ON INFINITESIMAL PROJECTIVE TRANSFORMATIONS OF THE TANGENT BUNDLE WITH THE COMPLETE LIFT OF A FINSLER METRIC

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Abstract. Let (M, g) be a Finsler manifold and TM_0 its slit tangent bundle with the complete lift metric $\tilde{g}$. In this paper, we prove that every infinitesimal complete lift projective transformation on $(TM_0, \tilde{g})$, is an infinitesimal affine transformation. Moreover, if (M, g) is a Landsberg manifold, then there is a one-to-one correspondence between infinitesimal complete lift projective transformations on $(TM_0, \tilde{g})$ and infinitesimal affine transformations on (M, g).

Key words and Phrases: Finsler manifold, Complete lift metric, Tangent bundle, Infinitesimal projective transformation.

Abstrak. Misalkan (M, g) adalah manifold Finsler dan TM_0 adalah bundel tangen slit-nya dengan metrik lift lengkap $\tilde{g}$. Pada makalah ini, kami membuktikan bahwa setiap transformasi projektif lift lengkap infinitesimal pada $(TM_0, \tilde{g})$, adalah transformasi afin infinitesimal. Lebih jauh, jika (M, g) adalah manifold Landsberg, maka terdapat korespondensi satu-satu antara transformasi projektif lift lengkap infinitesimal pada $(TM_0, \tilde{g})$ dengan transformasi afin infinitesimal pada (M, g).

Kata kunci: Manifold Finsler, Metric lift lengkap, Tangent bundle, Infinitesimal projective transformation.

1. Introduction

Let (M, g) be a Riemannian manifold and ϕ a transformation on M. Then, ϕ is called a projective transformation if it preserves the geodesics as point sets. Also, an affine transformation may be characterized as a projective transformation which preserves geodesics with the affine parameter.

Let V be a vector field on M and $\{\phi_t\}$ the local one-parameter group generated by V. Then, V is called an infinitesimal projective (affine) transformation on M if every ϕ_t is a projective (affine) transformation.

Let $\tilde{\phi}$ be a transformation of TM, the tangent bundle of M; then, $\tilde{\phi}$ is called a fiber-preserving transformation if it preserves the fibers. A vector field $\tilde{X}$ on TM with the local one-parameter group $\{\tilde{\phi}_t\}$ is called an infinitesimal fiber-preserving transformation on TM if each $\tilde{\phi}_t$ is a fiber-preserving transformation. Infinitesimal fiber-preserving transformations is an important class of infinitesimal transformations on TM which include infinitesimal complete lift transformations as a special subclass(refer to subsection 2.2 or [4]).

It is well-known that there are some lift metrics on TM as follows: complete lift metric or g_2, diagonal lift metric or Sasaki metric or $g_1 + g_3$, lift metric I+II or $g_1 + g_2$ and lift metic II+III or $g_2 + g_3$, where $g_1 := g_{ij}dx^i dx^j$, $g_2 := 2g_{ij}dx^i \delta y^j$ and $g_3 := g_{ij}\delta y^i \delta y^j$ are all bilinear differential forms defined globally on TM. For more details one can refer to [17].

The problems of exsisting special infinitesimal transformations on the tangent bundle of a Riemannian manifold with some lift metrics are considered by several authors, e.g. [5, 6, 7, 8, 9] and [12, 13, 14, 15]. The study shows that the special infinitesiaml transformations on TM might lead to some global results. For example, in [7] and [15], it is proved that if TM with the complete lift metric or the lift metric I+II admits an essential infinitesimal conformal[1] transformation, then M is isomorphic to the standard sphere. Also, it is proved in [12], that if TM with the complete lift metric admits a non-affine infinitesimal projective transformation, then M is locally flat. Therefore, it is meaningful to the study of the infinitesimal transformations on the tangent bundle TM.

Yamauchi in [14], proved the following theorem:

Theorem A: Let M be a non-Euclidean complete n-dimensional Riemannian manifold, and let TM be its tangent bundle with the complete lift metric. Then, every infinitesimal fiber-preserving projective transformation on TM is an affine one and it naturally induced an infinitesimal affine transformation on M.

Special infinitesimal transformations on the tangent bundle of a Finsler manifold are considered by many authors, e.g. [1, 2, 4, 10].

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[1]A vector field $\tilde{X}$ on $(TM, \tilde{g})$ is called an essential infinitesimal conformal transformation if there exsits a scalar function Ω on TM such that Ω depends only y^i with respect to the induced coordinate (x^i, y^i) on TM and $\pounds_{\tilde{X}}\tilde{g} = 2\Omega\tilde{g}$,[16].

In this paper, the infinitesimal fiber-preserving projective transformations on the tangent bundle of a Finsler manifold with the complete lift metric are considered. Then, as a special case, the infinitesimal complete lift projective transformations are studied and the following theorem is proved.

Theorem 1.1. *Let (M, g) be a C^∞ connected Finsler manifold and TM_0 its slit tangent bundle with the complete lift metric $\tilde{g}$. Then, every infinitesimal complete lift projective transformation on $(TM_0, \tilde{g})$ is affine and it naturally induced an infinitesimal affine transformation on (M, g).*

Thus, Theorem A is true for the Finsler manifold and the infinitesimal complete lift transformations. From Theorem 1.1, the following corollary can be immediately found.

Corollary 1.2. *The Lie algebra of complete lift projective vector field on $(TM_0, \tilde{g})$ is reduced to an affine one.*

The Landsberg manifolds form an important class of the Finsler manifolds which include the Berwald manifolds. In the next theorem, it is shown that the inverse of Theorem 1.1 is true for the Landsberg manifolds.

Theorem 1.3. *Let (M, g) be an n-dimensional Landsberg manifold and TM_0 its slit tangent bundle with a complete lift metric $\tilde{g}$. Then, every infinitesimal complete lift transformation V^c on $(TM_0, \tilde{g})$ is projective if and only if V is an infinitesimal affine transformation on (M, g).*

2. Preliminaries

Let M be a real n-dimensional C^∞ manifold and TM its tangent bundle. The elements of TM are denoted by (x, y) with $y \in T_x M$. Also $TM_0 = TM \setminus \{0\}$ be the slit tangent bundle of M. The natural projection $\pi : TM_0 \to M$ is given by $\pi(x, y) := x$. A *Finsler structure* on M is a function of $F : TM \to [0, \infty)$ with the following properties; (i) F is C^∞ on TM_0, (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM and (iii) the Hessian g of F^2 with elements $g_{ij}(x, y) := \frac{1}{2}[F^2(x, y)]_{y^i y^j}$ is positive-definite. In the sequel of this paper, a Finsler manifold with a Finsler structure F will be denoted by (M, g) instead of (M, F).

Let $V_v TM := ker\pi_*^v$ be the set of the vectors tangent to the fiber through $v \in TM_0$. Then, the vertical vector bundle on M is defined by $VTM := \bigcup_{v \in TM_0} V_v TM$. A non-linear connection or a horizontal distribution on TM_0 is a complementary distribution HTM for VTM on TTM_0. Therefore, by using a non-linear connection, the following decomposition is resulted:

$$TTM_0 = VTM \oplus HTM, \tag{1}$$

where HTM is a vector bundle completely determined by the non-linear differentiable functions $N^i{}_j(x, y)$ on TM, which is called coefficients of the non-linear

connection HTM. The pair (HTM, ∇) is called a Finsler connection on the manifold M, where ∇ a linear connection to VTM. Indeed, a Finsler connection is a triple $(N^i{}_j, F^i{}_{jk}, C^i{}_{jk})$ where $N^i{}_j$ are the coefficients of a nonlinear connection, $F^i{}_{jk}$ and $C^i{}_{jk}$ are the horizontal part and the vertical part of this connection, respectively[3].

Using the local coordinates (x^i, y^i) on TM we have the local field of frames field $\{X_i, X_{\bar{i}}\}$ on TTM. It is well known that a local field of frames $\{X_i, X_{\bar{i}}\}$ can be chosen so that it is adapted to the decomposition (1) i.e. $X_i \in \Gamma(HTM)$ and $X_{\bar{i}} \in \Gamma(VTM)$ set of vector fields on HTM and VTM, where

$$X_i := \frac{\partial}{\partial x^i} - N^j{}_i \frac{\partial}{\partial y^j}, \ X_{\bar{i}} := \frac{\partial}{\partial y^i},$$

where the indices $i, j, \ldots$ and $\bar{i}, \bar{j}, \ldots$ run over the range $1, \ldots, n$.

Analogous to Riemannian geometry, the following lemma in Finsler geometry was obtained by straightforward calculations.

Lemma 2.1. [14] *The Lie brackets of the adapted frame of TM satisfy the following*
　　(1) $[X_i, X_j] = R^h{}_{ij} X_{\bar{h}}$,
　　(2) $[X_i, X_{\bar{j}}] = X_{\bar{j}}(N^h{}_i) X_{\bar{h}}$,
　　(3) $[X_{\bar{i}}, X_{\bar{j}}] = 0$,
where $R^h{}_{ij} = X_j(N^h{}_i) - X_i(N^h{}_j)$.

For a Finsler connection $(N^i{}_j, F^i{}_{jk}, C^i{}_{jk})$, the curvature tensor has three component R, P and S, that are called hh-curvature, hv-curvature and vv-curvature, respectively. They are defined as follows:

$$R_k{}^h{}_{ij} := X_i(F^h{}_{kj}) - X_j(F^h{}_{ki}) + F^m{}_{kj} F^h{}_{mi} - F^m{}_{ki} F^h{}_{mj} + R^m{}_{ij} C^h{}_{jm},$$

$$P_k{}^h{}_{ij} := X_{\bar{k}}(F^h{}_{ij}) - X_j(C^h{}_{ik}) + F^m{}_{ki} C^h{}_{mj} - C^m{}_{kj} F^h{}_{mi} + X_{\bar{j}}(N^m{}_i) C^h{}_{jm},$$

$$S_k{}^h{}_{ij} := X_{\bar{j}}(C^h{}_{ki}) - X_{\bar{i}}(C^h{}_{kj}) + C^m{}_{ki} C^h{}_{mj} - C^m{}_{kj} C^h{}_{mi}.$$

Let (M, g) be a Finsler manifold, the geodesic of g satisfy the following system of differential equations

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i = G^i(x, y)$ are called the geodesic coefficients, which are given by

$$G^i = \frac{1}{4} g^{il} \left\{ [F^2]_{x^m y^l} y^m - [F^2]_{x^l} \right\}.$$

The differentiable functions $G^i{}_j := X_{\bar{j}}(G^i)$ determine a non-linear connection which is called the canonical nonlinear connection of Finsler manifold (M, g). In what follows, the canonical nonlinear connection $G^i{}_j$ will be used.

There are several Finsler connections on a Finsler manifold, which we present some of them. The Berwald connection of a Finsler manifold (M, g) is defined by

triple $(G^i{}_j, G^i{}_{jk}, 0)$, where $G^i{}_{jk} := X_{\bar{k}}(G^i{}_j)$. The hh-curvature $H_k{}^h{}_{ij}$ and hv-curvature $G_k{}^h{}_{ij}$ of the Berwald connection are obtained as follows:

$$H_k{}^h{}_{ij} = X_i(G^h{}_{kj}) - X_j(G^h{}_{ki}) + G^m{}_{kj}G^h{}_{mi} - G^m{}_{ki}G^h{}_{mj},$$

$$G_k{}^h{}_{ij} = X_{\bar{k}}(G^h{}_{ij}),$$

respectively. It is obvious that $R^h{}_{ij} = y^k H_k{}^h{}_{ij}$.

Also, the Cartan connection of a Finsler manifold (M, g) is defined by triple $(G^i{}_j, F^i{}_{jk}, C^i{}_{jk})$, where $F^i{}_{jk} := \frac{1}{2}g^{ih}\{X_j(g_{kh}) + X_k(g_{jh}) - X_h(g_{jk})\}$ and $C^i{}_{jk} := \frac{1}{2}g^{ih}X_{\bar{k}}(g_{jh})$.

A Finsler manifold (M, g) is called a Berwald manifold, if the geodesic spray coefficient G^i's are quadratic functions of y-coordinates in each tangent space, i.e.

$$X_{\bar{l}}X_{\bar{k}}X_{\bar{j}}(G^i) = 0, \quad \forall j, k, l.$$

In the other words, the Finsler manifold (M, g) is a Berwald metric if the hv-curvature tensor field of the Berwald connection vanishes.

A Finsler manifold (M, g) is called Landsberg manifold, if the geodesic spary coefficient G^i's satisfy the following equations [11]

$$y^m g_{im} X_{\bar{l}}X_{\bar{k}}X_{\bar{j}}(G^i) = 0, \quad \forall j, k, l.$$

It is obvious that every Brwald manifold is a Landsberg manifold.

By defining the tensor field $P^i{}_{jk} := G^i{}_{jk} - F^i{}_{jk}$, one can see that, a Finsler manifold (M, g) is a Landsberg manifold if and only if $P^i{}_{jk} = 0$. For more details, one can refer to [3]. In the following, all manifolds are supposed to be connected.

2.1. Infinitesimal fiber-preserving transformations. Let $\tilde{X}$ be a vector field on TM and $\{\tilde{\phi}_t\}$ the local one-parameter group of local transformations of TM generated by $\tilde{X}$. Then, $\tilde{X}$ is called a fiber-preserving vector field on TM if each $\tilde{\phi}_t$ is a fiber-preserving transformation of TM. From [13], we have the following lemma.

Lemma 2.2. *Let $\tilde{X}$ be a vector field on TM with components $(v^h, v^{\bar{h}})$ with respect to the adapted frame $\{X_h, X_{\bar{h}}\}$. Then $\tilde{X}$ is a fiber-preserving vector field on TM if and only if v^h are functions on M*

Therefore, every fiber-preserving vector field $\tilde{X}$ on TM induces a vector field $V = v^h \partial_h$ on M, where $\partial_i := \frac{\partial}{\partial x^i}$.

Let $\{dx^i, \delta y^i\}$ be the dual basis of $\{X_h, X_{\bar{h}}\}$, where $\delta y^i = dy^i - G^i{}_h dx^h$. Using a straight forward calculation similar to [14], one can obtain the following lemma.

Lemma 2.3. *Let $\tilde{X}$ be a fiber-preserving vector field of TM with the components $(v^h, v^{\bar{h}})$. Then, the Lie derivatives of the adapted frame and the dual basis are given as follows:*

(1) $\pounds_{\tilde{X}} X_h = -\partial_h v^m X_m + \{v^b R^m{}_{bh} - v^{\bar{b}} G^m{}_{bh} - X_h(v^{\bar{h}})\}X_{\bar{m}},$

(2) $\pounds_{\tilde{X}} X_{\bar{h}} = \{v^b G^m{}_{bh} - X_{\bar{h}}(v^{\bar{m}})\}X_{\bar{m}},$

(3) $\pounds_{\tilde{X}} dx^h = \partial_m v^h dx^m,$

(4) $\pounds_{\tilde{X}} \delta y^h = -\{v^b R^h{}_{bm} - v^{\bar{b}} G^h{}_{bm} - X_m(v^{\bar{h}})\} dx^m - \{v^b G^h{}_{bm} - X_{\bar{m}}(v^{\bar{h}})\} \delta y^m.$

For more details in infinitesimal transformations, one can refer to [8].

2.2. Complete Lift Vector Fields and Lie Derivative.

Let $V = v^i \partial_i$ be a vector field on M. Then, V induces an infinitesimal point transformation on M. This is naturally extended to a point transformation of the tangent bundle TM which is called extended point transformation. If $\{\varphi_t\}$ is the local 1-parameter group of M generated by V and $\tilde{\varphi}_t$ the extended point transformation of φ_t, then $\{\tilde{\varphi}_t\}$ induces a vector field V^c on TM which is called the complete lift of V, (c.f. [4]).

The complete lift vector field V^c of V can be written as $V^c = v^i X_i + y^j (F^i{}_{ja} v^a + \partial_j v^i) X_{\bar{i}}$. From the Lemma 2.2, it concluded that, the class of complete lift vector fields is a subclass of fiber-preserving vector fields.

Let V be a vector field on M and $\{\varphi_t\}$ the local one parameter group of local transformations of M generated by V. Take any tensor field S on M. Then, the Lie derivative $\pounds_V S$ of S with respect to V is a tensor field on M, defined by

$$\pounds_V S = \frac{\partial}{\partial t} \varphi_t^*(S)|_{t=0} = \lim_{t \to 0} \frac{\varphi_t^*(S) - S}{t},$$

on the domain of φ_t, where $\varphi_t^*(S)$ denotes the pull back of S by φ_t.

In local coordinates the Lie derivative of an arbitrary tensor, $T^k{}_i$, is given locally by[16];

$$\pounds_V T^k{}_i = v^a \partial_a T^k{}_i + y^a \partial_a v^b X_{\bar{b}}(T^k{}_i) - T^a{}_i \partial_a v^k + T^k{}_a \partial_i v^a.$$

Therefore

$$\pounds_V y^i = 0.$$

For the Lie derivatives of the adapted frame and the dual basis with respect to complete lift vector field V^c, the following lemma can be presented.

Lemma 2.4. [4] *Let (M, g) be a Finsler manifold, V a vector field on M and V^c its complete lift, then*

(1) $\pounds_{V^c} X_i = -\partial_i v^h X_h - \pounds_V G^h_i X_{\bar{h}},$

(2) $\pounds_{V^c} X_{\bar{i}} = -\partial_i v^h X_{\bar{h}},$

(3) $\pounds_{V^c} dx^h = \partial_m v^h dx^m,$

(4) $\pounds_{V^c} \delta y^h = \pounds_V G^h_m dx^m + \partial_m v^h \delta y^m.$

2.3. Infinitesimal projective transformations.

Let M be a Riemannian manifold. A vector field X on M is said to be an *infinitesimal projective transformation*, if there exists a 1-form θ on M such that

$$(\pounds_X \nabla)(Y, Z) = \theta(Y) Z + \theta(Z) Y,$$

or equivalently

$$\pounds_X(\nabla_Y Z) - \nabla_Y(\pounds_X Z) - \nabla_{[X,Y]} Z = \theta(Y) Z + \theta(Z) Y. \tag{2}$$

where ∇ is the Riemannian connection of M and $Y, Z \in \Gamma(M)$ the set of vector fields on M.

In Finsler geometry, a vector field V on (M, g) is called an infinitesimal projective transformation if there exists a function $\Psi(x, y)$ on TM such that $\mathcal{L}_V G^i = \Psi y^i$ [2]. Accordingly, the following relations can be resulted.

$$\mathcal{L}_V G^i_j = \Psi \delta^i_j + \Psi_j y^i,$$

$$\mathcal{L}_V G^i{}_{jk} = \Psi_k \delta^i_j + \Psi_j \delta^i_k + \Psi_{jk} y^i,$$

$$\mathcal{L}_V G^i{}_{jkl} = \Psi_{kl} \delta^i_j + \Psi_{jl} \delta^i_k + \Psi_{jk} \delta^i_l + \Psi_{jkl} y^i,$$

where $\Psi_j := X_{\bar{j}}(\Psi)$, $\Psi_{jk} := X_{\bar{k}}(\Psi_j)$ and $\Psi_{jkl} := X_{\bar{l}}(\Psi_{jk})$. If $\Psi = 0$, then it can be said that V is an infinitesimal affine transformation.

3. Riemannian Connection of TM_0 With the Complete Lift Metric

Let $g = (g_{ij}(x, y))$ be a Finsler metric on M. As we said that, there are several Riemannian or pseudo-Riemannian metrics on TM_0 which can be defined from g. They are called the lift metric of g. A one of such metrics is $\tilde{g} = 2g_{ij} dx^i \delta y^j$, which is called the complete lift metric, (c.f. [17]). Thus $(TM_0, \tilde{g})$ is a Riemannian manifold.

Let $\tilde{\nabla}$ be the Riemannian connection of TM_0 with respect to the complete lift metric $\tilde{g}$ and $\tilde{\Gamma}^A_{BC}$ the coefficients of $\tilde{\nabla}$, that is,

$$\tilde{\nabla}_{X_i} X_j = \tilde{\Gamma}^m_{ji} X_m + \tilde{\Gamma}^{\bar{m}}_{ji} X_{\bar{m}}, \quad \tilde{\nabla}_{X_i} X_{\bar{j}} = \tilde{\Gamma}^m_{\bar{j}i} X_m + \tilde{\Gamma}^{\bar{m}}_{\bar{j}i} X_{\bar{m}},$$

$$\tilde{\nabla}_{X_{\bar{i}}} X_j = \tilde{\Gamma}^m_{j\bar{i}} X_m + \tilde{\Gamma}^{\bar{m}}_{j\bar{i}} X_{\bar{m}}, \quad \tilde{\nabla}_{X_{\bar{i}}} X_{\bar{j}} = \tilde{\Gamma}^m_{\bar{j}\bar{i}} X_m + \tilde{\Gamma}^{\bar{m}}_{\bar{j}\bar{i}} X_{\bar{m}}, \tag{3}$$

where the coefficients $A, B, C, \dots$ run over the range $1, \dots, n, \bar{1}, \dots, \bar{n}$.

The following lemma is trivial.

Lemma 3.1. *We have the following equations*
(1) $\tilde{\nabla}_{X_i} dx^h = -\tilde{\Gamma}^h_{mi} dx^m - \tilde{\Gamma}^h_{\bar{m}i} \delta y^m$,
(2) $\tilde{\nabla}_{X_i} \delta y^h = -\tilde{\Gamma}^{\bar{h}}_{mi} dx^m - \tilde{\Gamma}^{\bar{h}}_{\bar{m}i} \delta y^m$,
(3) $\tilde{\nabla}_{X_{\bar{i}}} dx^h = -\tilde{\Gamma}^h_{m\bar{i}} dx^m - \tilde{\Gamma}^h_{\bar{m}\bar{i}} \delta y^m$,
(4) $\tilde{\nabla}_{X_{\bar{i}}} \delta y^h = -\tilde{\Gamma}^{\bar{h}}_{m\bar{i}} dx^m - \tilde{\Gamma}^{\bar{h}}_{\bar{m}\bar{i}} \delta y^m$.

Since the torsion tensor $T(X, Y)$ of $\tilde{\nabla}$ defined by $T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y]$ vanishes, the following relations can be given by means of Lemma 2.1 and the relation (3)

$$\tilde{\Gamma}^m_{ji} = \tilde{\Gamma}^m_{ij}, \quad \tilde{\Gamma}^{\bar{m}}_{ji} = \tilde{\Gamma}^{\bar{m}}_{ij} + R^m{}_{ij},$$

$$\tilde{\Gamma}^m_{\bar{j}i} = \tilde{\Gamma}^m_{i\bar{j}}, \quad \tilde{\Gamma}^{\bar{m}}_{\bar{j}i} = \tilde{\Gamma}^{\bar{m}}_{i\bar{j}} + G^m{}_{ji},$$

$$\tilde{\Gamma}^m_{\bar{j}\bar{i}} = \tilde{\Gamma}^m_{i\bar{j}}, \quad \tilde{\Gamma}^{\bar{m}}_{\bar{j}\bar{i}} = \tilde{\Gamma}^{\bar{m}}_{i\bar{j}}. \tag{4}$$

Lemma 3.2. *The connection coefficients $\tilde{\Gamma}^A_{BC}$ of $\tilde{\nabla}$ satisfy the following relations.*

$a)$ $\tilde{\Gamma}^h_{ji} = F^h_{ji} - P^h_{ji},$ $\quad b)$ $\tilde{\Gamma}^{\bar{h}}_{ji} = g^{hm}R_{imj},$

$c)$ $\tilde{\Gamma}^h_{\bar{j}i} = 0,$ $\qquad\qquad d)$ $\tilde{\Gamma}^h_{j\bar{i}} = 0,$

$e)$ $\tilde{\Gamma}^{\bar{h}}_{\bar{j}i} = G^h_{ji},$ $\qquad\quad f)$ $\tilde{\Gamma}^{\bar{h}}_{j\bar{i}} = 0,$

$g)$ $\tilde{\Gamma}^h_{\bar{j}\bar{i}} = 0,$ $\qquad\qquad h)$ $\tilde{\Gamma}^{\bar{h}}_{\bar{j}i} = 2C^h_{ji}.$

PROOF. By means of Lemma 3.1 and metric compatibility of $\tilde{\nabla}$, i.e., $\tilde{\nabla}\tilde{g} = 0$, we have

$0 = \tilde{\nabla}_{X_m}\tilde{g} = \tilde{\nabla}_{X_m}(2g_{ij}dx^i\delta y^j)$

$= -2g_{ia}\tilde{\Gamma}^{\bar{a}}_{jm}dx^idx^j + 2\{g_{aj}(F^a_{im} - \tilde{\Gamma}^a_{im}) + g_{ia}(F^a_{jm} - \tilde{\Gamma}^{\bar{a}}_{\bar{j}m})\}dx^i\delta y^j - 2g_{aj}\tilde{\Gamma}^a_{im}\delta y^i\delta y^j.$

and

$0 = \tilde{\nabla}_{X_{\bar{m}}}\tilde{g} = \tilde{\nabla}_{X_{\bar{m}}}(2g_{ij}dx^i\delta y^j)$

$= -2g_{ia}\tilde{\Gamma}^{\bar{a}}_{j\bar{m}}dx^idx^j + 2\{2C_{ijm} - g_{aj}\tilde{\Gamma}^a_{i\bar{m}} - g_{ia}\tilde{\Gamma}^{\bar{a}}_{\bar{j}\bar{m}}\}dx^i\delta y^j - 2g_{aj}\tilde{\Gamma}^a_{i\bar{m}}\delta y^i\delta y^j.$

It follows that

$$g_{ia}\tilde{\Gamma}^{\bar{a}}_{jm} + g_{ja}\tilde{\Gamma}^{\bar{a}}_{im} = 0, \tag{5}$$

$$g_{aj}(F^a_{im} - \tilde{\Gamma}^a_{im}) + g_{ia}(F^a_{jm} - \tilde{\Gamma}^{\bar{a}}_{\bar{j}m}) = 0, \tag{6}$$

$$g_{aj}\tilde{\Gamma}^a_{im} + g_{ai}\tilde{\Gamma}^a_{jm} = 0, \tag{7}$$

$$g_{ia}\tilde{\Gamma}^{\bar{a}}_{j\bar{m}} + g_{ja}\tilde{\Gamma}^{\bar{a}}_{i\bar{m}} = 0, \tag{8}$$

$$2C_{ijm} - g_{aj}\tilde{\Gamma}^a_{i\bar{m}} - g_{ia}\tilde{\Gamma}^{\bar{a}}_{\bar{j}\bar{m}} = 0, \tag{9}$$

$$g_{aj}\tilde{\Gamma}^a_{i\bar{m}} + g_{ai}\tilde{\Gamma}^a_{\bar{j}\bar{m}} = 0. \tag{10}$$

From (4) and (5), we have

$g_{ia}\tilde{\Gamma}^{\bar{a}}_{jm} = -g_{ja}\tilde{\Gamma}^{\bar{a}}_{im} = -g_{ja}(\tilde{\Gamma}^{\bar{a}}_{mi} + R^a_{mi}) = -R_{jmi} - g_{ja}\tilde{\Gamma}^{\bar{a}}_{mi}$

$\qquad = -R_{jmi} + g_{ma}(\tilde{\Gamma}^{\bar{a}}_{ij} + R^a_{ij}) = -R_{jmi} + R_{mij} + g_{ma}\tilde{\Gamma}^{\bar{a}}_{ij}$

$\qquad = -R_{jmi} + R_{mij} - g_{ia}\tilde{\Gamma}^{\bar{a}}_{mj} = -R_{jmi} + R_{mij} - g_{ia}(\tilde{\Gamma}^{\bar{a}}_{jm} + R^a_{jm})$

$\qquad = -R_{jmi} + R_{mij} - R_{ijm} - g_{ia}\tilde{\Gamma}^{\bar{a}}_{jm} = -g_{ia}\tilde{\Gamma}^{\bar{a}}_{jm} + 2R_{mij},$

which show the relation $b)$ in the lemma.

According to (4) and (10):

$g_{aj}\tilde{\Gamma}^a_{i\bar{m}} = -g_{ai}\tilde{\Gamma}^a_{\bar{j}\bar{m}} = -g_{ai}\tilde{\Gamma}^a_{\bar{m}\bar{j}} = g_{am}\tilde{\Gamma}^a_{ij} = g_{am}\tilde{\Gamma}^a_{ji} = -g_{aj}\tilde{\Gamma}^a_{\bar{m}\bar{i}} = -g_{aj}\tilde{\Gamma}^a_{i\bar{m}},$

thus, we get the relation $g)$ in the lemma.

From (4) and (6), we have

$g_{aj}(F^a_{im} - \tilde{\Gamma}^a_{im}) = -g_{ia}(F^a_{jm} - \tilde{\Gamma}^{\bar{a}}_{\bar{j}m}) = -g_{ia}(G^a_{jm} - P^a_{jm} - \tilde{\Gamma}^{\bar{a}}_{\bar{j}m})$

$\qquad\qquad = -g_{ia}(-P^a_{jm} - \tilde{\Gamma}^{\bar{a}}_{m\bar{j}}).$

From above and (8), it can be said that

$g_{aj}(F^a_{im} - \tilde{\Gamma}^a_{im}) + g_{aj}(F^a_{mi} - \tilde{\Gamma}^a_{mi}) = g_{ia}P^a_{jm} + g_{ma}P^a_{ji} = 2P_{jim},$

this shows the relations $a)$, $e)$ and $f)$.

From (7) and (9), it can be said that

$$g_{ai}\tilde{\Gamma}^a_{jm} = -g_{aj}\tilde{\Gamma}^a_{im} = -g_{aj}\tilde{\Gamma}^a_{m\bar{i}} = g_{ma}\tilde{\Gamma}^{\bar{a}}_{\bar{j}\bar{i}} - 2C_{mji}. \tag{11}$$

From (4), (9) and (11), the relations $c)$, $d)$ and $h)$ can be obtained. This completes the proof.

Remark 3.1. *From Lemma 3.2 and the relation (3), the following equations can be obtained.*

$$\tilde{\nabla}_{X_i} X_j = (F^h_{ji} - P^h_{ji})X_h + g^{hm}R_{imj}X_{\bar{h}}, \quad \tilde{\nabla}_{X_i} X_{\bar{j}} = G^h_{ji}X_{\bar{h}},$$
$$\tilde{\nabla}_{X_{\bar{i}}} X_j = 0, \quad \tilde{\nabla}_{X_{\bar{i}}} X_{\bar{j}} = 2C^h_{ji}X_{\bar{h}}. \tag{12}$$

It would be mentioned that, when (M, g) is a Riemannian manifold, then the relations (12) are reduced to

$$\tilde{\nabla}_{X_i} X_j = \Gamma^h_{ji}X_h + g^{hm}R_{imj}X_{\bar{h}}, \quad \tilde{\nabla}_{X_i} X_{\bar{j}} = \Gamma^h_{ji}X_{\bar{h}},$$
$$\tilde{\nabla}_{X_{\bar{i}}} X_j = 0, \quad \tilde{\nabla}_{X_{\bar{i}}} X_{\bar{j}} = 0, \tag{13}$$

where Γ^h_{ji} are the coefficients of the Riemannian connection of M.

4. MAIN RESULTS

Let (M, g) be a Finsler manifold and TM_0 its slit tangent bundle with complete lift metric $\tilde{g}$. Here, infinitesimal fiber-preserving projective transformations on $(TM_0, \tilde{g})$ are considered and the following proposition is proved.

Proposition 4.1. *Let (M, g) be a n-dimensional Finsler manifold and TM_0 its slit tangent bundle with the complete lift metric $\tilde{g}$. Then, every infinitesimal fiber-preserving projective transformation on $(TM_0, \tilde{g})$ induces an infinitesimal projective transformation on (M, g).*

PROOF. Let $\tilde{X}$ be an infinitesimal fiber-preserving projective transformation on TM_0. From (2), it can be concluded that there exists a 1-form Θ on TM_0 such that

$$\pounds_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\pounds_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z} = \Theta(\tilde{Y})\tilde{Z} + \Theta(\tilde{Z})\tilde{Y}, \tag{14}$$

where $\tilde{Y}$ and $\tilde{Z} \in \Gamma(TM)$. Let $\tilde{X} = v^h X_h + v^{\bar{h}}X_{\bar{h}}$ and $\Theta = \theta_i dx^i + \theta_{\bar{i}}\delta y^i$. We obtain the following.

$$\pounds_{\tilde{X}}\tilde{\nabla}_{X_{\bar{i}}} X_j - \tilde{\nabla}_{X_{\bar{i}}}\pounds_{\tilde{X}} X_j - \tilde{\nabla}_{[\tilde{X},X_{\bar{i}}]} X_j = \Theta(X_{\bar{i}})X_j + \Theta(X_j)X_{\bar{i}}, \tag{15}$$

$$\pounds_{\tilde{X}}\tilde{\nabla}_{X_{\bar{i}}} X_{\bar{j}} - \tilde{\nabla}_{X_{\bar{i}}}\pounds_{\tilde{X}} X_{\bar{j}} - \tilde{\nabla}_{[\tilde{X},X_{\bar{i}}]} X_{\bar{j}} = \Theta(X_{\bar{i}})X_{\bar{j}} + \Theta(X_{\bar{j}})X_{\bar{i}}, \tag{16}$$

$$\pounds_{\tilde{X}}\tilde{\nabla}_{X_i} X_j - \tilde{\nabla}_{X_i}\pounds_{\tilde{X}} X_j - \tilde{\nabla}_{[\tilde{X},X_i]} X_j = \Theta(X_i)X_j + \Theta(X_j)X_i. \tag{17}$$

By means of Lemma 2.1, Lemma 2.3, (12) and (15), the following relation is obtaind:

$$-\Big\{v^b H^h_{i\ bj} - X_{\bar{i}}(v^{\bar{b}})G^h_{bj} - v^{\bar{b}}G^h_{i\ bj} - X_{\bar{i}}X_j(v^{\bar{h}})$$
$$+ \big(v^b R^m_{bj} - v^{\bar{b}}G^m_{bj} - X_j(v^{\bar{m}})\big)C^h_{mi}\Big\}X_{\bar{h}} = \delta^h_j \theta_{\bar{i}}X_h + \delta^h_i \theta_j X_{\bar{h}}.$$

Thus, we obtain

$$\theta_{\bar{i}} = 0, \tag{18}$$

and

$$v^b H^h_{i\ bj} - X_{\bar{i}}(v^{\bar{b}})G^h_{bj} - v^{\bar{b}}G^h_{i\ bj} - X_{\bar{i}}X_j(v^{\bar{h}})$$
$$+ \left(v^b R^m_{bj} - v^{\bar{b}}G^m_{bj} - X_j(v^{\bar{m}})\right)C^h_{mi} = -\delta^h_i \theta_j. \tag{19}$$

By means of Lemma 2.1, Lemma 2.3, (12), (16) and (18), the following can be presented:

$$\tilde{X}(C^h_{ij}) + C^a_{ij}(v^b G^h_{ba} - X_{\bar{a}}(v^{\bar{h}})) - (v^b G^h_{i\ bj} - X_{\bar{i}}X_{\bar{j}}(v^{\bar{h}}))$$
$$- C^h_{ia}(v^b G^a_{bj} - X_{\bar{j}}(v^{\bar{a}})) - C^h_{aj}(v^b G^a_{bi} - X_{\bar{i}}(v^{\bar{a}})) = 0. \tag{20}$$

By means of Lemma2.1, Lemma2.3, (12) and (17) we have

$$\tilde{X}(F^h_{ji} - P^h_{ji}) - \partial_a v^h(F^a_{ji} - P^a_{ji}) + \partial_i \partial_j v^h$$
$$+ \partial_j v^a(F^h_{ai} - P^h_{ai}) + \partial_i v^a(F^h_{aj} - P^h_{aj}) = \delta^h_i \theta_j + \delta^h_j \theta_i, \tag{21}$$

and

$$(F^a_{ji} - P^a_{ji})\left[v^b R^h_{ba} - v^{\bar{b}}G^h_{ba} - X_a(v^{\bar{h}})\right] + X(g^{ht}R_{itj})$$
$$+ g^{at}R_{itj}\left[v^b G^h_{ba} - X_{\bar{a}}(v^{\bar{h}})\right] + \partial_j v^a g^{ht}R_{ita} - X_i\left[v^b R^h_{bj} - v^{\bar{b}}G^h_{bj} - X_j(v^{\bar{h}})\right]$$
$$- \left[v^b R^a_{bj} - v^{\bar{b}}G^a_{bj} - X_j(v^{\bar{a}})\right]G^h_{ai} + \partial_i v^b g^{ht}R_{btj} = 0. \tag{22}$$

From (21), it can be concluded that

$$\pounds_V G^h = \theta_i y^i y^h, \tag{23}$$

i.e. $V = v^h \partial_h$ is an infinitesimal projective transformation on (M, g). This completes the proof.

Proof of Theorem 1.1. Infinitesimal complete lift transformations is a subclass of infinitesimal fiber-preserving transformations. So, the similar method as in the proof of Proposition 4.1 is used. By means of Lemma 2.1, Lemma 2.4, relations (12), (15) and (18) the following can be given

$$\pounds_V G^h_{ji} + 2(\pounds_V G^t_j)C^h_{it} = \delta^h_i \theta_j.$$

Contracting by y^i

$$\pounds_V G^h_j = y^h \theta_j,$$

thus

$$(\pounds_V G^t_j)C^h_{it} = 0,$$

and, we obtain

$$\pounds_V G^h_{ji} = \delta^h_i \theta_j. \tag{24}$$

By means of Lemma 2.1 and Lemma 2.4 with the relations (12), (16) and (18):

$$\pounds_V C^h_{ji} = 0. \tag{25}$$

By taking in to account Lemma 2.1 and Lemma 2.4 with the relations (12) and (17), we can obtain

$$\pounds_V(F^h_{ji} - P^h_{ji}) = \delta^h_i \theta_j + \delta^h_j \theta_i, \tag{26}$$

and

$$\pounds_V G_t^h (P_{ji}^t - F_{ji}^t) + \partial_j v^a g^{ht} R_{ita} + \partial_i v^b g^{ht} R_{btj}$$
$$+ X_i(\pounds_V G_j^h) + G_{ti}^h(\pounds_V G_j^h) = 0.$$

Contracting (26) by $y^i y^j$ and using (24) leads to obtain

$$\pounds_V G^h = 0,$$

thus, $V = v^i \partial_i$ is an infinitesimal affine transformation on (M, g) and

$$\pounds_V G_j^h = \pounds_V G_{ji}^h = 0. \tag{27}$$

Substituting (27) in (24)

$$\delta_i^h \theta_j = 0.$$

Thus, $\theta_i = 0$, i.e. V^c is an infinitesimal affine transformation on $(TM, \tilde{g})$. This completes the proof.

Proof of Theorem 1.3. If V^c is an infinitesimal projective transformation then from Theorem 1.1, one can see that V is an affine. Let (M, g) be a Landsberg manifold then

$$F_{ji}^h = G_{ji}^h. \tag{28}$$

If V is an affine, then by use of (24), (26) and (28), it can be seen that V^c is an infinitesimal affine transformation on $(TM_0, \tilde{g})$. This completes the proof.

From Theorem 1.3, we have immediately the following remark.

Remark 4.1. *Let (M, g) be a Landsberg space, then, there is a one-to-one correspondence between complete lift projective vector fields on $(TM_0, \tilde{g})$ and affine vector fields on (M, g).*

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