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CONSISTENCY OF A KERNEL-TYPE ESTIMATOR OF THE INTENCITY OF THE CYCLIC POISSON PROCESS WITH THE LINEAR TREND

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Abstract. A consistent kernel-type nonparametric estimator of the intensity function of a cyclic Poisson process in the presence of linear trend is constructed and investigated. It is assumed that only a single realization of the Poisson process is observed in a bounded window. We prove that the proposed estimator is consistent when the size of the window indefinitely expands.

1. INTRODUCTION

Let N be a Poisson process on $[0, \infty)$ with (unknown) locally integrable intensity function λ. We assume that λ consists of two components, namely a cyclic (periodic) component with period $\tau > 0$ and a linear trend. In other words, for each point $s \in [0, \infty)$, we can write λ as

$$\lambda(s) = \lambda_c(s) + as, \tag{1}$$

where $\lambda_c(s)$ is (unknown) periodic function with (known) period τ and a denotes (unknown) slope of the linear trend. In this paper, we do not assume any parametric form of λ_c, except that it is periodic. That is, for each point $s \in [0, \infty)$ and all $k \in \mathbf{Z}$, with $\mathbf{Z}$ denotes the set of integers, we have

$$\lambda_c(s + k\tau) = \lambda_c(s). \tag{2}$$

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Here we consider a Poisson process on $[0, \infty)$ instead of, for example, on $\mathbf{R}$ because λ has to satisfy (1) and must be non negative. For the same reason we also restrict our attention to the case $a \geq 0$. The present paper (cf. also [2]) aims at extending previous work for the purely cyclic case, i.e. $a = 0$, (cf. [3], [4], [6], section 2.3 of [7]) to the more general model (1).

Suppose now that, for some $\omega \in \Omega$, a single realization $N(\omega)$ of the Poisson process N defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ (cf. (1)) is observed, though only within a bounded interval $W_n = [0, n] \subset [0, \infty)$. Our goal in this paper is to construct a consistent (general) kernel-type estimator of λ_c at a given point $s \in [0, \infty)$ using only a single realization $N(\omega)$ of the Poisson process N observed in interval $W_n = [0, n]$.

There are many practical situations where we have to use only a single realization for estimating intensity of a cyclic Poisson process. A review of such applications can be seen in [3], and a number of them can also be found in [1], [5], [7], [9] and [10].

We will assume throughout that s is a Lebesgue point of λ, that is we have

$$\lim_{h \downarrow 0} \frac{1}{2h} \int_{-h}^{h} |\lambda(s + x) - \lambda(s)| dx = 0 \tag{3}$$

(eg. see [11], p.107-108), which automatically means that s is a Lebesgue point of λ_c as well.

Note that, since λ_c is a periodic function with period τ, the problem of estimating λ_c at a given point $s \in [0, \infty)$ can be reduced into a problem of estimating λ_c at a given point $s \in [0, \tau)$. Hence, for the rest of this paper, we will assume that $s \in [0, \tau)$.

Note also that, the meaning of the asymptotic $n \to \infty$ in this paper is somewhat different from the classical one. Here n does not denote our sample size, but it denotes the length of the interval of observations. The size of our samples is a random variable denoted by $N([0, n])$.

2. Construction of the estimator and results

Let $K : \mathbf{R} \to \mathbf{R}$ be a real valued function, called *kernel*, which satisfies the following conditions: (K1) K is a probability density function, (K2) K is bounded, and (K3) K has (closed) support $[-1, 1]$. Let also h_n be a sequence of positive real numbers converging to 0, that is,

$$h_n \downarrow 0, \tag{1}$$

as $n \to \infty$.

Using the introduced notations, we may define the estimators of respectively a and λ_c at a given point $s \in [0, \tau)$ as follows

$$\hat{a}_n := \frac{2N([0, n])}{n^2}, \tag{2}$$

and

$$\hat{\lambda}_{c,n,K}(s) := \frac{1}{\ln(\frac{n}{\tau})} \sum_{k=1}^{\infty} \frac{1}{kh_n} \int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right) N(dx) - \hat{a}_n \left(s + \frac{n}{\ln(\frac{n}{\tau})}\right). \quad (3)$$

The estimator given in (3) is a generalization of the estimator discussed and investigated in Helmers and Mangku [2] for the case that the period τ is known. A general kernel-type estimator of the intensity of a purely cyclic Poisson process (i.e. $a = 0$) was proposed and studied in Helmers, Mangku and Zitikis ([3], [4]).

If we are interested in estimating $\lambda(s)$ at a given point s, then $\lambda(s)$ can be estimated by

$$\hat{\lambda}_{n,K}(s) = \hat{\lambda}_{c,n,K}(s) + \hat{a}_n s. \quad (4)$$

To obtain the estimator $\hat{a}_n$ of a, it suffices to note that

$$\mathbf{E}N([0,n]) = \frac{a}{2}n^2 + \mathcal{O}(n),$$

as $n \to \infty$, which directly yields the estimator given in (2). Note also that, if N were a Poisson proses with intensity function $\lambda(s) = as$, then $\hat{a}_n$ would be the maximum likelihood estimator of a (see [8]).

Next we describe the idea behind the construction of the kernel-type estimator $\hat{\lambda}_{c,n,K}(s)$ of $\lambda_c(s)$. By (1) and (2) we have that, for any point s and $k \in \mathbf{N}$ ($\mathbf{N}$ denotes the set of natural numbers),

$$\lambda_c(s) = \lambda_c(s+k\tau) = \lambda(s+k\tau) - a(s+k\tau). \quad (5)$$

Let $B_h(x) := [x-h, x+h]$ and $L_n := \sum_{k=-\infty}^{\infty} k^{-1}\mathbf{I}(s+k\tau \in [0,n])$. By (5), we can write

$$\begin{aligned}
\lambda_c(s) &= \frac{1}{L_n}\sum_{k=1}^{\infty}\frac{1}{k}\left(\lambda_c(s+k\tau)\right)\mathbf{I}(s+k\tau \in [0,n]) \\
&= \frac{1}{L_n}\sum_{k=1}^{\infty}\frac{1}{k}\left(\lambda(s+k\tau) - a(s+k\tau)\right)\mathbf{I}(s+k\tau \in [0,n]) \\
&= \frac{1}{L_n}\sum_{k=1}^{\infty}\frac{1}{k}\left(\lambda(s+k\tau)\right)\mathbf{I}(s+k\tau \in [0,n]) - as \\
&\quad - \frac{a\tau}{L_n}\sum_{k=1}^{\infty}\mathbf{I}(s+k\tau \in [0,n]). \qquad (6)
\end{aligned}$$

By (1) and the assumption that s is a Lebesgue point of λ, we have

$$\begin{aligned}
\lambda_c(s) &\approx \frac{1}{L_n}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{|B_{h_n}(s+k\tau)|}\int_{B_{h_n}(s+k\tau)\cap[0,n]}\lambda(x)dx - as - \frac{an}{L_n} \\
&= \frac{1}{L_n}\sum_{k=1}^{\infty}\frac{1}{k}\frac{\mathbf{E}N(B_{h_n}(s+k\tau)\cap[0,n])}{2h_n} - a\left(s+\frac{n}{L_n}\right) \\
&\approx \frac{1}{L_n}\sum_{k=1}^{\infty}\frac{1}{k}\frac{N(B_{h_n}(s+k\tau)\cap[0,n])}{2h_n} - a\left(s+\frac{n}{L_n}\right). \qquad (7)
\end{aligned}$$

In the first $\approx$ in (7) we also have used the fact that

$$\frac{a\tau}{L_n}\sum_{k=1}^{\infty}\mathbf{I}(s+k\tau\in[0,n]) = \frac{a\tau}{L_n}\left(\frac{n}{\tau}+\mathcal{O}(1)\right) = \frac{an}{L_n}+\mathcal{O}\left(\frac{1}{L_n}\right)\approx\frac{an}{L_n}.$$

From the second $\approx$ in (7) and by noting that $L_n \sim \ln(n/\tau)$ as $n\to\infty$, we see that

$$\bar{\lambda}_{c,n}(s) = \frac{1}{\ln(\frac{n}{\tau})}\sum_{k=1}^{\infty}\frac{1}{k}\frac{N([s+k\tau-h_n, s+k\tau+h_n]\cap[0,n])}{2h_n} - a\left(s+\frac{n}{\ln(\frac{n}{\tau})}\right) \qquad (8)$$

can be viewed as an estimator of $\lambda_c(s)$, provided the slope a of the linear trend to be known. If a is unknown, we replace a by $\hat{a}_n$ (cf. (2)) and one obtains the estimator of $\lambda_c(s)$ given by

$$\hat{\lambda}_{c,n}(s) = \frac{1}{\ln(\frac{n}{\tau})}\sum_{k=1}^{\infty}\frac{1}{k}\frac{N([s+k\tau-h_n, s+k\tau+h_n]\cap[0,n])}{2h_n} - \hat{a}_n\left(s+\frac{n}{\ln(\frac{n}{\tau})}\right). \qquad (9)$$

Now note that the estimator $\hat{\lambda}_{c,n}(s)$ given in (9) is a special case of the estimator $\hat{\lambda}_{c,n,K}(s)$ in (3), that is in (9) we use the uniform kernel $\bar{K} = \frac{1}{2}\mathbf{I}_{[-1,1]}(.)$. Replacing this uniform kernel by a general kernel K, we then obtain the estimator of λ_c given in (3).

In Helmers and Mangku [2] has been proved the following lemma.

Lema 1. *Suppose that the intensity function λ satisfies (1) and is locally integrable. Then we have*

$$\mathbf{E}(\hat{a}_n) = a + \frac{2\theta}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \qquad (10)$$

and

$$Var(\hat{a}_n) = \frac{2a}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \qquad (11)$$

as $n \to \infty$, where $\theta = \tau^{-1} \int_0^\tau \lambda_c(s)ds$, the global intensity of the periodic component λ_c. Hence $\hat{a}_n$ is a consistent estimator of a. Its MSE (mean-squared-error) is given by

$$MSE\left(\hat{a}_n\right) = \frac{4\theta^2 + 2a}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \tag{12}$$

as $n \to \infty$.

Our main results are presented in the following theorem and corollary.

Theorem 1. *Suppose that the intensity function λ satisfies (1) and is locally integrable. If the kernel K satisfies conditions $(K1), (K2), (K3)$, and h_n satisfies assumptions (1) and*

$$h_n \ln n \to \infty, \tag{13}$$

then

$$\hat{\lambda}_{c,n,K}(s) \xrightarrow{p} \lambda_c(s), \tag{14}$$

as $n \to \infty$, provided s is a Lebesgue point of λ_c. In other words, $\hat{\lambda}_{c,n,K}(s)$ is a consistent estimator of $\lambda_c(s)$. In addition, the MSE of $\hat{\lambda}_{c,n,K}(s)$ converges to 0, as $n \to \infty$.

We note that, Lemma 1 and Theorem 1 together imply the following result.

Corollary 1. *Suppose that the intensity function λ satisfies (1) and is locally integrable. If the kernel K satisfies conditions $(K1), (K2), (K3)$, and h_n satisfies assumptions (1) and (13), then*

$$\hat{\lambda}_{n,K}(s) \xrightarrow{p} \lambda(s), \tag{15}$$

as $n \to \infty$, provided s is a Lebesgue point of λ. In other words, $\hat{\lambda}_{n,K}(s)$ in (4) is a consistent estimator of $\lambda(s)$. In addition, the MSE of $\hat{\lambda}_{n,K}(s)$ converges to 0, as $n \to \infty$.

3. Proofs of Theorem 1

To prove Theorem 1, it suffices to verify the following two lemmas.

Lemma 2. (Asymptotic unbiasedness) *Suppose that the intensity function λ satisfies (1) and is locally integrable. If the kernel K satisfies conditions $(K1), (K2), (K3)$, and h_n satisfies assumptions (1) and (13), then*

$$\mathbf{E}\hat{\lambda}_{c,n,K}(s) \to \lambda_c(s), \tag{1}$$

as $n \to \infty$, provided s is a Lebesgue point of λ_c.

Lemma 3. (Convergence of the variance) *Suppose that the intensity function λ satisfies (1) and is locally integrable. If the kernel K satisfies conditions $(K1), (K2), (K3)$, and h_n satisfies assumptions (1) and (13), then*

$$Var\left(\hat{\lambda}_{c,n,K}(s)\right) \;\to\; 0, \tag{2}$$

as $n \to \infty$, provided s is a Lebesgue point of λ_c.

Proof of Lemma 2

Note that

$$\mathbf{E}\hat{\lambda}_{c,n,K}(s) = \frac{1}{\ln(\frac{n}{\tau})}\sum_{k=1}^{\infty}\frac{1}{kh_n}\int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right)\mathbf{E}N(dx) - \left(s+\frac{n}{\ln(\frac{n}{\tau})}\right)\mathbf{E}\hat{a}_n. \tag{3}$$

First we consider the first term on the r.h.s. of (3). This term can be written as

$$\begin{aligned}
&\frac{1}{\ln(\frac{n}{\tau})}\sum_{k=1}^{\infty}\frac{1}{kh_n}\int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)dx \\
&= \frac{1}{\ln(\frac{n}{\tau})}\sum_{k=1}^{\infty}\frac{1}{kh_n}\int_{\mathbf{R}} K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)\mathbf{I}(x\in[0,n])dx.
\end{aligned} \tag{4}$$

By a change of variable and using (1) and (2), we can write the r.h.s. of (4) as

$$\begin{aligned}
&\frac{1}{\ln(\frac{n}{\tau})}\sum_{k=1}^{\infty}\frac{1}{kh_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau\in[0,n])dx \\
&= \frac{1}{\ln(\frac{n}{\tau})}\sum_{k=1}^{\infty}\frac{1}{kh_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda_c(x+s)\mathbf{I}(x+s+k\tau\in[0,n])dx \\
&\quad+\frac{1}{\ln(\frac{n}{\tau})}\sum_{k=1}^{\infty}\frac{1}{kh_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)a(x+s+k\tau)\mathbf{I}(x+s+k\tau\in[0,n])dx.
\end{aligned} \tag{5}$$

We will first show that the first term on the r.h.s. of (3), that is the r.h.s. of (5), is equal to

$$\lambda_c(s)+as+\frac{an}{\ln(\frac{n}{\tau})}+o(1), \tag{6}$$

as $n \to \infty$, by showing that the first term on the r.h.s. of (5) is equal to $\lambda_c(s)+o(1)$ and its second term is equal to $as + an/\ln(n/\tau) + o(1)$, as $n \to \infty$. To check this, note that the first term on the r.h.s. of (5) is equal to

$$\begin{aligned}
& \frac{1}{\ln(\frac{n}{\tau})}\sum_{k=1}^{\infty}\frac{1}{kh_n}\int_{\mathbf{R}}K\left(\frac{x}{h_n}\right)(\lambda_c(x+s)-\lambda_c(s))\,\mathbf{I}(x+s+k\tau\in[0,n])dx \\
& +\frac{\lambda_c(s)}{\ln(\frac{n}{\tau})}\sum_{k=1}^{\infty}\frac{1}{kh_n}\int_{\mathbf{R}}K\left(\frac{x}{h_n}\right)\mathbf{I}(x+s+k\tau\in[0,n])dx \\
= & \frac{1}{h_n\ln(\frac{n}{\tau})}\int_{\mathbf{R}}K\left(\frac{x}{h_n}\right)(\lambda_c(x+s)-\lambda_c(s))\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{I}(x+s+k\tau\in[0,n])dx \\
& +\frac{\lambda_c(s)}{h_n\ln(\frac{n}{\tau})}\int_{\mathbf{R}}K\left(\frac{x}{h_n}\right)\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{I}(x+s+k\tau\in[0,n])dx. \qquad (7)
\end{aligned}$$

Using the fact that

$$\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{I}(x+s+k\tau\in[0,n])=\ln(\frac{n}{\tau})+\mathcal{O}(1), \qquad (8)$$

as $n\to\infty$ uniformly in $x\in[-h_n,h_n]$, the r.h.s. of (7) can be written as

$$\begin{aligned}
= & \frac{1}{h_n\ln(\frac{n}{\tau})}\int_{\mathbf{R}}K\left(\frac{x}{h_n}\right)(\lambda_c(x+s)-\lambda_c(s))\left(\ln(\frac{n}{\tau})+\mathcal{O}(1)\right)dx \\
& +\frac{\lambda_c(s)}{h_n\ln(\frac{n}{\tau})}\int_{\mathbf{R}}K\left(\frac{x}{h_n}\right)\left(\ln(\frac{n}{\tau})+\mathcal{O}(1)\right)dx \\
= & \int_{\mathbf{R}}K\left(\frac{x}{h_n}\right)\frac{1}{h_n}(\lambda_c(x+s)-\lambda_c(s))\,dx \\
& +\lambda_c(s)\int_{\mathbf{R}}K(x)dx+\mathcal{O}\left(\frac{1}{h_n\ln n}\right), \qquad (9)
\end{aligned}$$

as $n\to\infty$. Since s is a Lebesque of λ_c (cf. (3)) and the kernel K satisfies conditions (K2) and (K3), it easily seen that the first term on the r.h.s. of (9) is $o(1)$, as $n\to\infty$. By the assumption: $\int_{\mathbf{R}}K(x)dx=1$ (cf. (K1)), the second term on the r.h.s. of (9) is equal to $\lambda_c(s)$. A simple argument using assumption (13) shows that the third term on the r.h.s. of (9) is $o(1)$, as $n\to\infty$. Hence, the first term on the r.h.s. of (5) is equal to $\lambda_c(s)+o(1)$, as $n\to\infty$.

Next we show that the second term on the r.h.s. of (5) is equal to $as+an/\ln(n/\tau)+o(1)$, as $n\to\infty$. To verify this, note that this term can be written as

$$
\begin{aligned}
& \frac{a}{\ln(\frac{n}{\tau})} \sum_{k=1}^{\infty} \frac{1}{kh_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) x \mathbf{I}(x+s+k\tau \in [0,n]) dx \\
& + \frac{as}{\ln(\frac{n}{\tau})} \sum_{k=1}^{\infty} \frac{1}{kh_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \mathbf{I}(x+s+k\tau \in [0,n]) dx \\
& + \frac{a\tau}{\ln(\frac{n}{\tau})} \sum_{k=1}^{\infty} \frac{1}{kh_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) k\mathbf{I}(x+s+k\tau \in [0,n]) dx \\
= \; & \frac{a}{h_n \ln(\frac{n}{\tau})} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) x \sum_{k=1}^{\infty} \frac{1}{k} \mathbf{I}(x+s+k\tau \in [0,n]) dx \\
& + \frac{as}{h_n \ln(\frac{n}{\tau})} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \sum_{k=1}^{\infty} \frac{1}{k} \mathbf{I}(x+s+k\tau \in [0,n]) dx \\
& + \frac{a\tau}{h_n \ln(\frac{n}{\tau})} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \sum_{k=1}^{\infty} \mathbf{I}(x+s+k\tau \in [0,n]) dx. \qquad (10)
\end{aligned}
$$

Using (8), and the fact that

$$
\sum_{k=1}^{\infty} \mathbf{I}(x+s+k\tau \in [0,n]) = \frac{n}{\tau} + \mathcal{O}(1),
$$

as $n \to \infty$ uniformly in $x \in [-h_n, h_n]$, the quantity in (10) can be written as

$$
\begin{aligned}
& \frac{a}{h_n \ln(\frac{n}{\tau})} \left(\ln(\frac{n}{\tau}) + \mathcal{O}(1)\right) \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) x dx \\
& \qquad + \frac{as}{h_n \ln(\frac{n}{\tau})} \left(\ln(\frac{n}{\tau}) + \mathcal{O}(1)\right) \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) dx \\
& \qquad\qquad + \frac{a\tau}{h_n \ln(\frac{n}{\tau})} \left(\frac{n}{\tau} + \mathcal{O}(1)\right) \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) dx. \qquad (11)
\end{aligned}
$$

Since K is bounded and $\int_{-1}^{1} x dx = 0$, the first term of (11) is equal to zero. A simple calculation shows that the second term of (11) is equal to $as + o(1)$ and the third term of (11) is equal to $an/\ln(n/\tau) + o(1)$ as $n \to \infty$. Hence, we have that the second term on the r.h.s. of (5) is equal to $as + an/\ln(n/\tau) + o(1)$ as $n \to \infty$. Combining this with the previous result, we obtain (6).

Finally we consider the second term on the r.h.s. of (3). By (10) of Lemma 1, this term can be computed as follows

$$
-\left(s + \frac{n}{\ln n}\right)\left(a + \frac{2\theta}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) = -as - \frac{an}{\ln(\frac{n}{\tau})} + o(1), \qquad (12)
$$

as $n \to \infty$. Combining (6) and (12) we obtain (1). This completes the proof of Lemma 2.

Proof of Lemma 3

The variance of $\hat{\lambda}_{c,n,K}(s)$ can be computed as follows

$$Var\left(\hat{\lambda}_{c,n,K}(s)\right) = Var\left(\frac{1}{\ln(\frac{n}{\tau})}\sum_{k=1}^{\infty}\frac{1}{kh_n}\int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right)N(dx)\right)$$
$$+Var\left(\hat{a}_n\left(s+\frac{n}{\ln(\frac{n}{\tau})}\right)\right)$$
$$+2Cov\left(\frac{1}{\ln(\frac{n}{\tau})}\sum_{k=1}^{\infty}\frac{1}{kh_n}\int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right)N(dx), \hat{a}_n\left(s+\frac{n}{\ln(\frac{n}{\tau})}\right)\right). \tag{13}$$

We will prove Lemma 3 by showing that each term on the r.h.s. of (13) is $o(1)$ as $n\to\infty$.

First we check that the first term on the r.h.s. of (13) is $o(1)$ as $n\to\infty$. To do this, we argue as follows. By (1), for sufficiently large n, we have that the intervals $[s+k\tau-h_n, s+k\tau+h_n]$ and $[s+j\tau-h_n, s+j\tau+h_n]$ are not overlap for all $k\neq j$. This implies, for all $k\neq j$,

$$K\left(\frac{x-(s+k\tau)}{h_n}\right)N(dx) \text{ and } K\left(\frac{x-(s+j\tau)}{h_n}\right)N(dx)$$

are independent. Hence, the variance in the first term on the r.h.s. of (13) can be computed as follows

$$\begin{aligned}
&\frac{1}{(h_n\ln(\frac{n}{\tau}))^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^n K^2\left(\frac{x-(s+k\tau)}{h_n}\right)Var(N(dx))\\
=\;&\frac{1}{(h_n\ln(\frac{n}{\tau}))^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^n K^2\left(\frac{x-(s+k\tau)}{h_n}\right)\mathbf{E}N(dx)\\
=\;&\frac{1}{(h_n\ln(\frac{n}{\tau}))^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^n K^2\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)dx.
\end{aligned}\tag{14}$$

By a change of variable and using (1) and (2), the r.h.s. of (14) can be written as

$$\begin{aligned}
&\frac{1}{(h_n\ln(\frac{n}{\tau}))^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)\lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau\in[0,n])dx\\
=\;&\frac{1}{(h_n\ln(\frac{n}{\tau}))^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)\lambda_c(x+s)\mathbf{I}(x+s+k\tau\in[0,n])dx\\
&+\frac{1}{(h_n\ln(\frac{n}{\tau}))^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)a(x+s+k\tau)\mathbf{I}(x+s+k\tau\in[0,n])dx.
\end{aligned}\tag{15}$$

The first term on the r.h.s. of (15) is equal to

$$\begin{aligned} & \frac{1}{(h_n \ln(\frac{n}{\tau}))^2} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)(\lambda_c(x+s)-\lambda_c(s)+\lambda_c(s)) \sum_{k=1}^{\infty} \frac{1}{k^2} \mathbf{I}(x+s+k\tau \in [0,n])dx \\ = & \frac{1}{(h_n \ln(\frac{n}{\tau}))^2} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)(\lambda_c(x+s)-\lambda_c(s)) \sum_{k=1}^{\infty} \frac{1}{k^2} \mathbf{I}(x+s+k\tau \in [0,n])dx \\ & + \frac{\lambda_c(s)}{(h_n \ln(\frac{n}{\tau}))^2} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right) \sum_{k=1}^{\infty} \frac{1}{k^2} \mathbf{I}(x+s+k\tau \in [0,n])dx. \qquad (16) \end{aligned}$$

Note that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \mathbf{I}(x+s+k\tau \in [0,n]) = \mathcal{O}(1), \qquad (17)$$

as $n \to \infty$, uniformly in $x \in [-h_n, h_n]$. Since the kernel K is bounded and has support in $[-1,1]$, by (3) and (17) we see that the first term on the r.h.s. of (16) is of order $o((\ln n)^{-2}(h_n)^{-1})) = o(1)$, as $n \to \infty$ (cf. (13)). A similar argument shows that the second term on the r.h.s. of (16) is of order $\mathcal{O}((h_n \ln n)^{-2}) = o(1)$, as $n \to \infty$.

Next we consider the second term on the r.h.s. of (15). This term can be written as

$$\begin{aligned} & \frac{a}{(h_n \ln(\frac{n}{\tau}))^2} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)(x+s) \sum_{k=1}^{\infty} \frac{1}{k^2} \mathbf{I}(x+s+k\tau \in [0,n])dx \\ & + \frac{a\tau}{(h_n \ln(\frac{n}{\tau}))^2} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right) \sum_{k=1}^{\infty} \frac{1}{k} \mathbf{I}(x+s+k\tau \in [0,n])dx. \qquad (18) \end{aligned}$$

By (17), the first term of (18) reduces to

$$\begin{aligned} & \mathcal{O}(1) \frac{a}{(h_n \ln(\frac{n}{\tau}))^2} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)(x+s)dx \\ & = \mathcal{O}(1) \frac{a}{(\ln(\frac{n}{\tau}))^2 h_n} \int_{\mathbf{R}} K^2(x)(xh_n+s)dx \\ & = \mathcal{O}\left(\frac{1}{(\ln(\frac{n}{\tau}))^2 h_n}\right) = o(1), \qquad (19) \end{aligned}$$

as $n \to \infty$. By a similar argument and using (8), we see that the second term of (18) is of order $\mathcal{O}((h_n \ln n)^{-1}) = o(1)$, as $n \to \infty$ (cf. (13)). Hence we have proved that the first term on the r.h.s. of (13) is $o(1)$, as $n \to \infty$.

Next we consider the second term on the r.h.s. of (13). By (11) of Lemma 1, this term can be computed as follows

$$\left(s + \frac{n}{\ln(\frac{n}{\tau})}\right)^2 Var\,(\hat{a}_n) = \left(s^2 + \frac{n^2}{(\ln(\frac{n}{\tau}))^2} + \frac{2sn}{\ln(\frac{n}{\tau})}\right)\left(\frac{2a}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right) = o(1), \quad (20)$$

as $n \to \infty$.

Finally, we consider the third term on the r.h.s. of (13). Since the first and second terms on the r.h.s. of (13) are both of order $o(1)$ as $n \to \infty$, by Cauchy-Schwarz, it easily seen that the third term on the r.h.s. of (13) is $o(1)$ as $n \to \infty$. Therefore, all terms on the r.h.s. of (13) are indeed of order $o(1)$ as $n \to \infty$, which imply (2). This completes the proof of Lemma 3.

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