^2 d\mu - A\int_M Rf^2\Delta\phi d\mu \\
&\quad +[\frac{3}{2}A-1+(n-2)\rho]\int_M R|\nabla f|^2 d\mu \\
&\quad +(2A-1+n\rho)\int_M R(-\lambda f^2 + cRf^2)d\mu \\
&\quad +2\int_M R_{ij}f_i f_j d\mu + 2c\int_M |Ric|^2 f^2 d\mu - 2c\rho\int_M R^2 f^2 d\mu.
\end{aligned}$$

Here in the last equality we have used (6). □

In theorem (2.2) if ϕ is a constant function, we can get the evolution for the first eigenvalue of the geometric operator $-\Delta + cR$ under the Ricci-Bourguignon flow (1), which studied in [12].

Remark: In theorem (2.2), we assume that ϕ does not dependent on the time t. If ϕ is depend to t then the eigenvalue $\lambda(t)$ introduced in (6) and (7) satisfies

$$\begin{aligned}
\frac{d}{dt}\lambda(t) &= 2A\int_M R|\frac{1}{2}\nabla f - f\nabla\phi|^2 d\mu - A\int_M Rf^2\Delta\phi d\mu + \int_M f_i(\phi_t)_i f d\mu \\
&\quad +[\frac{3}{2}A-1+(n-2)\rho]\int_M R|\nabla f|^2 d\mu - \lambda(2A-1+n\rho)\int_M Rf^2 d\mu \\
&\quad +2\int_M R_{ij}f_i f_j d\mu + 2c\int_M |Ric|^2 f^2 d\mu \qquad (17)\\
&\quad +c(2A-1+(n-2)\rho)\int_M R^2 f^2 d\mu
\end{aligned}$$

where $A = c(1-2(n-1)\rho)$. Because the evolution equation of eigenvalues will have an additional term $\int_M f_i(\phi_t)_i f d\mu$.

In the following we show that some quantity dependent on the eigenvalue of geometric operator (6) are monotonic along the Ricci-Bourguignon flow. Not that the scalar curvature under the Ricci-Bourguignon flow evolves by

$$\frac{\partial R}{\partial t} = (1-2(n-1)\rho)\Delta R + 2|Ric|^2 - 2\rho R^2,$$

by $|Ric|^2 \leq R^2$ we have

$$\frac{\partial R}{\partial t} \leq (1-2(n-1)\rho)\Delta R + 2(1-\rho)R^2.$$

Let $\sigma(t)$ be the solution to the ODE $y' = 2(1-\rho)y^2$ with initial value $\alpha = \max_{x\in M} R(0)$. By the maximum principle, we have

$$R(t) \leq \sigma(t) = \left(-2(1-\rho)t + \frac{1}{\alpha}\right)^{-1} \tag{18}$$

on $[0, T')$, where $T' = \min T, \frac{1}{2(1-\rho)\alpha}$. Also, the inequality $|Ric|^2 \geq \frac{R^2}{n}$ result that

$$\frac{\partial R}{\partial t} \geq (1 - 2(n-1)\rho)\Delta R + 2(\frac{1}{n} - \rho)R^2.$$

we assume that $\gamma(t)$ be the solution to the ODE $y' = 2(\frac{1}{n} - \rho)y^2$ with initial value $\beta = \min_{x\in M} R(0)$. Then the maximum principle implies that

$$R(t) \geq \gamma(t) = \frac{n\beta}{n - 2(1-n\rho)\beta t} \quad \text{on} \quad [0, T). \tag{19}$$

Theorem 2.3. *Let $(M, g(t))$ be a solution of the Ricci-Bourguignon flow (1) for $t \in [0, T]$ on a closed n-dimensional manifold M and $\rho < \frac{1}{2(n-1)}$ with nonnegative scalar curvature. Let the Ricci curvature operator be a nonnegative along the Ricci-Bourguignon flow and scalar curvature satisfies*

$$R \geq \frac{1 - 2(n-1)\rho}{2A - 1 + (n-2)\rho}\Delta\phi, \quad in\ M \times [0, T]. \tag{20}$$

If $\lambda(t)$ is the first eigenvalue of (2) then for $c \geq \frac{2(1-(n-2)\rho)}{3(1-2(n-1)\rho)}$, the quantity

$$e^{\int_0^t [-(2A+n\rho)\gamma(\tau)+\sigma(\tau)]d\tau}\lambda(t)$$

is nondecreasing under the Ricci-Bourguignon flow on $[0, T')$ where $A = c(1-2(n-1)\rho)$ and $\sigma(t)$ and $\gamma(t)$ introduced in (18) and (19), respectively.

Proof. According to hypothesis of the theorem the Ricci curvature operator is nonnegative along the Ricci-Bourguignon flow and on the other hand in [4], shown that the nonnegative of the scalar curvature is preserved along the Ricci-Bourguignon flow. Therefore (10) and (20) imply that for $\rho < \frac{1}{2(n-1)}$ and $c \geq \frac{2(1-(n-2)\rho)}{3(1-2(n-1)\rho)}$ we get

$$\frac{d}{dt}\lambda(t) \geq \lambda(2A - 1 + n\rho)\int_M Rf^2 d\mu \geq \lambda[(2A + n\rho)\gamma(t) - \sigma(t)]. \tag{21}$$

in last inequality we used $\int_M f^2 d\mu = 1$. Hence the theorem follows from the last inequality. □

Theorem 2.4. *Let $g(t)$, $t \in [0, T)$ be a solution to the Ricci-Bourguignon flow (1) on a closed Riemannian manifold M^n with nonnegative scalar curvature and the scalar curvature satisfies*

$$R \geq \frac{2(n-1)}{4c(n-1) - n + 2}\Delta\phi, \quad in\ M \times [0, T). \tag{22}$$

Suppose that $n\geq 3$ and the Ricci curvature satisfies

$$|Ric-\frac{1}{4c-1}\nabla\nabla\phi|^2\geq\frac{4c}{(4c-1)^2}|\nabla\nabla\phi|^2,\quad in\ M\times[0,T) \tag{23}$$

where $c>\frac{n-2}{2(n-1)}$ and $\phi\in C^\infty(M)$ satisfies the heat equation

$$\frac{\partial\phi}{\partial t}=\Delta\phi. \tag{24}$$

Then for $\rho\leq 0$ the quantity $e^{-\int_0^t(-\rho n\gamma(\tau)+4\rho c(n-1)\sigma(\tau))d\tau}\lambda(t)$ is nondecreasing along the Ricci-Bourguignon flow, where $\sigma(t)$ and $\gamma(t)$ introduced in (18) and (19), respectively.

Proof. From (12) and (17), we have

$$\begin{aligned}\frac{d}{dt}\lambda(t) &= -2\rho\int_M\Big[-Rf\Delta f+Rf\phi_if_i+\frac{n-2}{2}\nabla_iRf_if\\ &\quad +c(n-1)\Delta Rf^2+cR^2f^2\Big]d\mu \\ &\quad +\int_M\left[-2R_{ij}f_{ij}f+2R_{ij}\phi_if_jf+c\Delta Rf^2+2c|Ric|^2f^2+f_i(\phi_t)_if\right]d\mu\end{aligned} \tag{25}$$

we set

$$I=\int_M\left[-Rf\Delta f+Rf\phi_if_i+\frac{n-2}{2}\nabla_iRf_if+c(n-1)\Delta Rf^2+cR^2f^2\right]d\mu$$

and

$$II=\int_M\left[-2R_{ij}f_{ij}f+2R_{ij}\phi_if_jf+c\Delta Rf^2+2c|Ric|^2f^2+f_i(\phi_t)_if\right]d\mu.$$

Notice that, using (13) and (14) we can rewrite I as follow:

$$\begin{aligned} I &= c(n-1)\int_M R|\nabla f-f\nabla\phi|^2d\mu+[-\frac{n}{2}c+2c^2(n-1)+c]\int_M R^2f^2d\mu\\ &\quad +[-(\frac{n}{2}-1)+c(n-1)]\int_M R|\nabla f|^2d\mu-[-\frac{n}{2}+2c(n-1)]\lambda\int_M Rf^2d\mu\\ &\quad -c(n-1)\int_M Rf^2\Delta\phi d\mu,\end{aligned} \tag{26}$$

on the other hand, in [10], has been shown that

$$\begin{aligned} II &= \frac{1}{2}\int_M|R_{ij}+\psi_{ij}|^2e^{-\psi}d\mu+\frac{4c-1}{2}\int_M|Ric|^2e^{-\psi}d\mu\\ &\quad +\int_M(\psi_{ij}\phi_{ij}+\frac{1}{2}\psi_i(\Delta\phi)_i)e^{-\psi}d\mu-\frac{1}{2}\int_M\psi_i(\phi_t)_ie^{-\psi}d\mu\end{aligned} \tag{27}$$

where $f^2=e^{-\psi}$ for some smooth function ψ. Therefore if $\frac{\partial\phi}{\partial t}=\Delta\phi$ then we have

$$-\frac{1}{2}\int_M\psi_i(\phi_t)_ie^{-\psi}d\mu=-\frac{1}{2}\int_M\psi_i(\Delta\phi)_ie^{-\psi}d\mu. \tag{28}$$

Combining (25), (26), (27) and (28) we arrive at

$$\begin{aligned}
\frac{d}{dt}\lambda(t) &= \frac{1}{2}\int_M |R_{ij}+\psi_{ij}|^2 e^{-\psi}d\mu + \frac{4c-1}{2}\int_M |Ric|^2 e^{-\psi}d\mu + \int_M \psi_{ij}\phi_{ij}e^{-\psi}d\mu \\
&\quad -2\rho c(n-1)\int_M R|\frac{1}{2}\nabla\psi - \nabla\phi|^2 e^{-\psi}d\mu \\
&\quad -2\rho[-\frac{n}{2}c + 2c^2(n-1) + c]\int_M R^2 e^{-\psi}d\mu \qquad (29)\\
&\quad +\frac{\rho}{2}[(\frac{n}{2}-1) - c(n-1)]\int_M R|\nabla\psi|^2 e^{-\psi}d\mu \\
&\quad +2\rho[-\frac{n}{2} + 2c(n-1)]\lambda\int_M R\,e^{-\psi}d\mu + 2\rho c(n-1)\int_M Re^{-\psi}\Delta\phi d\mu \\
&= \frac{1}{2}\int_M |R_{ij}+\psi_{ij}+\phi_{ij}|^2 e^{-\psi}d\mu \\
&\quad +\frac{4c-1}{2}\int_M (|R_{ij} - \frac{1}{4c-1}\phi_{ij}|^2 - \frac{4c}{(4c-1)^2}|\phi_{ij}|^2)e^{-\psi}d\mu \\
&\quad -2\rho c(n-1)\int_M R|\frac{1}{2}\nabla\psi - \nabla\phi|^2 e^{-\psi}d\mu \\
&\quad -2\rho[-\frac{n}{2}c + 2c^2(n-1) + c]\int_M R^2 e^{-\psi}d\mu \\
&\quad +\frac{\rho}{2}[(\frac{n}{2}-1) - c(n-1)]\int_M R|\nabla\psi|^2 e^{-\psi}d\mu \\
&\quad +2\rho[-\frac{n}{2} + 2c(n-1)]\lambda\int_M R\,e^{-\psi}d\mu + 2\rho c(n-1)\int_M Re^{-\psi}\Delta\phi d\mu.
\end{aligned}$$

The nonnegative scalar curvature is preserved along the Ricci-Bourguignon flow, then (22) and (23) for $\rho \leq 0$ and $c > \frac{n-2}{2(n-1)}$ imply that

$$\frac{d}{dt}\lambda(t) \geq 2\rho[-\frac{n}{2} + 2c(n-1)]\lambda\int_M R\,e^{-\psi}d\mu \geq [-\rho n\gamma(t) + 4\rho c(n-1)\sigma(t)]\,\lambda \quad (30)$$

the last inequality complete the proof of theorem. □

If we assume that ϕ is constant function and $\rho = 0$ then (29) implies that for $c \geq \frac{1}{4}$ the eigenvalues is strictly increasing under the Ricci-Bourguignon flow, where this result funded by Cao in [3].

References

[1] Bourguignon, J.P., "Ricci curvature and Einstein metrics, Global differential geometry and global analysis" (Berlin,1979) Lecture nots in Math. vol. 838, Springer, Berlin, 1981, 42-63.

[2] Cao, X.D., "Eigenvalues of $(-\Delta + \frac{R}{2})$ on manifolds with nonnegative curvature operator", *Math. Ann.* **337:2**(2007), 435-441.

[3] Cao, X.D., "First eigenvalues of geometric operators under the Ricci flow",*Proc. Amer. Math. Soc.*, **136**(2008), 4075-4078.

[4] Catino, G., Cremaschi, L., Djadli, Z., Mantegazza, C., and Mazzieri, L., "The Ricci-Bourguignon flow", *Pacific J. Math.*, (2015).

[5] Cerbo, L.F.D., "Eigenvalues of the Laplacian under the Ricci flow", *Rendiconti di Mathematica, Serie VII*, **27**(2007), 183-195.

[6] Cheng, Q.-M., and Yang, H.C., "Estimates on eigenvalues of Laplacian",*Math. Ann.*, **331**(2005), 445-460.

[7] Fang, S., and Yang, F., "First eigenvalues of geometric operators under the Yamabe flow", *Bull. Korean Math. Soc.*, **53**(2016), 1113-1122.

[8] Li, J.F., "Eigenvalues and energy functionals with monotonicity formula under Ricci flow", *Math. Ann.*, (2007), 338, 927-946.

[9] Perelman, G., "The entropy formula for the Ricci flow and its geometric applications", (2002), ArXiv: 0211159.

[10] Wen, F.S., Feng, X.H., and Peng, Z., "Evolution and monotonicity of eigenvalues under the Ricci flow",*Sci. China Math.*, **58**(2015),no. 8, 1737-1744.

[11] Wu, J.Y., "First eigenvalue monotonicity for the p-Laplace operator under the Ricci flow", *Acta mathematica Sinica, English senes*, **27:8**(2011), 1591-1598.

[12] Zeng, F., He, Q., and Chen, B., "Monotonicity of eigenvalues of geometric operators along the Ricci-Bourguignon flow", Arxiv, 152.08158v1.