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INTERVAL OSCILLATION CRITERIA FOR HIGHER ORDER NEUTRAL NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract. Some oscillation criteria for nth order neutral differential equations with deviating arguments of the form

$$[r(t)|(y(t)+p(t)y(\tau(t)))^{(n-1)}|^{\alpha-1}(y(t)+p(t)y(\tau(t)))^{(n-1)}]' + \sum_{i=1}^{m} q_i(t)f_i(y(\sigma_i(t))) = 0$$

n even are established. New oscillation criteria are different from most known ones in the sense that they based on a class of new function $H(t,s)$ defined in the sequel. The results are sharper than some previous results which can be seen by the examples at the end of this paper.

1. INTRODUCTION

In this paper we consider the oscillation behavior of solutions of the n-th order neutral differential equations of the form

$$[r(t)|(y(t)+p(t)y(\tau(t)))^{(n-1)}|^{\alpha-1}(y(t)+p(t)y(\tau(t)))^{(n-1)}]' + \sum_{i=1}^{m} q_i(t)f_i(y(\sigma_i(t))) = 0, \quad (1)$$

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where $t \geq t_0$, $n \geq 2$ is even integer, $\alpha > 0$ are constant. In this paper, we assume that

(I_1) $p(t) \in C([t_0,\infty);[0,1)), q_i(t) \in C([t_0,\infty);[0,\infty)), f_i \in C(R;R), i = 1,2,\cdots,m,$ $t \in [t_0,\infty)$.

(I_2) $r(t) \in C^1([t_0,\infty);(0,\infty)), r'(t) \geq 0, R_1(t) := \int_{t_0}^{t} \frac{ds}{r^{\frac{1}{\alpha}}(s)} \to \infty (t \to \infty)$.

(I_3) $\frac{f_i(x)}{|x|^{\alpha-1}x} \geq \beta_i > 0$ for $x \neq 0, \beta_i$ are constants, $i = 1,2,\cdots,m$.

(I_4) $\tau(t), \sigma_i(t) \in C^1([t_0,\infty);[0,\infty))$, $\tau(t) \leq t, \sigma_i(t) \leq t$, $\sigma'(t) > 0$ for $t \geq t_0$ and $\lim\limits_{t\to\infty} \sigma_i(t) = \lim\limits_{t\to\infty} \sigma(t) = \lim\limits_{t\to\infty} \tau(t) = \infty, i = 1,2,\cdots,m$, where $\sigma(t) \leq \min\{\sigma_1(t), \sigma_2(t), \cdots, \sigma_m(t), \frac{t}{2}\}$.

By a solution of Eq. (1), we mean a function $y(t) \in C^{n-1}([T_x,\infty);R)$ for some $T_x \geq t_0$ which has the property that

$$r(t)|(y(t)+p(t)y(\tau(t)))^{(n-1)}|^{\alpha-1}(y(t)+p(t)y(\tau(t)))^{(n-1)} \in C^1([T_x,\infty);R)$$

and satisfies Eq. (1) on $[T_x,\infty)$.

A nontrivial solution of Eq. (1) is called oscillatory if it has arbitrary large zero. Otherwise, it is called nonoscillatory. Eq. (1) is called oscillatory if all of its solutions are oscillatory.

If $p(t) = 0, r(t) = 1, m = 1$, then Eq. (1) becomes

$$(|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t))' + q(t)f(x(\sigma(t))) = 0 \tag{2}$$

and the related equations have been studied by Agarwal et. al. [2], Xu et. al. [15].

Eq. (1) with $n = 2, p(t) = 0, m = 1$, namely, the equation

$$[r(t)|x'(t)|^{\alpha-1}x'(t)]' + q(t)f(x(\sigma(t))) = 0 \tag{3}$$

and related equations have been investigated by Dzurina and Stavroulakis [4], Sun and Meng [14], Mirzov [10-12], Elbert [5,6] Agarwal et. al. [1], Chern et. al. [3], Li [7], Zhuang and Li [19].

Recently, Xu and Meng [16-18] have studied the oscillation properties of Eq. (1) for $n = 2$. Very recently Meng and Xu [8,9] have investigated the oscillation properties for higher order neutral differential equations.

Motivated by the idea of Li [7], by using averaging functions and inequality, in this paper we obtain several new interval criteria for oscillation, that is, criteria are given by the behavior of Eq. (1) (or of r, p and q_i) only on a sequence of subintervals of $[t,\infty)$. Our results improve and extend the results of Li [7] and Zhuang and Li [19]. In order to prove our Theorems, we use the function class X to study the oscillatory of Eq. (1). We say that a function $H = H(t,s)$ belongs to the function class X, if $H \in C(D;R_+)$, where $D = \{(t,s) : t_0 \leq s \leq t < \infty\}$, which satisfies $H(t,t) = 0, H(t,s) > 0$ for $t > s$, and has partial derivative $\frac{\partial H}{\partial s}$ and $\frac{\partial H}{\partial t}$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t,s)\sqrt{H(t,s)}, \frac{\partial H}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}, \tag{4}$$

where $h_1(t,s), h_2(t,s)$ are locally nonnegative continuous functions on D.

2. MAIN RESULTS

First, we give the following lemmas for our results.

Lemma 2.1. *[13] Let $u(t) \in C^n([t_0,\infty); R^+)$. If $u^{(n)}(t)$ is eventually of one sign for all large t, say $t_1 > t_0$, then there exist a $t_x > t_0$ and an integer $l, 0 \le l \le n$, with $n+l$ even for $u^{(n)}(t) \ge 0$ or $n+l$ odd for $u^{(n)}(t) \le 0$ such that $l > 0$ implies that $u^{(k)}(t) > 0$ for $t > t_x, k = 0,1,2,\cdots,l-1$, and $l \le n-1$ implies that $(-1)^{l+k}u^{(k)}(t) > 0$ for $t > t_x, k = l, l+1,\cdots,n-1$.*

Lemma 2.2. *[13] If the function $u(t)$ is as in Lemma 2.1 and $u^{(n-1)}(t)u^{(n)}(t) \le 0$ for $t > t_x$, then there exists a constant $\theta, 0 < \theta < 1$, such that*

$$u(t) \ge \frac{\theta}{(n-1)!}t^{n-1}u^{(n-1)}(t) \textit{ for all large } t.$$

and

$$u'(\frac{t}{2}) \ge \frac{\theta}{(n-2)!}t^{n-2}u^{(n-1)}(t) \textit{ for all large } t.$$

Lemma 2.3. *Suppose that $y(t)$ is an eventually positive solution of Eq. (1), let*

$$z(t) = y(t) + p(t)y(\tau(t)), \tag{5}$$

then there exists a number $t_1 \ge t_0$ such that

$$z(t) > 0, z'(t) > 0, z^{(n-1)}(t) > 0 \textit{ and } z^{(n)}(t) \le 0, t \ge t_1. \tag{6}$$

Proof. Since $y(t)$ is an eventually positive solution of (1), from (I_4), there exists a number $t_1 \ge t_0$ such that

$$y(t) > 0, y(\tau(t)) > 0, y(\sigma_i(t)) > 0, t \ge t_1. \tag{7}$$

Noting that $p(t) \geq 0$, we have $z(t) > 0, t \geq t_1$ and from $(I_1), (I_3)$ we have

$$(r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t))' = -\sum_{i=1}^{m} q_i(t)f_i(y(\sigma_i(t))) \leq 0, t \geq t_1.$$

So $r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$ is decreasing and $z^{(n-1)}(t)$ is eventually of one sign. we claim that

$$z^{(n-1)}(t) \geq 0 \quad \text{for} \quad t \geq t_1. \tag{8}$$

Otherwise, if there exist a $\widetilde{t_1} \geq t_1$ such that $z^{(n-1)}(\widetilde{t_1}) < 0$, then for all $t \geq \widetilde{t_1}$,

$$r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t) \leq r(\widetilde{t_1})|z^{(n-1)}(\widetilde{t_1})|^{\alpha-1}z^{(n-1)}(\widetilde{t_1}) = -C(C>0), \tag{9}$$

then we have $-z^{(n-1)}(t) \geq \left(\frac{C}{r(t)}\right)^{\frac{1}{\alpha}}, t \geq \widetilde{t_1}$, integrating the above inequality from $\widetilde{t_1}$ to t, we have

$$z^{(n-2)}(t) \leq z^{(n-2)}(\widetilde{t_1}) - C^{\frac{1}{\alpha}}(R(t) - R(\widetilde{t_1})).$$

Letting $t \to \infty$, from (I_2), we get $\lim\limits_{t\to\infty} z^{(n-2)}(t) = -\infty$, which implies $z^{(n-1)}(t)$ and $z^{(n-2)}(t)$ are negative for all large t, from Lemma 2.1, no two consecutive derivatives can be eventually negative, for this would imply that $\lim\limits_{t\to\infty} z(t) = -\infty$, a contradiction. Hence $z^{(n-1)}(t) \geq 0$ for $t \geq t_1$. from Eq. (1) and $(I_1), (I_2)$ we have

$$\begin{aligned}
&\alpha r(t)(z^{(n-1)}(t))^{\alpha-1}z^{(n)}(t) = [r(t)(z^{(n-1)}(t))^{\alpha}]' - r'(t)(z^{(n-1)}(t))^{\alpha} \\
= \quad & -\sum_{i=1}^{m} q_i(t)f_i(y(\sigma_i(t))) - r'(t)(z^{(n-1)}(t))^{\alpha} \leq 0, t \geq t_1,
\end{aligned}$$

this implies that $z^{(n)}(t) \leq 0, t \geq t_1$. From Lemma 2.1 again (note n is even), we have $z'(t) > 0, t \geq t_1$. This completes the proof.

Theorem 2.1. *Assume that there exist a positive, nondecreasing function $\rho(t) \in C^1([t_0,\infty))$ such that for any constant $M > 0$, some $H \in X$ and for each sufficient large $T_0 \geq t_0$, there exist increasing divergent sequences of positive numbers $\{a_n\}, \{b_n\}, \{c_n\}$ with $T_0 \leq a_n < c_n < b_n$ such that*

$$\begin{aligned}
& \frac{1}{H(c_n,a_n)}\int_{a_n}^{c_n} H(s,a_n)\rho(s)C_1(s)ds + \frac{1}{H(b_n,c_n)}\int_{c_n}^{b_n} H(b_n,s)\rho(s)C_1(s)ds \\
> \quad & \frac{1}{H(c_n,a_n)}\int_{a_n}^{c_n}\left[h_1(s,a_n) + \sqrt{H(s,a_n)}\frac{\rho'(s)}{\rho(s)}\right]^{\alpha+1}\frac{C_2(s)}{H^{\frac{\alpha-1}{2}}(s,a_n)}ds \\
+ \quad & \frac{1}{H(b_n,c_n)}\int_{c_n}^{b_n}\left[h_2(b_n,s) + \sqrt{H(b_n,s)}\frac{\rho'(s)}{\rho(s)}\right]^{\alpha+1}\frac{C_2(s)}{H^{\frac{\alpha-1}{2}}(b_n,s)}ds,
\end{aligned} \tag{10}$$

where

$$C_1(t) = \sum_{i=1}^{m} \beta_i q_i(t)(1 - p(\sigma_i(t)))^\alpha, C_2(t) = \frac{r(t)\rho(t)}{M^\alpha(\alpha+1)^{\alpha+1}(\sigma'(t)\sigma^{n-2}(t))^\alpha},$$

then every solution of Eq. (1) is oscillatory.

Proof. Suppose the contrary, let $y(t)$ is a nonoscillatory solution of Eq. (1), without loss of generality we assume

$$y(t) > 0, y(\tau(t)) > 0 \quad \text{for} \quad t \geq t_1 \geq t_0.$$

Then

$$z(t) = y(t) + p(t)y(\tau(t)) > 0 \quad \text{for} \quad t \geq t_1 \geq t_0. \tag{11}$$

From Lemma 2.3, there exists $t_2 \geq t_1$ such that

$$z(t) > 0, z'(t) > 0, z^{(n-1)}(t) > 0 \;\text{ and }\; z^{(n)}(t) \leq 0, t \geq t_2. \tag{12}$$

It is easy to check that we can apply Lemma 2.2 and conclude that there exist $0 < \theta < 1$ and $t_3 > t_2$ such that

$$\begin{aligned} z'(\sigma(t)) &\geq \frac{\theta}{(n-2)!}(2\sigma(t))^{n-2} z^{(n-1)}(2\sigma(t)) \\ &\geq \frac{\theta}{(n-2)!} 2^{n-2}\sigma^{n-2}(t) z^{(n-1)}(t) = M\sigma^{n-2}(t) z^{(n-1)}(t), t \geq t_3, \end{aligned} \tag{13}$$

where $M = \dfrac{\theta}{(n-2)!}2^{n-2}$.

From (5), we have

$$y(t) = z(t) - p(t)y(\tau(t)) \geq z(t) - p(t)z(\tau(t)) \geq z(t)(1 - p(t)), t \geq t_3. \tag{14}$$

Since $\lim_{t\to\infty} \sigma(t) = \infty$, there exists $t_4 \geq t_3$ such that $\sigma(t) \geq t_3, t \geq t_4$, so

$$y(\sigma(t)) \geq z(\sigma(t))(1 - p(\sigma(t))), t \geq t_4. \tag{15}$$

By (I_3) and (15) we get

$$f_i(y(\sigma_i(t))) \geq \beta_i y^\alpha(\sigma_i(t)) \geq \beta_i z^\alpha(\sigma_i(t))(1 - p(\sigma_i(t)))^\alpha, t \geq t_4. \tag{16}$$

From (1), (16), we get

$$\begin{aligned} 0 &= [r(t)(z^{(n-1)}(t))^{\alpha}]' + \sum_{i=1}^{m} \beta_i q_i(t) f_i(y(\sigma_i(t))) \\ &\geq [r(t)(z^{(n-1)}(t))^{\alpha}]' + \sum_{i=1}^{m} \beta_i q_i(t) z^{\alpha}(\sigma_i(t))(1-p(\sigma_i(t)))^{\alpha} \\ &\geq [r(t)(z^{(n-1)}(t))^{\alpha}]' + \sum_{i=1}^{m} \beta_i q_i(t) z^{\alpha}(\sigma(t))(1-p(\sigma_i(t)))^{\alpha}, t \geq t_4. \end{aligned} \tag{17}$$

Let

$$w(t) = \rho(t) \frac{r(t)(z^{(n-1)}(t))^{\alpha}}{z^{\alpha}(\sigma(t))}, t \geq t_4, \tag{18}$$

clearly, $w(t) > 0$, from (13), (17) and (18) we get

$$\begin{aligned} w'(t) = &\frac{\rho'(t)}{\rho(t)} w(t) + \rho(t) \frac{[r(t)(z^{(n-1)}(t))^{\alpha}]'}{z^{\alpha}(\sigma(t))} \\ &- \rho(t) \frac{r(t)(z^{(n-1)}(t))^{\alpha} \alpha z^{\alpha-1}(\sigma(t)) z'(\sigma(t)) \sigma'(t)}{z^{2\alpha}(\sigma(t))} \\ \leq &\frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) \sum_{i=1}^{m} \beta_i q_i(t)(1-p(\sigma_i(t)))^{\alpha} \\ &- \alpha \sigma'(t) \rho(t) \frac{r(t)(z^{(n-1)}(t))^{\alpha} z'(\sigma(t))}{z^{\alpha+1}(\sigma(t))}, \\ \leq &\frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) \sum_{i=1}^{m} \beta_i q_i(t)(1-p(\sigma_i(t)))^{\alpha} \\ &- \alpha M \sigma'(t) \sigma^{n-2}(t) \frac{w^{\frac{\alpha+1}{\alpha}}(t)}{(r(t)\rho(t))^{\frac{1}{\alpha}}} \\ \leq &\frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) C_1(t) - \alpha M \sigma'(t) \sigma^{n-2}(t) \frac{w^{\frac{\alpha+1}{\alpha}}(t)}{(r(t)\rho(t))^{\frac{1}{\alpha}}}. \end{aligned}$$

Then from above inequality we have

$$\rho(t) C_1(t) \leq -w'(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \alpha M \sigma'(t) \sigma^{n-2}(t) \frac{w^{\frac{\alpha+1}{\alpha}}(t)}{(r(t)\rho(t))^{\frac{1}{\alpha}}}. \tag{19}$$

Multiplying (19) by $H(s,t)$, integrating it with respect s from t to c_n and using (4) we get that

$$\begin{aligned}
\int_t^{c_n} H(s,t)\rho(s)C_1(s)ds \le & -\int_t^{c_n} w'(s)H(s,t)ds + \int_t^{c_n} H(s,t)\frac{\rho'(s)}{\rho(s)}w(s)ds \\
& -\alpha M \int_t^{c_n} H(s,t)\sigma'(s)\sigma^{n-2}(s)\frac{w^{\frac{\alpha+1}{\alpha}}(s)}{(r(s)\rho(s))^{\frac{1}{\alpha}}}ds \\
= & -H(c_n,t)w(c_n) + \int_t^{c_n} w(s)h_1(s,t)\sqrt{H(s,t)}ds \\
& + \int_t^{c_n} H(s,t)\frac{\rho'(s)}{\rho(s)}w(s)ds \\
& -\alpha M \int_t^{c_n} H(s,t)\sigma'(s)\sigma^{n-2}(s)\frac{w^{\frac{\alpha+1}{\alpha}}(s)}{(r(s)\rho(s))^{\frac{1}{\alpha}}}ds \\
= & -H(c_n,t)w(c_n) \\
& + \int_t^{c_n} \left[h_1(s,t)\sqrt{H(s,t)} + H(s,t)\frac{\rho'(s)}{\rho(s)}\right] w(s)ds \\
& -\alpha M \int_t^{c_n} H(s,t)\sigma'(s)\sigma^{n-2}(s)\frac{w^{\frac{\alpha+1}{\alpha}}(s)}{(r(s)\rho(s))^{\frac{1}{\alpha}}}ds
\end{aligned}$$

Set

$$F(w) = \left[h_1\sqrt{H} + H\frac{\rho'}{\rho}\right] w - \alpha M H\sigma'\sigma^{n-2}\frac{w^{\frac{\alpha+1}{\alpha}}}{(r\rho)^{\frac{1}{\alpha}}},$$

by simple calculate, we can get that when

$$w = \frac{\left(h_1\sqrt{H} + H\frac{\rho'}{\rho}\right)^{\alpha} r\rho}{[M(\alpha+1)H\sigma'\sigma^{n-2}]^{\alpha}},$$

$F(w)$ has the maximum value

$$\frac{\left(h_1\sqrt{H} + H\frac{\rho'}{\rho}\right)^{\alpha+1} r\rho}{[M(\alpha+1)H\sigma'\sigma^{n-2}]^{\alpha}(\alpha+1)},$$

that is

$$F(w) \le F_{max}(w) = \left(h_1 + \sqrt{H}\frac{\rho'}{\rho}\right)^{\alpha+1} H^{\frac{1-\alpha}{2}} C_2(s),$$

from above inequality, we get

$$\begin{aligned}
& \int_t^{c_n} H(s,t)\rho(s)C_1(s)ds \le -H(c_n,t)w(c_n) \\
+ & \int_t^{c_n} \left(h_1(s,t) + \sqrt{H(s,t)}\frac{\rho'(s)}{\rho(s)}\right)^{\alpha+1} H^{\frac{1-\alpha}{2}}(s,t)C_2(s)ds.
\end{aligned}$$

Letting $t \to a_n^+$ in the above, we obtain

$$\begin{aligned}
&\frac{1}{H(c_n,a_n)}\int_{a_n}^{c_n} H(s,a_n)\rho(s)C_1(s)ds \le -w(c_n)\\
&\quad+\frac{1}{H(c_n,a_n)}\int_{a_n}^{c_n}\left[h_1(s,a_n)+\sqrt{H(s,a_n)}\frac{\rho'(s)}{\rho(s)}\right]^{\alpha+1}\frac{C_2(s)}{H^{\frac{\alpha-1}{2}}(s,a_n)}ds.
\end{aligned} \tag{20}$$

Multiplying (19) by $H(t,s)$, integrating it with respect s from c_n to t, using (4) and by simple calculate we get that

$$\begin{aligned}
\int_{c_n}^t H(t,s)\rho(s)C_1(s)ds &\le -\int_{c_n}^t w'(s)H(t,s)ds+\int_{c_n}^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds\\
&\quad-\alpha M\int_{c_n}^t H(t,s)\sigma'(s)\sigma^{n-2}(s)\frac{w^{\frac{\alpha+1}{\alpha}}(s)}{(r(s)\rho(s))^{\frac{1}{\alpha}}}ds\\
&=H(t,c_n)w(c_n)-\int_{c_n}^t w(s)h_2(t,s)\sqrt{H(t,s)}ds\\
&\quad+\int_{c_n}^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds\\
&\quad-\alpha M\int_{c_n}^t H(t,s)\sigma'(s)\sigma^{n-2}(s)\frac{w^{\frac{\alpha+1}{\alpha}}(s)}{(r(s)\rho(s))^{\frac{1}{\alpha}}}ds\\
&\le H(t,c_n)w(c_n)\\
&\quad+\int_{c_n}^t\left[h_2(t,s)\sqrt{H(t,s)}+H(t,s)\frac{\rho'(s)}{\rho(s)}\right]w(s)ds\\
&\quad-\alpha M\int_{c_n}^t H(t,s)\sigma'(s)\sigma^{n-2}(s)\frac{w^{\frac{\alpha+1}{\alpha}}(s)}{(r(s)\rho(s))^{\frac{1}{\alpha}}}ds\\
&\le H(t,c_n)w(c_n)\\
&\quad+\int_{c_n}^t\left[h_2(t,s)+\sqrt{H(t,s)}\frac{\rho'(s)}{\rho(s)}\right]^{\alpha+1}H^{\frac{1-\alpha}{2}}(t,s)C_2(s)ds.
\end{aligned}$$

Letting $t \to b_n^+$ in the above, we obtain

$$\begin{aligned}
&\frac{1}{H(b_n,c_n)}\int_{c_n}^{b_n} H(b_n,s)\rho(s)C_1(s)ds \le w(c_n)\\
&\quad+\frac{1}{H(b_n,c_n)}\int_{c_n}^{b_n}\left[h_2(b_n,s)+\sqrt{H(b_n,s)}\frac{\rho'(s)}{\rho(s)}\right]^{\alpha+1}\frac{C_2(s)}{H^{\frac{\alpha-1}{2}}(b_n,s)}ds.
\end{aligned} \tag{21}$$

Adding (20) and (21) we have the inequality

$$\begin{aligned} & \frac{1}{H(c_n,a_n)}\int_{a_n}^{c_n} H(s,a_n)\rho(s)C_1(s)ds + \frac{1}{H(b_n,c_n)}\int_{c_n}^{b_n} H(b_n,s)\rho(s)C_1(s)ds \\ \le\ & \frac{1}{H(c_n,a_n)}\int_{a_n}^{c_n}\left[h_1(s,a_n)+\sqrt{H(s,a_n)}\frac{\rho'(s)}{\rho(s)}\right]^{\alpha+1}\frac{C_2(s)}{H^{\frac{\alpha-1}{2}}(s,a_n)}ds \\ +\ & \frac{1}{H(b_n,c_n)}\int_{c_n}^{b_n}\left[h_2(b_n,s)+\sqrt{H(b_n,s)}\frac{\rho'(s)}{\rho(s)}\right]^{\alpha+1}\frac{C_2(s)}{H^{\frac{\alpha-1}{2}}(b_n,s)}ds, t \ge t_4. \end{aligned} \tag{22}$$

Which contradict the assumption (10). Thus, the claim holds, i.e., no nontrivial solution of Eq. (1) can be eventually positive. Therefore, every solution of Eq. (1) is oscillatory.

We can easily see that the following result is true.

Theorem 2.2. *If there exists a positive, nondecreasing function $\rho(t) \in C^1([t_0,\infty))$, such that for any constant $M > 0$,*

$$\limsup_{t\to\infty}\int_l^t\left[H(s,l)\rho(s)C_1(s)-\left(h_1(s,l)+\sqrt{H(s,l)}\frac{\rho'(s)}{\rho(s)}\right)^{\alpha+1}\frac{C_2(s)}{H^{\frac{\alpha-1}{2}}(s,l)}\right]ds > 0 \tag{23}$$

and

$$\limsup_{t\to\infty}\int_l^t\left[H(t,s)\rho(s)C_1(s)-\left(h_2(t,s)+\sqrt{H(t,s)}\frac{\rho'(s)}{\rho(s)}\right)^{\alpha+1}\frac{C_2(s)}{H^{\frac{\alpha-1}{2}}(t,s)}\right]ds > 0 \tag{24}$$

hold, where $C_1(t), C_2(t)$ is defined as in Theorem 2.1, then every solution of Eq. (1) is oscillatory.

Proof. For any $T \ge t_0$, let $a_n = T$, in (23) we choose $l = a_n$, then there exist $c_n > a_n$ such that

$$\int_{a_n}^{c_n}\left[H(s,a_n)\rho(s)C_1(s)-\left(h_1(s,a_n)+\sqrt{H(s,a_n)}\frac{\rho'(s)}{\rho(s)}\right)^{\alpha+1}\frac{C_2(s)}{H^{\frac{\alpha-1}{2}}(s,a_n)}\right]ds > 0. \tag{25}$$

In (24) we choose $l = c_n$, then there exist $b_n > c_n$ such that

$$\int_{c_n}^{b_n}\left[H(b_n,s)\rho(s)C_1(s)-\left(h_2(b_n,s)+\sqrt{H(b_n,s)}\frac{\rho'(s)}{\rho(s)}\right)^{\alpha+1}\frac{C_2(s)}{H^{\frac{\alpha-1}{2}}(b_n,s)}\right]ds > 0. \tag{26}$$

Combining (25) and (26) we obtain (10). The conclusion thus comes from Theorem 2.1. the proof is complete.

Remark If we take $p(t) = 0, n = 2, m = 1, f(x) = |x|^{\alpha-1}x$, then Theorem 2.1 and Theorem 2.2 reduce to Theorem 2.1 and Theorem 2.2 of Li [7], respectively. If $r(t) = 1, n = 2, \alpha = 1$, then Theorem 2.1 and Theorem 2.2 reduce to Theorem 2.1 and Theorem 2.2 of Zhuang and Li [19], respectively. For the case where $H := H(t-s) \in X$, we have $h_1(t-s) = h_2(t-s)$ and denote them by $h(t-s)$. The subclass of X containing such $H(t-s)$ is denoted by X_1, applying Theorem 2.1 to X_1 we obtain the following theorem.

Theorem 2.3. *If for each $T \geq t_0$ and any constant $M > 0$, there exists a positive, nondecreasing function $\rho(t) \in C^1([t_0, \infty)), H \in X_1$ and $a_n, c_n \in R$ such that $T \leq a_n < c_n$ and*

$$\begin{aligned} &\int_{a_n}^{c_n} H(s-a_n)\left[\rho(s)C_1(s) + \rho(2c_n - s)C_1(2c_n - s)\right] ds \\ > &\int_{a_n}^{c_n} \left[h(s-a_n) + \sqrt{H(s-a_n)}\frac{\rho'(s)}{\rho(s)}\right]^{\alpha+1} \frac{C_2(s)}{H^{\frac{\alpha-1}{2}}(s-a_n)} ds \\ + &\int_{a_n}^{c_n} \left[h(s-a_n) + \sqrt{H(s-a_n)}\frac{\rho'(2c_n - s)}{\rho(2c_n - s)}\right]^{\alpha+1} \frac{C_2(2c_n - s)}{H^{\frac{\alpha-1}{2}}(s-a_n)} ds \end{aligned} \tag{27}$$

hold, where $C_1(t), C_2(t)$ is defined as in Theorem 2.1, then Eq. (1) is oscillatory.

Proof. Let $b_n = 2c_n - a_n$, then $H(b_n - c_n) = H(c_n - a_n) = H(\frac{b_n - a_n}{2})$ and for any $g \in L[a_n, b_n]$,we have $\int_{c_n}^{b_n} g(s)ds = \int_{a_n}^{c_n} g(2c_n - s)ds$, hence,

$$\int_{c_n}^{b_n} H(b_n - s)\rho(s)C_1(s)ds = \int_{a_n}^{c_n} H(s-a_n)\rho(2c_n - s)C_1(2c_n - s)ds,$$

$$\begin{aligned} &\int_{c_n}^{b_n} \left(h_2(b_n - s) + \sqrt{H(b_n - s)}\frac{\rho'(s)}{\rho(s)}\right)^{\alpha+1} \frac{C_2(s)}{H^{\frac{\alpha-1}{2}}(b_n - s)} ds \\ = &\int_{a_n}^{c_n} \left(h(s-a_n) + \sqrt{H(s-a_n)}\frac{\rho'(2c_n - s)}{\rho(2c_n - s)}\right)^{\alpha+1} \frac{C_2(2c_n - s)}{H^{\frac{\alpha-1}{2}}(s-a_n)} ds. \end{aligned}$$

So that (27) holds implies that (10) holds for $H \in X_1$, and therefore, Eq. (1) is oscillatory by Theorem 2.1.

From above oscillation criteria, we can obtain different sufficient conditions for oscillation of all solutions of Eq. (1) by different choices of $H(t, s)$. Now we choose $H(t,s) = (t-s)^\lambda, t \geq s \geq t_0$, where $\lambda > \alpha$ is a constant. Then $H \in X_1$ and $h(t-s) = \lambda(t-s)^{\frac{\lambda}{2}-1}$, based on the above results we obtain the following corollary.

Corollary 2.1. *Every solution of Eq. (1) is oscillatory provided that for any constant $M > 0$, there exist a positive, nondecreasing function $\rho(t) \in C^1([t_0,\infty))$ such that for each $l \geq t_0$ and for some $\lambda > \alpha$, the following two inequalities hold:*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-\alpha}} \int_l^t \left[(s-l)^\lambda \rho(s) C_1(s) - C_2(s)(s-l)^{\lambda-\alpha-1}\left(\lambda + \frac{\rho'(s)}{\rho(s)}(s-l)\right)^{\alpha+1}\right] ds > 0,$$

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-\alpha}} \int_l^t \left[(t-s)^\lambda \rho(s) C_1(s) - C_2(s)(t-s)^{\lambda-\alpha-1}\left(\lambda + \frac{\rho'(s)}{\rho(s)}(t-s)\right)^{\alpha+1}\right] ds > 0.$$

where $C_1(t), C_2(t)$ is defined as in Theorem 2.1.

Define

$$R(t) = \int_l^t \frac{ds}{r^{\frac{1}{\alpha}}(s)}, t \geq l \geq t_0$$

and let

$$H(t,s) = [R(t) - R(s)]^\lambda, t \geq t_0,$$

where $\lambda > \alpha$ is constant.

If we take $\rho(t) = 1$, then by Theorem 2.2 we have the following important oscillation criterion.

Theorem 2.1. *Assume that $\lim_{t\to\infty} R(t) = \infty$, then every solution of Eq.(1.1) is oscillatory provided that for any constant $M > 0$, there exist a positive, nondecreasing function $\rho(t) \in C^1([t_0,\infty))$ such that for each $l \geq t_0$ and for some $\lambda > \alpha$, the following two inequalities hold:*

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t [(R(s) - R(l)]^\lambda C_1(s)(\sigma'(s)\sigma^{n-2}(s))^\alpha ds > \frac{\lambda^{\alpha+1}}{M^\alpha(\alpha+1)^{\alpha+1}(\lambda-\alpha)}, \tag{28}$$

and

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t [(R(t) - R(s)]^\lambda C_1(s)(\sigma'(s)\sigma^{n-2}(s))^\alpha ds > \frac{\lambda^{\alpha+1}}{M^\alpha(\alpha+1)^{\alpha+1}(\lambda-\alpha)}, \tag{29}$$

where $C_1(t), C_2(t)$ is defined as in Theorem 2.1.

Proof. By assumption, we have

$$h_1(t,s) = h_2(t,s) = \lambda[(R(t) - R(s)]^{\frac{\lambda-2}{2}} \frac{1}{r^{\frac{1}{\alpha}}(t)},$$

noting that

$$\int_l^t \frac{h_1(s,l)^{\alpha+1}C_2(s)}{H^{\frac{\alpha-1}{2}}(s,l)}(\sigma'(s)\sigma^{n-2}(s))^\alpha ds = \int_l^t \frac{\lambda^{\alpha+1}[(R(s)-R(l)]^{\lambda-\alpha-1}r(s)}{r^{\frac{\alpha+1}{\alpha}}(s)M^\alpha(\alpha+1)^{\alpha+1}}$$
$$= \frac{\lambda^{\alpha+1}[(R(t)-R(l)]^{\lambda-\alpha}}{(\lambda-\alpha)M^\alpha(\alpha+1)^{\alpha+1}},$$

and

$$\int_l^t \frac{h_2(t,s)^{\alpha+1}C_2(s)}{H^{\frac{\alpha-1}{2}}(t,s)}(\sigma'(s)\sigma^{n-2}(s))^\alpha ds = \int_l^t \frac{\lambda^{\alpha+1}[(R(t)-R(s)]^{\lambda-\alpha-1}r(s)}{r^{\frac{\alpha+1}{\alpha}}(s)M^\alpha(\alpha+1)^{\alpha+1}}$$
$$= \frac{\lambda^{\alpha+1}[(R(t)-R(l)]^{\lambda-\alpha}}{(\lambda-\alpha)M^\alpha(\alpha+1)^{\alpha+1}},$$

in view of $\lim_{t\to\infty} R(t)=\infty$, we have

$$\lim_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_l^t \frac{h_1(s,l)^{\alpha+1}C_2(s)}{H^{\frac{\alpha-1}{2}}(s,l)}(\sigma'(s)\sigma^{n-2}(s))^\alpha ds = \frac{\lambda^{\alpha+1}}{M^\alpha(\alpha+1)^{\alpha+1}(\lambda-\alpha)}, \tag{30}$$

and

$$\lim_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_l^t \frac{h_2(t,s)^{\alpha+1}C_2(s)}{H^{\frac{\alpha-1}{2}}(t,s)}(\sigma'(s)\sigma^{n-2}(s))^\alpha ds = \frac{\lambda^{\alpha+1}}{M^\alpha(\alpha+1)^{\alpha+1}(\lambda-\alpha)}. \tag{31}$$

From (28) and (30), we have that

$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_l^t\left[(R(s)-R(l))^\lambda C_1(s)-\frac{h_1(s,l)^{\alpha+1}C_2(s)}{H^{\frac{\alpha-1}{2}}(s,l)}\right](\sigma'(s)\sigma^{n-2}(s))^\alpha ds =$$
$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_l^t (R(s)-R(l))^\lambda C_1(s)(\sigma'(s)\sigma^{n-2}(s))^\alpha ds - \frac{\lambda^{\alpha+1}}{M^\alpha(\alpha+1)^{\alpha+1}(\lambda-\alpha)} > 0, \tag{32}$$

i.e., (23) holds. Similarly, (29) and (31) imply that (24) holds. By Theorem 2.2, every solution of Eq. (1) is oscillatory.

This complete the proof.

Example Consider the following equation :

$$[|(x(t)+(1-e^{-\mu t})x(t-\pi))^{(n-1)}|^{\alpha-1}(x(t)+(1-e^{-\mu t})x(t-\pi))^{(n-1)}]'$$
$$+ \frac{\beta}{t^{\alpha(n-1)+1}}e^{\gamma\alpha\mu t}|x(\gamma t)|^{\alpha-1}x(\gamma t)=0, t\geq 1, \tag{33}$$

where $n \geq 2$ is even and $\alpha > 0, \beta > 0, \mu \geq 0, 0 < \gamma \leq 1$.

Here $p(t) = 1 - e^{-\mu t}, q(t) = \dfrac{\beta e^{\gamma\alpha\mu t}}{t^{\alpha(n-1)+1}}, m = 1, \sigma_1(t) = \gamma t$.

Then $R(t) = \int_1^t dt = t - 1, R'(t) = 1, \lim_{t\to\infty} R(t) = \infty, \sigma_1'(t) = \gamma$, if $0 < \gamma \leq \dfrac{1}{2}$, then $\sigma(t) = \gamma t$. if $\dfrac{1}{2} < \gamma \leq 1$, then $\sigma(t) = \dfrac{t}{2}$.

For $\rho(t) \equiv 1, \lambda > \alpha$. If $0 < \gamma \leq \dfrac{1}{2}$, then $\sigma(t) = \sigma_1(t) = \gamma t$,

$$\begin{aligned}
& \lim_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t [(R(s) - R(l)]^\lambda C_1(s)(\sigma'(s)\sigma^{n-2}(s))^\alpha ds \\
= \;& \lim_{t\to\infty} \frac{1}{(t-1)^{\lambda-\alpha}} \int_l^t (s-l)^\lambda \frac{\beta e^{\lambda\alpha\mu s}}{s^{\alpha(n-1)+1}} (1 - p(\gamma s))^\alpha (\gamma(\gamma s)^{n-2})^\alpha ds \\
= \;& \lim_{t\to\infty} \frac{1}{(t-1)^{\lambda-\alpha}} \int_l^t (s-l)^\lambda \frac{\beta\gamma^{\alpha(n-1)}}{s^{\alpha+1}} ds \\
= \;& \lim_{t\to\infty} \frac{(t-l)^\lambda}{(\lambda-\alpha)(t-1)^{\lambda-\alpha-1}} \frac{1}{t^{\alpha+1}} \beta\gamma^{\alpha(n-1)} = \frac{\beta\gamma^{\alpha(n-1)}}{\lambda-\alpha}.
\end{aligned} \tag{34}$$

Next, we will prove that

$$\begin{aligned}
& \int_l^t [(R(t) - R(s)]^\lambda C_1(s)(\sigma'(s)\sigma^{n-2}(s))^\alpha ds \\
\geq \;& \int_l^t [(R(s) - R(l)]^\lambda C_1(s)(\sigma'(s)\sigma^{n-2}(s))^\alpha ds.
\end{aligned} \tag{35}$$

Let

$$\begin{aligned}
G(t) = & \int_l^t \{[(R(t) - R(s)]^\lambda - [(R(s) - R(l)]^\lambda\} C_1(s)(\sigma'(s)\sigma^{n-2}(s))^\alpha ds \\
= \;& \int_l^t \{(t-s)^\lambda - (s-l)^\lambda\} \frac{\beta}{s^{\alpha(n-1)+1} e^{\mu\alpha\gamma s}} \gamma^{\alpha(n-1)} s^{\alpha(n-2)} ds \\
= \;& \beta\gamma^{\alpha(n-1)} \int_l^t \{(t-s)^\lambda - (s-l)^\lambda\} \frac{1}{s^{\alpha+1} e^{\mu\alpha\gamma s}} ds,
\end{aligned}$$

then $G(l) = 0$, and for $t \geq l$,

$$\begin{aligned}
G'(t) = & \beta\gamma^{\alpha(n-1)} \int_l^t \lambda(t-s)^{\lambda-1} \frac{1}{s^{\alpha+1} e^{\mu\alpha\gamma s}} ds - (t-l)^\lambda \frac{1}{t^{\alpha+1} e^{\mu\alpha\gamma t}} \\
\geq \;& \beta\gamma^{\alpha(n-1)} \frac{1}{t^{\alpha+1} e^{\mu\alpha\gamma t}} \left[\int_l^t \lambda(t-s)^{\lambda-1} ds - (t-l)^\lambda \right] \\
= \;& \frac{\beta\gamma^{\alpha(n-1)}}{t^{\alpha+1} e^{\mu\alpha\gamma t}} \left[-(t-s)^\lambda |_l^t - (t-l)^\lambda \right] = 0.
\end{aligned}$$

Hence $G(t) \geq G(l) = 0$ for $t \geq l$, i.e., (2.31) holds. By (34) and (35), we have

$$\lim_{t\to\infty} \frac{1}{(t-1)^{\lambda-\alpha}} \int_l^t [(R(t) - R(s)]^\lambda C_1(s)(\sigma'(s)\sigma^{n-2}(s))^\alpha ds > \frac{\beta\gamma^{\alpha(n-1)}}{\lambda - \alpha}.$$

Then for $\beta > \dfrac{(\frac{\alpha}{\alpha+1})^{\alpha+1}}{M^\alpha \gamma^{\alpha(n-1)}}$, there exists $\lambda > \alpha$ such that

$$\frac{M^\alpha \gamma^{\alpha(n-1)}\beta}{\lambda - \alpha} > \frac{\lambda^{\alpha+1}}{(\lambda-\alpha)(\alpha+1)^{\alpha+1}} > \frac{\alpha^{\alpha+1}}{(\lambda-\alpha)(\alpha+1)^{\alpha+1}},$$

this means that

$$\frac{\gamma^{\alpha(n-1)}\beta}{\lambda - \alpha} > \frac{\lambda^{\alpha+1}}{M^\alpha(\lambda-\alpha)(\alpha+1)^{\alpha+1}},$$

so that (28) and (29) hold for the same λ. Applying Theorem 2.4, we fined (33) is oscillatory for $\beta > \dfrac{(\frac{\alpha}{\alpha+1})^{\alpha+1}}{M^\alpha \gamma^{\alpha(n-1)}}$. If $\frac{1}{2} < \gamma \leq 1$, then $\sigma(t) = \frac{t}{2}$, use the same method above, we can get (33) is oscillatory for $\beta > \dfrac{(\frac{\alpha}{\alpha+1})^{\alpha+1}}{M^\alpha (\frac{1}{2})^{\alpha(n-1)}}$. However, the main results of [2, 15] fail to apply to (33), since $\mu \neq 0$.

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