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ON THE DIOPHANTINE EQUATION $2^x + 17^y = z^2$

Julius Fergy T. Rabago

Department of Mathematics and Computer Science
College of Science
University of the Philippines Baguio
Baguio City 2600, Benguet, Philippines
e-mail: jfrabago@gmail.com

Abstract. We show that the Diophantine equation $2^x + 17^y = z^2$ has exactly five solutions (x, y, z) in positive integers. The only solutions are $(3, 1, 5)$, $(5, 1, 7)$, $(6, 1, 9)$, $(7, 3, 71)$ and $(9, 1, 23)$. This note, in turn, addresses an open problem proposed by Sroysang in [10].

Key words and Phrases: Diophantine equation, integer solution.

Abstrak. Pada paper ini kami memperlihatkan bahwa persamaan Diophantine $2^x + 17^y = z^2$ mempunyai tepat lima solusi (x, y, z) dalam bilangan-bilangan bulat positif. Solusinya adalah $(3, 1, 5)$, $(5, 1, 7)$, $(6, 1, 9)$, $(7, 3, 71)$ dan $(9, 1, 23)$. Hasil ini berkaitan dengan open problem yang diusulkan oleh Sroysang di [10].

Kata kunci: Persamaan Diophantine, solusi bilangan bulat.

1. Introduction

Several Diophantine equations of type $a^x + b^y = c^z$ have been of interest in previous decades, see, e.g., [1]–[12]. Most of these studies focused on the case when b is prime. For instance, in [1], Acu showed that $(3, 0, 3)$ and $(2, 1, 3)$ are the only solutions to the equation $2^x + 5^y = z^2$ in the set of non-negative integers $\mathbb{N}_0$. Meanwhile, in [4], Suvarnamani et al. were able to show through Mihăilescu's Theorem [3] that the equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solution in $\mathbb{N}_0$. Quite recently, as motivated by the results delivered in [6, 8, 9], Qi and Li [5] examined the solvability of the equation $8^x + p^y = z^2$ in $\mathbb{N}$ for fixed prime p. In [12], on the other hand, a classification of solutions $(b, x, y, z) \in \mathbb{N}^4$ of the equation $2^x + b^y = c^z$ was given by Yu and Li. For instance, it was shown that the equation

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$2^x + b^y = c^z$ admits a solution for $x > 1$, $y = 1$, $2|z$ and $2^x < b^{50/13}$. A particular solution to $2^x + b^y = c^z$ under the previously mentioned assumptions, however, was beyond the scope of the study presented in [12]. In this note, as inspired by these aforementioned works, we shall show using certain results on exponential Diophantine equations that the equation $2^x + 17^y = z^2$ has exactly five solutions (x, y, z) in $\mathbb{N}^3$. More precisely, we prove that the only solution $(x, y, z) \in \mathbb{N}^3$ to $2^x + 17^y = z^2$ are $(3, 1, 5)$, $(5, 1, 7)$, $(6, 1, 9)$, $(7, 3, 71)$ and $(9, 1, 23)$. We emphasize that the problem we are considering here is in fact an open question raised by Sroysang in [10]. More precisely, Sroysang asked the set of all solutions (x, y, z) of the equation $2^x + 17^y = z^2$, for non-negative integers x, y and z. Consequently, this work addresses the solution to this open problem put forward in [10]. We formally state and prove our main result in the next section.

2. Main Result

Our main result reads as follows.

Theorem 2.1. *The only solution $(x, y, z) \in \mathbb{N}^3$ to the equation $2^x + 17^y = z^2$ are $(3, 1, 5)$, $(5, 1, 7)$, $(6, 1, 9)$, $(7, 3, 71)$ and $(9, 1, 23)$.*

In proving the above theorem, we need the following results from [2, 3, 6, 12].

Lemma 2.2 ([3])**.** *The equation $a^x - b^y = 1$, $a, b, x, y \in \mathbb{N}$, $\min\{a, b, x, y\} > 1$, has the only solution $(a, b, x, y) = (3, 2, 2, 3)$.*

Lemma 2.3 ([6])**.** *The only solution $(x, y, z) \in \mathbb{N}_0^3$ to the equation $8^x + 17^y = z^2$ are $(1, 0, 3)$, $(1, 1, 5)$, $(2, 1, 9)$ and $(3, 1, 23)$.*

Lemma 2.4 ([2])**.** *The equation $2^x + b^y = z^2$, $b, x, y, z \in \mathbb{N}$, $\gcd(b, z) = 1$, $b > 1$, $x > 1$, $y \geq 3$, has the only solution $(b, x, y, z) = (17, 7, 3, 71)$.*

Lemma 2.5 ([12])**.** *The equation $2^x + b^y = c^z$ admits a solution for $x > 1$, $y = 1$, $2|z$ and $2^x < b^{50/13}$.*

Now we prove Theorem 2.1 as follows.

Proof of Theorem 2.1. Let $x, y, z \in \mathbb{N}$. First, we determine the parity of x, y and z. Since $x \geq 1$, then obviously z is odd. Moreover, it is clear that the equation $2^x + 17^y = z^2$ may admit a solution (x, y, z) in $\mathbb{N}^3$ provided y is odd, otherwise we'll obtain a contradiction (cf. [6]). Similarly, except for the possibility that x may take the number 6 as its value, x must always be odd. If not, we get $17^y = (z + 2^{x'})(z - 2^{x'})$, $x' = x/2$, and so $17^\beta - 17^\alpha = 2^{x'+1}$, where $\alpha + \beta = y$. Evidently, α cannot be at least the unity since, otherwise, $17(17^{\beta-1} - 17^{\alpha-1}) = 2^{x'+1}$ which is not possible. Meanwhile, if $\alpha = 0$, then $17^y - 1 = 2^{x'+1}$, or equivalently $17^y - 2^{x'+1} = 1$. For $y > 1$, employing Lemma

2.2, this equation admits no solution in $\mathbb{N}$. On the other hand, if $y = 1$, then we get $2^{x'+1} = 2^4$. This yields the value $x' = 3$, and $x = 6$, giving us the solution $(x, y, z) = (6, 1, 9)$. So, except when $x = 6$, x is always odd. Suppose now that $2^x + 17^y = z^2$ holds true for some triples $(x, y, z) \in \mathbb{N}^3$, where x, y and z are all odd. We consider two cases: (C.1) $3|x$ and (C.2) $3 \nmid x$. Hereafter we assume $k \in \mathbb{N}$.

Case 1. If $x = 3k$, $2 \nmid k$, then we have $2^{3k} + 17^y = z^2$, or equivalently $8^k + 17^y = z^2$. In view of Lemma 2.3, the only solution $(k, y, z) \in \mathbb{N}^3$ are $(1, 1, 5)$ and $(3, 1, 23)$. Hence, we get $(3, 1, 5)$ and $(9, 1, 23)$ as the only solutions to $2^x + 17^y = z^2$ in $\mathbb{N}$, for x divisible by three.

Case 2. Now, assume that $3 \nmid x$. First, we suppose that $x = 1$. Then, we have $2 + 17^y = z^2$. Note that $17^y \equiv 1 \pmod 4$. Hence, taking modulo 4, we get $2 + 17^y \equiv 3 \pmod 4$ while $z^2 \equiv 1 \pmod 4$, a contradiction. Therefore, $x > 1$. Apparently, $17 \nmid z$ because the congruence $2^x \equiv 0 \pmod{17}$ is impossible. So, for $y \geq 3$, the only solution we get is $(x, y, z) = (7, 3, 71)$ because of Lemma 2.4. Now, we are left with the possibility that $y = 1$. Since z has quadratic exponent, we know from Lemma 2.5 that the equation $2^x + 17 = z^2$ may admit a solution in $\mathbb{N}$ such that

$$x < \frac{50 \log 17}{13 \log 2} < 16.$$

Since the bound for x is small, one can effectively use a simple mathematical program to find whether there is any integer x on the interval $(1, 16)$ that makes the quantity $\sqrt{2^x + 17}$ an integer. Nevertheless, the values of x that could satisfy the equation $2^x + 17 = z^2$ may be obtained theoretically, and this we show as follows.

Rewriting $2^x + 17 = z^2$ as $3[(2^x + 1)/3] = (z + 4)(z - 4)$, we see that z must be at least 5. We know that z is odd, so $z = 2l + 1$ for some integer $l \geq 2$. It follows that $2^x + 17 = (2l + 1)^2 = 4l^2 + 4l + 1$, or equivalently $4l^2 + 4l - 2^x = 16$. Suppose that l is even, say $l = 2^s m$ for some $s, m \in \mathbb{N}$ where m is odd. Then,

$$2^x (2^{2s+2-x} m^2 + 2^{s+2-x} m - 1) = 2^4. \tag{1}$$

Recall that $x > 1$, $3 \nmid x$ and x is odd, so x must be at least 5. Thus, from (1), we get a contradiction, and so l cannot be an even integer. Hence, l is odd. For $x \geq 5$ and l odd, we have

$$2^x = 4l^2 + 4l - 16 \quad \Longleftrightarrow \quad 2^2(2^{x-4} + 1) = l(l + 1).$$

Now, l being odd implies that $l + 1 = 4$, or equivalently $l = 3$. On the other hand, we get $2^{x-4} = 2$, from which we obtain $x = 5$. Finally, this give us the solution $(x, y, z) = (5, 1, 7)$.

In concluding, the only solution (x, y, z) in $\mathbb{N}^3$ to the equation $2^x + 17^y = z^2$ are $(3, 1, 5)$, $(5, 1, 7)$, $(6, 1, 9)$, $(7, 3, 71)$ and $(9, 1, 23)$.

Corollary 2.6. *Let $n \in \mathbb{N} \setminus \{1\}$. Then, the Diophantine equation $2^x + 17^y = w^{2n}$ has a unique solution in positive integers, i.e., $(n, x, y, z) = (1, 6, 1, 3)$.*

Proof. Let $n > 1$ be a natural number and suppose that the equation $2^x + 17^y = (w^n)^2$ has a solution in positive integers. We let $z = w^n$, then we have $2^x + 17^y = z^2$.

By Theorem 2.1, $z \in \{5, 7, 9, 23, 71\}$. Hence, $w^n = 5, 7, 9, 23$ or 71. The case when $w^n = 5, 7, 23$ and 71 are only possible when $n = 1$. This contradicts the assumption that $n > 1$. On the other hand, the equation $w^n = 9$ implies that $w = 3$ and $n = 2$. Thus, $2^x + 31^y = w^{2n}$ has a unique solution $(n, x, y, z) = (1, 6, 1, 3)$ in $\mathbb{N}^4$.

We end our discussion with the following remark.

Remark 1. If we allow x, y or z in Theorem 2.1 to be zero, then $(x, y, z) = (3, 0, 3)$ is a solution to $2^x + 17^y = z^2$ because $2^3 + 17^0 = 9 = 3^2$. Meanwhile, x can never be zero since the equation $17^y = z^2 - 1$ will lead to a contradiction. Indeed, for y, z in $\mathbb{N}$, we have $17^\alpha(17^{\beta-\alpha} - 1) = 17^\beta - 17^\alpha = (z + 1) - (z - 1) = 2$, where $\alpha + \beta = y$. Evidently, $\alpha = 0$, and we get $17^y = 3$ which is clearly impossible. Thus, we have the unique solution $(x, y, z) = (3, 0, 3)$ in $\mathbb{N}_0^3$, with at least one of x, y and z is zero, to the equation $2^x + 17^y = z^2$. This result, together with Theorem 2.1, answers completely the question raised by Sroysang in [10].

Appendix

An Alternative Proof to Lemma 2.3. Lemma 2.3, which was originally proposed as open problem in [8], has been proven by the author in [6] independently from the approach presented here. Nevertheless, the complete set of solution to the equation $8^x + 17^y = z^2$ in $\mathbb{N}_0$ can be obtained using Lemma 2.4 and Lemma 2.5 without any difficulty. Indeed, suppose $8^x + 17^y = z^2$ admits a solution $(x, y, z) \in \mathbb{N}^3$. Clearly, $17 \nmid z$, and so by Lemma 2.4 we only need to consider the case when $y \in \{1, 2\}$. However, the case $y = 2$ is impossible since the equation $8^x + 17^2 = z^2$ would imply that $2^\alpha(2^{\beta-\alpha} - 1) = 2 \cdot 17$. This equation, in turn, would mean that $\alpha = 1$, and so $2^{3x-1} = 2^4$ or equivalently, $x = 5/3$ which is a obviously a contradiction to the assumption that $x \in \mathbb{N}$. This leaves us to consider the case when $y = 1$. Now, from Lemma 2.5, we see that $2^{3x} < 17^{50/13}$. A quick computation gives the bound $1 \le x < 5.24$. Since the upper bound for x is small, then we can manually test each $x \in \{1, 2, 3, 4, 5\}$ to see which of these quantities give an integer value for $\sqrt{8^x + 17}$. However, for $x = 2x'$, $x' \in \mathbb{N}$, we get $z^2 - (2^{3x'})^2 = 17$ which implies that $2^{3x'+1} = 2^4$, giving us $x' = 1$. Therefore, the only possible even value for x is 2, eliminating the possibility that $x = 4$. So, we have $(x, y, z) = (2, 1, 9)$. The remaining possibility is easily verified by direct substitution, leaving the value $x = 5$ inadmissible. Therefore, we obtain the other two solutions $(1, 1, 5)$ and $(3, 1, 23)$ for $(x, y, z) \in \mathbb{N}^3$. Now, expanding the set of solutions to $\mathbb{N}_0^3$ yields the only additional solution $(x, y, z) = (1, 0, 3)$. This completes the proof of Lemma 2.3 in an alternative fashion.

References

[1] Acu, D., On a Diophantine equation $2^x + 5^y = z^2$, *Gen. Math.*, **15**:4 (2007), 145-148.

[2] Ivorra, W., Sur les équations $x^p + 2^\beta y^p = z^2$ et $x^p + 2^\beta y^p = 2z^2$, *Acta Arith.*, **108**:4 (2003), 327-338.

[3] Mihăilescu, P., Primary cyclotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.*, **572** (2004), 167-195.

[4] Suvarnamani, A., Singta, A. and Chotchaisthit, S., On two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, *Sci. Technol. RMUTT J.*, **1** (2011), 25-28.

[5] Qi, L. and Li, X., The Diophantine equation $8^x + p^y = z^2$, *Sci. World J.*, Vol. 2015, Article ID 306590, 3 pages.

[6] Rabago, J.F.T., On an open problem by B. Sroysang, *Konuralp J. Math.*, **1**:2 (2013), 30-32.

[7] Rabago, J.F.T., A note on an open problem by B. Sroysang, *Sci. Technol. RMUTT J.*, **3**:1 (2013), 41-43.

[8] Sroysang, B., More on the Diophantine equation $8^x + 19^y = z^2$, *Int. J. Pure Appl. Math.*, **81**:4 (2012), 601-604.

[9] Sroysang, B., On the Diophantine equation $31^x + 32^y = z^2$, *Int. J. Pure Appl. Math.*, **81**:4 (2012), 609-612.

[10] Sroysang, B., More on Diophantine equation $2^x + 19^y = z^2$, *Int. J. Pure Appl. Math.*, **88**:1 (2013), 157-160.

[11] Sroysang, B., More on the Diophantine equation $8^x + 13^y = z^2$, *Int. J. Pure Appl. Math.*, **90**:1 (2014), 69-72.

[12] Yu, Y. and Li, X., The exponential Diophantine equation $2^x + b^y = c^z$, *Sci. World J.*, Vol. 2014, Article ID 401816, 3 pages.