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FUZZY TRANSLATIONS OF A FUZZY SET IN UP-ALGEBRAS

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Abstract. In this paper, we apply the concept of fuzzy translations of a fuzzy set to UP-algebras. For any fuzzy set μ in a UP-algebra, the concepts of fuzzy α-translations of μ of type I and of fuzzy β-translations of μ of type II are introduced, their basic properties are investigated and some useful examples are discussed. The concepts of prime fuzzy sets and of weakly prime fuzzy sets in UP-algebras are also studied. Moreover, we discuss the concepts of extensions and of intensions of a fuzzy set in UP-algebras.

Key words and Phrases: UP-algebra, fuzzy set, fuzzy translation, extension, intension.

Abstrak. Di dalam paper ini, diaplikasikan konsep translasi fuzzy dari suatu himpunan fuzzy pada aljabar-UP. Untuk sebarang himpunan fuzzy μ di suatu aljabar-UP, konsep α-translasi fuzzy dari μ dengan tipe I dan konsep β-translasi dari μ dengan tipe II diperkenalkan, kemudian diinvestigasi beberapa sifat dasarnya, dan didiskusikan beberapa contoh penting. Konsep himpunan fuzzy prima dan himpunan fuzzy prima lemah di aljabar-UP juga dikaji di dalam paper ini. Lebih jauh, didiskusikan juga konsep ekstensi dan intensi dari suatu himpunan fuzzy di aljabar-UP.

Kata kunci: aljabar-UP, himpunan fuzzy, translasi fuzzy, ekstensi, intensi.

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1. Introduction and Preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [12], BCI-algebras [13], BCH-algebras [10], K-algebras [7], KU-algebras [21], SU-algebras [18] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [13] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [12, 13] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

A fuzzy subset f of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [25] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the concept of fuzzy sets by Zadeh [25], several researches were conducted on the generalizations of the notion of fuzzy set and application to many logical algebras such as: In 2001, Lele, Wu, Weke, Mamadou and Njock [20] studied fuzzy ideals and weak ideals in BCK-algebras. Jun [14] introduced the notion of Q-fuzzy subalgebras of BCK/BCI-algebras. In 2002, Jun, Roh and Kim [17] studied fuzzy B-algebras in B-algebras. In 2004, Jun [15] introduced the concept of (α, β)-fuzzy ideals of BCK/BCI-algebras. In 2005, Akram and Dar [2] introduced the notions of T-fuzzy subalgebras and of T-fuzzy H-ideals in BCI-algebras. In 2007, Akram and Dar [3] introduced the notion of fuzzy ideals in K-algebras. In 2008, Akram [1] introduced the notion of bifuzzy ideals of K-algebras. In 2012, Sitharselvam, Priya and Ramachandran [23] introduced the notion of anti Q-fuzzy KU-ideals of KU-algebras. In 2016, Somjanta, Thuekaew, Kumpeangkeaw and Iampan [24] introduced and studied fuzzy UP-subalgebras and fuzzy UP-ideals of UP-algebras.

Moreover, fuzzy sets were extended to fuzzy translations in many logical algebras such as: In 2009, Lee, Jun and Doh [19] investigated relations among fuzzy translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications of fuzzy subalgebras in BCK/BCI-algebras. In 2011, Jun [16] discussed fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy ideals in BCK/BCI-algebras. In 2013, Chandramouleeswaran, Muralikrishna and Srinivasan [6] introduced the notions of fuzzy translations and fuzzy multiplications of fuzzy sets in BF-algebras. In 2014, Ansari and Chandramouleeswaran [4] introduced the notions of fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy β-ideals of β-algebras. In 2015, Senapati, Bhowmik, Pal and Davvaz [22] discussed the fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy H-ideals in BCK/BCI-algebras. Bhowmik, Senapati [5] introduced the notions of fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy subalgebras in BG-algebras. In 2016, Hashemi [9] introduced the notions of fuzzy translations of fuzzy associative ideals in BCK/BCI-algebras.

In this paper, we apply the concept of fuzzy translations of a fuzzy set to UP-algebras. For any fuzzy set μ in a UP-algebra, the concepts of fuzzy α-translations of μ of type I and of fuzzy β-translations of μ of type II are introduced, their basic properties are investigated and some useful examples are discussed. The concepts of prime fuzzy sets and of weakly prime fuzzy sets in UP-algebras are also studied. Moreover, we discuss the concepts of extensions and of intensions of a fuzzy set in UP-algebras.

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 1.1. [11] *An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra if it satisfies the following axioms: for any $x, y, z \in A$,*

(UP-1): $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(UP-2): $0 \cdot x = x$,
(UP-3): $x \cdot 0 = 0$, *and*
(UP-4): $x \cdot y = y \cdot x = 0$ *implies* $x = y$.

From [11], we know that the notion of UP-algebras is a generalization of KU-algebras [21].

Example 1.2. [11] *Let X be a universal set. Define a binary operation $\cdot$ on the power set of X by putting $A \cdot B = B \cap A' = A' \cap B = B - A$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); \cdot, \emptyset)$ is a UP-algebra and we shall call it the power UP-algebra of type 1.*

Example 1.3. [11] *Let X be a universal set. Define a binary operation $*$ on the power set of X by putting $A * B = B \cup A' = A' \cup B$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); *, X)$ is a UP-algebra and we shall call it the power UP-algebra of type 2.*

Example 1.4. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	3
3	0	1	2	0

Then $(A; \cdot, 0)$ is a UP-algebra.

In what follows, let A and B denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1.5. [11] *In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,*

(1) $x \cdot x = 0$,
(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,
(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,
(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,
(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*
(7) $x \cdot (y \cdot y) = 0$.

On a UP-algebra $A = (A; \cdot, 0)$, we define a binary relation $\leq$ on A [11] as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0.$$

Definition 1.6. [11] *A subset S of A is called a UP-subalgebra of A if the constant 0 of A is in S, and $(S; \cdot, 0)$ itself forms a UP-algebra.*

Proposition 1.7. [11] *A nonempty subset S of a UP-algebra $A = (A; \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Definition 1.8. [24] *A subset F of A is called a UP-filter of A if it satisfies the following properties:*

(1) the constant 0 of A is in F, and
(2) for any $x, y \in A, x \cdot y \in F$ and $x \in F$ imply $y \in F$.

Definition 1.9. [11] *A subset B of A is called a UP-ideal of A if it satisfies the following properties:*

(1) the constant 0 of A is in B, and
(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Definition 1.10. *A subset C of A is called a strongly UP-ideal of A if it satisfies the following properties:*

(1) the constant 0 of A is in C, and
(2) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$ imply $x \in C$.

Definition 1.11. [25] *A fuzzy set in a nonempty set X (or a fuzzy subset of X) is an arbitrary function $f \colon X \to [0, 1]$ where $[0, 1]$ is the unit segment of the real line.*

Definition 1.12. [24] *A fuzzy set f in A is called a fuzzy UP-subalgebra of A if it satisfies the following property: for any $x, y \in A$,*

$$f(x \cdot y) \geq \min\{f(x), f(y)\}.$$

By Proposition 1.5 1, we have $f(0) = f(x \cdot x) \geq \min\{f(x), f(x)\} = f(x)$ for all $x \in A$.

Definition 1.13. [24] *A fuzzy set f in A is called a fuzzy UP-filter of A if it satisfies the following properties: for any $x, y \in A$,*

(1) $f(0) \geq f(x)$, *and*
(2) $f(y) \geq \min\{f(x \cdot y), f(x)\}$.

Definition 1.14. [24] *A fuzzy set f in A is called a fuzzy UP-ideal of A if it satisfies the following properties: for any $x, y, z \in A$,*

(1) $f(0) \geq f(x)$, *and*
(2) $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\}$.

Definition 1.15. *A fuzzy set f in A is called a fuzzy strongly UP-ideal of A if it satisfies the following properties: for any $x, y, z \in A$,*

(1) $f(0) \geq f(x)$, and
(2) $f(x) \geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$.

For example, we see that μ_7 and μ_8 are fuzzy strongly UP-ideal of A in Example 3.23.

2. Fuzzy α-Translation of a Fuzzy Set

In this section, we study the basic properties of UP-subalgebras (resp. UP-filters, UP-ideals, strongly UP-ideals) and fuzzy UP-subalgebras (resp. fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) of a UP-algebra, and study the concept of prime and weakly prime of subsets and of fuzzy sets of a UP-algebra.

The proof of Theorems 2.1, 2.2, 2.4, and 2.6 can be verified easily.

Theorem 2.1. *A subset C of A is a strongly UP-ideal of A if and only if $C = A$.*

Theorem 2.2. *Every UP-filter of A is a UP-subalgebra.*

Example 2.3. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Hence, $(A; \cdot, 0)$ is a UP-algebra. Let $S = \{0, 2\}$. Then S is a UP-subalgebra of A. Since $2 \cdot 3 = 2 \in S$, but $3 \notin S$, we have S is not a UP-filter of A.

Theorem 2.4. *Every UP-ideal of A is a UP-filter.*

Example 2.5. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Hence, $(A; \cdot, 0)$ is a UP-algebra. Let $F = \{0, 1\}$. Then F is a UP-filter of A. Since $2 \cdot (1 \cdot 3) = 0 \in F$, $1 \in F$ but $2 \cdot 3 = 2 \notin F$, we have F is not a UP-ideal of A.

Theorem 2.6. *Every strongly UP-ideal of A is a UP-ideal.*

Example 2.7. *Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	2	0	4
4	0	0	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. Let $B = \{0, 1, 3\}$. Then B is UP-ideal of A. Since $(2 \cdot 0) \cdot (2 \cdot 2) = 0 \in B$, $0 \in B$ but $2 \notin B$, we have B is not a strongly UP-ideal of A.

By Theorems 2.2, 2.4, and 2.6 and Examples 2.3, 2.5, and 2.7, we have that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals.

Definition 2.8. [24] *A nonempty subset B of A is called a prime subset of A if it satisfies the following property: for any $x, y \in A$,*

$$x \cdot y \in B \text{ implies } x \in B \text{ or } y \in B.$$

Definition 2.9. [24] *A UP-subalgebra (resp. UP-filter, UP-ideal, strongly UP-ideal) B of A is called a prime UP-subalgebra (resp. prime UP-filter, prime UP-ideal, prime strongly UP-ideal) of A if it is a prime subset of A.*

Definition 2.10. *A nonempty subset B of A is called a weakly prime subset of A if it satisfies the following property: for any $x, y \in A$ and $x \neq y$,*

$$x \cdot y \in B \text{ implies } x \in B \text{ or } y \in B.$$

Example 2.11. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Hence, $(A; \cdot, 0)$ is a UP-algebra. Let $B = \{0, 1\}$. Then B is a weakly prime subset of A. Since $2 \cdot 2 = 0 \in B$, but $2 \notin B$, we have B is not prime subset of A.

Definition 2.12. *A UP-subalgebra (resp. UP-filter, UP-ideal, strongly UP-ideal) B of A is called a weakly prime UP-subalgebra (resp. weakly prime UP-filter, weakly prime UP-ideal, weakly prime strongly UP-ideal) of A if it is a weakly prime subset of A.*

The proof of Theorems 2.13, 2.14, 2.15, 2.17, and 2.19 and Lemma 2.21 can be verified easily.

Theorem 2.13. *Let S be a subset of A. Then the following statements are equivalent:*

(1) S is a prime UP-subalgebra (resp. prime UP-filter, prime UP-ideal, prime strongly UP-ideal) of A,
(2) $S = A$, and
(3) S is a strongly UP-ideal of A.

Theorem 2.14. *A fuzzy set f in A is constant if and only if it is a fuzzy strongly UP-ideal of A.*

Theorem 2.15. *Every fuzzy UP-filter of A is a fuzzy UP-subalgebra.*

Example 2.16. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	2
2	0	0	0	1
3	0	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [0, 1]$ as follows:

$$f(0) = 1, f(1) = 0.6, f(2) = 0.4, \text{ and } f(3) = 0.1.$$

Then f is a fuzzy UP-subalgebra of A. Since $f(2) = 0.4 < 0.6 = \min\{f(1\cdot 2), f(1)\}$, we have f is not a fuzzy UP-filter of A.

Theorem 2.17. *Every fuzzy UP-ideal of A is a fuzzy UP-filter.*

Example 2.18. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [0, 1]$ as follows:

$$f(0) = 1, f(1) = 0.2, f(2) = 0.1, \text{ and } f(3) = 0.1.$$

Then f is a fuzzy UP-filter of A. Since $f(2\cdot 3) = 0.1 < 0.2 = \min\{f(2\cdot(1\cdot 3)), f(1)\}$, we have f is not a fuzzy UP-ideal of A.

Theorem 2.19. *Every fuzzy strongly UP-ideal of A is a fuzzy UP-ideal.*

Example 2.20. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [0, 1]$ as follows:

$$f(0) = 0.6, f(1) = 0.4, f(2) = 0.3, \text{ and } f(3) = 0.2.$$

Then f is a fuzzy UP-ideal of A. Since $f(1) = 0.4 < 0.6 = \min\{f((1 \cdot 0) \cdot (1 \cdot 1)), f(0)\}$, we have f is not a fuzzy strongly UP-ideal of A.

By Theorems 2.15, 2.17, and 2.19 and Examples 2.16, 2.18, and 2.20, we have that the notion of fuzzy UP-subalgebras is a generalization of fuzzy UP-filters, the notion of fuzzy UP-filters is a generalization of fuzzy UP-ideals, and the notion of fuzzy UP-ideals is a generalization of fuzzy strongly UP-ideals.

Lemma 2.21. *If f is a fuzzy UP-filter of A, then it is order preserving.*

Definition 2.22. [24] *A fuzzy set f in A is called a prime fuzzy set in A if it satisfies the following property: for any $x, y \in A$,*

$$f(x \cdot y) \leq \max\{f(x), f(y)\}.$$

Definition 2.23. [24] *A fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) f of A is called a prime fuzzy UP-subalgebra (resp. prime fuzzy UP-filter, prime fuzzy UP-ideal, prime fuzzy strongly UP-ideal) of A if it is a prime fuzzy set in A.*

Definition 2.24. *A fuzzy set f in A is called a weakly prime fuzzy set in A if it satisfies the following property: for any $x, y \in A$ and $x \neq y$,*

$$f(x \cdot y) \leq \max\{f(x), f(y)\}.$$

Example 2.25. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a mapping $f: A \to [0, 1]$ as follows:

$$f(0) = 0.4, f(1) = 0.3, f(2) = 0.2, \text{ and } f(3) = 0.1.$$

Then f is a weakly prime fuzzy set of A. Since $f(1 \cdot 1) = f(0) = 0.4 > \max\{f(1)), f(1)\} = 0.3$, we have f is not a prime fuzzy set of A.

Definition 2.26. *A fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) f of A is called a weakly prime fuzzy UP-subalgebra (resp. weakly prime fuzzy UP-filter, weakly prime fuzzy UP-ideal, weakly prime fuzzy strongly UP-ideal) of A if it is a weakly prime fuzzy set in A.*

The proof of Theorems 2.27, and 2.29 can be verified easily.

Theorem 2.27. *Let f be a fuzzy set in A. Then the following statements are equivalent:*

(1) f is a prime fuzzy UP-subalgebra (resp. prime fuzzy UP-filter, prime fuzzy UP-ideal, prime fuzzy strongly UP-ideal) of A,

(2) f is a constant fuzzy set in A, and
(3) f is a fuzzy strongly UP-ideal of A.

Example 2.28. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a fuzzy set $\mu_1 \colon A \to [0, 1]$ in A as follows:

$$\mu_1(0) = 0.6, \mu_1(1) = 0.5, \mu_1(2) = 0.5, \text{ and } \mu_1(3) = 0.5.$$

Then μ_1 is weakly prime fuzzy UP-subalgebras of A. Since $\mu_1(1 \cdot 1) = 0.6 > 0.5 = \max\{\mu_1(1)), \mu_1(1)\}$, we have μ_1 is not a prime fuzzy UP-subalgebras of A.

We define a fuzzy set $\mu_2 \colon A \to [0, 1]$ in A as follows:

$$\mu_2(0) = 0.6, \mu_2(1) = 0.6, \mu_2(2) = 0.5, \text{ and } \mu_2(3) = 0.5.$$

Then μ_2 is weakly prime fuzzy UP-filter of A. Since $\mu_2(2 \cdot 2) = 0.6 > 0.5 = \max\{\mu_2(2)), \mu_2(2)\}$, we have μ_2 is not a prime fuzzy UP-filter of A.

We define a fuzzy set $\mu_3 \colon A \to [0, 1]$ in A as follows:

$$\mu_3(0) = 0.6, \mu_3(1) = 0.6, \mu_3(2) = 0.6, \text{ and } \mu_3(3) = 0.5.$$

Then μ_3 is weakly prime fuzzy UP-ideal of A. Since $\mu_3(3 \cdot 3) = 0.6 > 0.5 = \max\{\mu_3(3)), \mu_3(3)\}$, we have μ_3 is not a prime fuzzy UP-ideal of A.

Theorem 2.29. *For UP-algebras, the notions of weakly prime fuzzy strongly UP-ideals and of prime fuzzy strongly UP-ideals coincide.*

Definition 2.30. [8] *The inclusion "$\subseteq$" is defined by setting, for any fuzzy sets μ_1 and μ_2 in A,*

$$\mu_1 \subseteq \mu_2 \Leftrightarrow \mu_1(x) \leq \mu_2(x) \text{ for all } x \in A.$$

We say that μ_2 is a fuzzy extension of μ_1 [4], *and μ_1 is a fuzzy intension of μ_2.*

For any fuzzy set μ in A, we let $\dagger = 1 - \sup\{\mu(x) \mid x \in A\}$.

Definition 2.31. *Let μ be a fuzzy set in A and let $\alpha \in [0, \dagger]$. A mapping $\mu_\alpha^\dagger \colon A \to [0, 1]$ defined by*

$$\mu_\alpha^\dagger(x) = \mu(x) + \alpha \text{ for all } x \in A$$

is said to be a fuzzy α-translation of μ of type I or, in short, a fuzzy α-translation of μ.

Example 2.32. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	2
2	0	0	0	1
3	0	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a mapping $\mu\colon A \to [0, 1]$ as follows:

$$\mu(0) = 0.7, \mu(1) = 0.5, \mu(2) = 0.3, \text{ and } \mu(3) = 0.1.$$

Then $\dagger = 1 - \sup\{0.7, 0.5, 0.3, 0.1\} = 1 - 0.7 = 0.3$. Let $\alpha = 0.2 \in [0, 0.3]$. Then a mapping $\mu^{\dagger}_{\alpha}\colon A \to [0, 1]$ defined by

$$\mu^{\dagger}_{\alpha}(0) = 0.9, \mu^{\dagger}_{\alpha}(1) = 0.7, \mu^{\dagger}_{\alpha}(2) = 0.5, \text{ and } \mu^{\dagger}_{\alpha}(3) = 0.3.$$

The proof of Theorems 2.33, 2.34, 2.35, 2.36, 2.37, 2.38, 2.39, and 2.40 can be verified easily.

Theorem 2.33. *If μ is a fuzzy UP-subalgebra of A, then the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy UP-subalgebra of A for all $\alpha \in [0, \dagger]$.*

Theorem 2.34. *If there exists $\alpha \in [0, \dagger]$ such that the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy UP-subalgebra of A, then μ is a fuzzy UP-subalgebra of A.*

Theorem 2.35. *If μ is a fuzzy UP-filter of A, then the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy UP-filter of A for all $\alpha \in [0, \dagger]$.*

Theorem 2.36. *If there exists $\alpha \in [0, \dagger]$ such that the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy UP-filter of A, then μ is a fuzzy UP-filter of A.*

Theorem 2.37. *If μ is a fuzzy UP-ideal of A, then the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy UP-ideal of A for all $\alpha \in [0, \dagger]$.*

Theorem 2.38. *If there exists $\alpha \in [0, \dagger]$ such that the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy UP-ideal of A, then μ is a fuzzy UP-ideal of A.*

Theorem 2.39. *If μ is a fuzzy strongly UP-ideal of A, then the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy strongly UP-ideal of A for all $\alpha \in [0, \dagger]$.*

Theorem 2.40. *If there exists $\alpha \in [0, \dagger]$ such that the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy strongly UP-ideal of A, then μ is a fuzzy strongly UP-ideal of A.*

Theorem 2.41. *If μ is a fuzzy UP-filter (resp. fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, then the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is order preserving for all $\alpha \in [0, \dagger]$.*

Proof. It follows from Theorem 2.35 (resp. Theorems 2.37 and 2.17, Theorems 2.39 and 2.19) and Lemma 2.21. □

Theorem 2.42. *If there exists $\alpha \in [0, \dagger]$ such that the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy UP-filter (resp. fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, then μ is order preserving.*

Proof. It follows from Theorem 2.36 (resp. Theorems 2.38 and 2.17, Theorems 2.40 and 2.19) and Lemma 2.21. □

Example 2.43. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [0, 1]$ as follows:

$$f(0) = 1, f(1) = 0.2, f(2) = 0.3, \text{ and } f(3) = 0.4.$$

Then f is a fuzzy UP-subalgebra of A. Since $f(1) = 0.2 < \min\{f(2\cdot1)), f(2)\} = 0.3$, we have f is not a fuzzy UP-filter of A. Since $3 \leq 2$ but $f(3) = 0.4 > 0.3 = f(2)$, we have f is not order preserving.

The proof of Theorems 2.44, 2.45, 2.46, and 2.47 can be verified easily.

Theorem 2.44. *If μ is a prime fuzzy set in A, then the fuzzy α-translation $\mu_\alpha^\dagger$ of μ is a prime fuzzy set in A for all $\alpha \in [0, \dagger]$.*

Theorem 2.45. *If there exists $\alpha \in [0, \dagger]$ such that the fuzzy α-translation $\mu_\alpha^\dagger$ of μ is a prime fuzzy set in A, then μ is a prime fuzzy set in A.*

Theorem 2.46. *If μ is a weakly prime fuzzy set in A, then the fuzzy α-translation $\mu_\alpha^\dagger$ of μ is a weakly prime fuzzy set in A for all $\alpha \in [0, \dagger]$.*

Theorem 2.47. *If there exists $\alpha \in [0, \dagger]$ such that the fuzzy α-translation $\mu_\alpha^\dagger$ of μ is a weakly prime fuzzy set in A, then μ is a weakly prime fuzzy set in A.*

Theorem 2.48. *If μ is a weakly prime fuzzy UP-subalgebra (resp. weakly prime fuzzy UP-filter, weakly prime fuzzy UP-ideal, weakly prime fuzzy strongly UP-ideal) of A, then the fuzzy α-translation $\mu_\alpha^\dagger$ of μ is a weakly prime fuzzy UP-subalgebra (resp. weakly prime fuzzy UP-filter, weakly prime fuzzy UP-ideal, weakly prime fuzzy strongly UP-ideal) of A for all $\alpha \in [0, \dagger]$.*

Proof. It follows from Theorems 2.33 (resp. Theorem 2.35, Theorem 2.37, Theorem 2.39) and 2.46. □

Theorem 2.49. *If there exists $\alpha \in [0, \dagger]$ such that the fuzzy α-translation $\mu_\alpha^\dagger$ of μ is a weakly prime fuzzy UP-subalgebra (resp. weakly prime fuzzy UP-filter, weakly prime fuzzy UP-ideal, weakly prime fuzzy strongly UP-ideal) of A, then μ is a weakly prime fuzzy UP-subalgebra (resp. weakly prime fuzzy UP-filter, weakly prime fuzzy UP-ideal, weakly prime fuzzy strongly UP-ideal) of A.*

Proof. It follows from Theorems 2.34 (resp. Theorem 2.36, Theorem 2.38, Theorem 2.40) and 2.47. □

Note 2.50. *If μ is a fuzzy set in A and $\alpha \in [0, \dagger]$, then $\mu^{\dagger}_{\alpha}(x) = \mu(x) + \alpha \geq \mu(x)$ for all $x \in A$. Hence, the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy extension of μ for all $\alpha \in [0, \dagger]$.*

Lemma 2.51. *Let μ and ν be fuzzy sets in A. If $\nu \supseteq \mu^{\dagger}_{\beta}$ for $\beta \in [0, \dagger]$, there exists $\alpha \in [0, \dagger]$ with $\alpha \geq \beta$ such that $\nu \supseteq \mu^{\dagger}_{\alpha} \supseteq \mu^{\dagger}_{\beta}$.*

Proof. Assume that $\nu \supseteq \mu^{\dagger}_{\beta}$ for $\beta \in [0, \dagger]$. Then $\nu(x) \geq \mu^{\dagger}_{\beta}(x)$ for all $x \in A$. Putting $\alpha = \beta + \inf_{x \in A}\{\nu(x) - \mu^{\dagger}_{\beta}(x)\}$. Then

$$\begin{aligned}
\inf_{x \in A}\{\nu(x) - \mu^{\dagger}_{\beta}(x)\} &= \inf_{x \in A}\{\nu(x) - (\mu(x) + \beta)\} \\
&\leq \inf_{x \in A}\{1 - (\mu(x) + \beta)\} \\
&= 1 + \inf_{x \in A}\{-\mu(x) - \beta\} \\
&= 1 + \inf_{x \in A}\{-\mu(x)\} - \beta \\
&= 1 - \sup_{x \in A}\{\mu(x)\} - \beta \\
&= \dagger - \beta,
\end{aligned}$$

so $\alpha = \beta + \inf_{x \in A}\{\nu(x) - \mu^{\dagger}_{\beta}(x)\} \leq \beta + \dagger - \beta = \dagger$. Thus $\alpha \in [0, \dagger]$ and $\alpha \geq \beta$, so $\mu^{\dagger}_{\alpha} \supseteq \mu^{\dagger}_{\beta}$. Now, for all $x \in A$, we have

$$\begin{aligned}
\mu^{\dagger}_{\alpha}(x) &= \mu(x) + \alpha \\
&= \mu(x) + \beta + \inf_{x \in A}\{\nu(x) - \mu^{\dagger}_{\beta}(x)\} \\
&= \mu^{\dagger}_{\beta}(x) + \inf_{x \in A}\{\nu(x) - \mu^{\dagger}_{\beta}(x)\} \\
&\leq \mu^{\dagger}_{\beta}(x) + \nu(x) - \mu^{\dagger}_{\beta}(x) \\
&= \nu(x),
\end{aligned}$$

so $\nu \supseteq \mu^{\dagger}_{\alpha}$. Hence, $\nu \supseteq \mu^{\dagger}_{\alpha} \supseteq \mu^{\dagger}_{\beta}$ for some $\alpha \in [0, \dagger]$ with $\alpha \geq \beta$. □

Definition 2.52. *Let μ_1 and μ_2 be two fuzzy sets in A and $\mu_1 \subseteq \mu_2$. If μ_1 is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, then μ_2 is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, and we say that μ_2 is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) extension of μ_1.*

Theorem 2.53. *If μ is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, then the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) extension of μ for all $\alpha \in [0, \dagger]$.*

Proof. It follows from Theorem 2.33 (resp. Theorem 2.35, Theorem 2.37, Theorem 2.39) and Note 2.50. □

Theorem 2.54. *If μ is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, then the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) extension of the fuzzy β-translation $\mu^{\dagger}_{\beta}$ of μ for all $\alpha, \beta \in [0, \dagger]$ with $\alpha \geq \beta$.*

Proof. It follows from Theorem 2.33 (resp. Theorem 2.35, Theorem 2.37, Theorem 2.39). □

Theorem 2.55. *Let μ be a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A and $\beta \in [0, \dagger]$. For every fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) extension ν of the fuzzy β-translation $\mu^{\dagger}_{\beta}$ of μ, there exists $\alpha \in [0, \dagger]$ with $\alpha \geq \beta$ such that ν is the fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) extension of the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ.*

Proof. It follows from Theorem 2.33 (resp. Theorem 2.35, Theorem 2.37, Theorem 2.39) and Lemma 2.51. □

3. Fuzzy β-Translation of a Fuzzy Set

For any fuzzy set μ in A, we let $\ddagger = \inf\{\mu(x) \mid x \in A\}$.

Definition 3.1. *Let μ be a fuzzy set in A and let $\beta \in [0, \ddagger]$. A mapping $\mu^{\ddagger}_{\beta} \colon A \to [0, 1]$ defined by*

$$\mu^{\ddagger}_{\beta}(x) = \mu(x) - \beta \text{ for all } x \in A$$

is said to be a fuzzy β-translation of μ of type II or, in short, a fuzzy β-translation of μ.

Example 3.2. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	2
2	0	0	0	1
3	0	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a mapping $\mu \colon A \to [0, 1]$ as follows:

$$\mu(0) = 0.7, \mu(1) = 0.5, \mu(2) = 0.3, \text{ and } \mu(3) = 0.1.$$

Then $\ddagger = \inf\{0.7, 0.5, 0.3, 0.1\} = 0.1$. Let $\beta = 0.05 \in [0, 0.1]$. Then a mapping $\mu^{\ddagger}_{\beta} \colon A \to [0, 1]$ defined by

$$\mu^{\ddagger}_{\beta}(0) = 0.65, \mu^{\ddagger}_{\beta}(1) = 0.45, \mu^{\ddagger}_{\beta}(2) = 0.25, \text{ and } \mu^{\ddagger}_{\beta}(3) = 0.05.$$

The proof of Theorems 3.3, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, and 3.11 can be verified easily.

Theorem 3.3. *If μ is a fuzzy UP-subalgebra of A, then the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a fuzzy UP-subalgebra of A for all $\beta \in [0, \ddagger]$.*

Example 3.4. *In Example 2.16, a mapping $f\colon A \to [0,1]$ defined by*

$$f(0) = 1, f(1) = 0.6, f(2) = 0.4, \text{ and } f(3) = 0.1$$

is a fuzzy UP-subalgebra of A. Then $\ddagger = \inf\{1, 0.6, 0.4, 0.1\} = 0.1$. For all $\beta \in [0, 0.1]$, we can show that a mapping $\mu^{\ddagger}_{\beta}\colon A \to [0,1]$ is a fuzzy UP-subalgebra of A.

Theorem 3.5. *If there exists $\beta \in [0, \ddagger]$ such that the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a fuzzy UP-subalgebra of A, then μ is a fuzzy UP-subalgebra of A.*

Theorem 3.6. *If μ is a fuzzy UP-filter of A, then the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a fuzzy UP-filter of A for all $\beta \in [0, \ddagger]$.*

Theorem 3.7. *If there exists $\beta \in [0, \ddagger]$ such that the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a fuzzy UP-filter of A, then μ is a fuzzy UP-filter of A.*

Theorem 3.8. *If μ is a fuzzy UP-ideal of A, then the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a fuzzy UP-ideal of A for all $\beta \in [0, \ddagger]$.*

Theorem 3.9. *If there exists $\beta \in [0, \ddagger]$ such that the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a fuzzy UP-ideal of A, then μ is a fuzzy UP-ideal of A.*

Theorem 3.10. *If μ is a fuzzy strongly UP-ideal of A, then the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a fuzzy strongly UP-ideal of A for all $\beta \in [0, \ddagger]$.*

Theorem 3.11. *If there exists $\beta \in [0, \ddagger]$ such that the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a fuzzy strongly UP-ideal of A, then μ is a fuzzy strongly UP-ideal of A.*

Theorem 3.12. *If μ is a fuzzy UP-filter (resp. fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, then the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is order preserving for all $\beta \in [0, \ddagger]$.*

Proof. It follows from Theorem 3.6 (resp. Theorems 3.8 and 2.17, Theorems 3.10 and 2.19) and Lemma 2.21. □

Theorem 3.13. *If there exists $\beta \in [0, \dagger]$ such that the fuzzy β-translation $\mu^{\dagger}_{\beta}$ of μ is a fuzzy UP-filter (resp. fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, then μ is order preserving.*

Proof. It follows from Theorem 3.7 (resp. Theorems 3.9 and 2.17, Theorems 3.11 and 2.19) and Lemma 2.21. □

The proof of Theorems 3.14, 3.15, 3.16, and 3.17 can be verified easily.

Theorem 3.14. *If μ is a prime fuzzy set in A, then the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a prime fuzzy set in A for all $\beta \in [0, \ddagger]$.*

Theorem 3.15. *If there exists $\beta \in [0, \ddagger]$ such that the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a prime fuzzy set in A, then μ is a prime fuzzy set in A.*

Theorem 3.16. *If μ is a weakly prime fuzzy set in A, then the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a weakly prime fuzzy set in A for all $\beta \in [0, \ddagger]$.*

Theorem 3.17. *If there exists $\beta \in [0, \ddagger]$ such that the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a weakly prime fuzzy set in A, then μ is a weakly prime fuzzy set in A.*

Theorem 3.18. *If μ is a weakly prime fuzzy UP-subalgebra (resp. weakly prime fuzzy UP-filter, weakly prime fuzzy UP-ideal, weakly prime fuzzy strongly UP-ideal) of A, then the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a weakly prime fuzzy UP-subalgebra (resp. weakly prime fuzzy UP-filter, weakly prime fuzzy UP-ideal, weakly prime fuzzy strongly UP-ideal) of A for all $\beta \in [0, \ddagger]$.*

Proof. It follows from Theorems 3.3 (resp. Theorem 3.6, Theorem 3.8, Theorem 3.10) and 3.16. □

Theorem 3.19. *If there exists $\beta \in [0, \ddagger]$ such that the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a weakly prime fuzzy UP-subalgebra (resp. weakly prime fuzzy UP-filter, weakly prime fuzzy UP-ideal, weakly prime fuzzy strongly UP-ideal) of A, then μ is a weakly prime fuzzy UP-subalgebra (resp. weakly prime fuzzy UP-filter, weakly prime fuzzy UP-ideal, weakly prime fuzzy strongly UP-ideal) of A.*

Proof. It follows from Theorems 3.5 (resp. Theorem 3.7, Theorem 3.9, Theorem 3.11) and 3.17. □

Note 3.20. *If μ is a fuzzy set in A and $\beta \in [0, \ddagger]$, then $\mu^{\ddagger}_{\beta}(x) = \mu(x) - \beta \leq \mu(x)$ for all $x \in A$. Hence, the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a fuzzy intension of μ for all $\beta \in [0, \ddagger]$.*

Lemma 3.21. *Let μ and ν be fuzzy sets in A. If $\nu \subseteq \mu^{\ddagger}_{\beta}$ for $\beta \in [0, \ddagger]$, there exists $\alpha \in [0, \ddagger]$ with $\alpha \geq \beta$ such that $\nu \subseteq \mu^{\ddagger}_{\alpha} \subseteq \mu^{\ddagger}_{\beta}$.*

Proof. Assume that $\nu \subseteq \mu^{\ddagger}_{\beta}$ for $\beta \in [0, \ddagger]$. Then $\nu(x) \leq \mu^{\ddagger}_{\beta}(x)$ for all $x \in A$. Putting $\alpha = \beta + \inf_{x \in A}\{\mu^{\ddagger}_{\beta}(x) - \nu(x)\}$. Then

$$\begin{aligned}
\inf_{x \in A}\{\mu^{\ddagger}_{\beta}(x) - \nu(x)\} &\leq \inf_{x \in A}\{\mu^{\ddagger}_{\beta}(x)\} \\
&= \inf_{x \in A}\{(\mu(x) - \beta)\} \\
&= \inf_{x \in A}\{\mu(x)\} - \beta \\
&= \ddagger - \beta,
\end{aligned}$$

so $\alpha = \beta + \inf_{x\in A}\{\mu^{\ddagger}_{\beta}(x) - \nu(x)\} \leq \beta + \ddagger - \beta = \ddagger$. Thus $\alpha \in [0, \ddagger]$ and $\alpha \geq \beta$, so $\mu^{\dagger}_{\alpha} \subseteq \mu^{\dagger}_{\beta}$. Now, for all $x \in A$, we have

$$\begin{aligned}
\mu^{\ddagger}_{\alpha}(x) &= \mu(x) - \alpha \\
&= \mu(x) - (\beta + \inf_{x\in A}\{\mu^{\ddagger}_{\beta}(x) - \nu(x)\}) \\
&= \mu(x) - \beta - \inf_{x\in A}\{\mu^{\ddagger}_{\beta}(x) - \nu(x)\} \\
&= \mu^{\ddagger}_{\beta}(x) + \sup_{x\in A}\{\nu(x) - \mu^{\ddagger}_{\beta}(x)\} \\
&\geq \mu^{\ddagger}_{\beta}(x) + \nu(x) - \mu^{\ddagger}_{\beta}(x) \\
&= \nu(x),
\end{aligned}$$

so $\nu \subseteq \mu^{\ddagger}_{\alpha}$. Hence, $\nu \subseteq \mu^{\ddagger}_{\alpha} \subseteq \mu^{\ddagger}_{\beta}$ for some $\alpha \in [0, \ddagger]$ with $\alpha \geq \beta$. □

Definition 3.22. *Let μ_1 and μ_2 be two fuzzy sets in A and $\mu_1 \subseteq \mu_2$. If μ_2 is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, then μ_1 is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, and we say that μ_1 is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) intension of μ_2.*

Example 3.23. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define two fuzzy sets $\mu_1 \colon A \to [0,1]$ and $\mu_2 \colon A \to [0,1]$ in A as follows:

$$\begin{aligned}
&\mu_1(0) = 0.6, \mu_1(1) = 0.4, \mu_1(2) = 0.3, \text{ and } \mu_1(3) = 0.2, \\
&\mu_2(0) = 0.9, \mu_2(1) = 0.7, \mu_2(2) = 0.6, \text{ and } \mu_2(3) = 0.5.
\end{aligned}$$

Then $\mu_1 \subseteq \mu_2$, and μ_1 and μ_2 are fuzzy UP-subalgebras of A. Hence, μ_1 is a fuzzy UP-subalgebra intension of μ_2, and μ_2 is a fuzzy UP-subalgebra extension of μ_1. We define two fuzzy sets $\mu_3 \colon A \to [0,1]$ and $\mu_4 \colon A \to [0,1]$ in A as follows:

$$\begin{aligned}
&\mu_3(0) = 0.5, \mu_3(1) = 0.3, \mu_3(2) = 0.1, \text{ and } \mu_3(3) = 0.5, \\
&\mu_4(0) = 0.8, \mu_4(1) = 0.6, \mu_4(2) = 0.4, \text{ and } \mu_4(3) = 0.8.
\end{aligned}$$

Then $\mu_3 \subseteq \mu_4$, and μ_3 and μ_4 are fuzzy UP-filter of A. Hence, μ_3 is a fuzzy UP-filter intension of μ_4, and μ_4 is a fuzzy UP-filter extension of μ_3. We define two fuzzy sets $\mu_5 \colon A \to [0,1]$ and $\mu_6 \colon A \to [0,1]$ in A as follows:

$$\begin{aligned}
&\mu_5(0) = 0.6, \mu_5(1) = 0.4, \mu_5(2) = 0.3, \text{ and } \mu_5(3) = 0.2, \\
&\mu_6(0) = 0.9, \mu_6(1) = 0.7, \mu_6(2) = 0.6, \text{ and } \mu_6(3) = 0.3.
\end{aligned}$$

Then $\mu_5 \subseteq \mu_6$, and μ_5 and μ_6 are fuzzy UP-ideal of A. Hence, μ_5 is a fuzzy UP-ideal intension of μ_6, and μ_6 is a fuzzy UP-ideal extension of μ_5. We define two fuzzy sets $\mu_7 \colon A \to [0,1]$ and $\mu_8 \colon A \to [0,1]$ in A as follows:

$$\mu_7(x) = 0.5 \text{ and } \mu_8(x) = 0.8. \text{ for all } x \in A.$$

Then $\mu_7 \subseteq \mu_8$, and μ_7 and μ_8 are fuzzy strongly UP-ideal of A. Hence, μ_7 is a fuzzy strongly UP-ideal intension of μ_8, and μ_8 is a fuzzy strongly UP-ideal extension of μ_7.

Theorem 3.24. *If μ is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, than the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) intension of μ for all $\beta \in [0, \ddagger]$.*

Proof. It follows from Theorem 3.3 (resp. Theorem 3.6, Theorem 3.8, Theorem 3.10) and Note 3.20. □

Theorem 3.25. *If μ is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, then the fuzzy α-translation $\mu^{\ddagger}_{\alpha}$ of μ is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) intension of the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ for all $\alpha, \beta \in [0, \ddagger]$ with $\alpha \geq \beta$.*

Proof. It follows from Theorem 3.3 (resp. Theorem 3.6, Theorem 3.8, Theorem 3.10). □

Theorem 3.26. *Let μ be a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A and $\beta \in [0, \ddagger]$. For every fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) intension ν of the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ, there exists $\alpha \in [0, \ddagger]$ with $\alpha \geq \beta$ such that ν is the fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) intension of the fuzzy α-translation $\mu^{\ddagger}_{\alpha}$ of μ.*

Proof. It follows from Theorem 3.3 (resp. Theorem 3.6, Theorem 3.8, Theorem 3.10) and Lemma 3.21. □

Theorem 3.27. *If there exists $\alpha \in [0, \dagger]$ such that the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, then the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A for all $\beta \in [0, \ddagger]$.*

Proof. It follows from Theorems 2.34 and 3.3 (resp. Theorems 2.36 and 3.6, Theorems 2.38 and 3.8, Theorems 2.40 and 3.10). □

Theorem 3.28. *If there exists $\beta \in [0, \ddagger]$ such that the fuzzy β-translation $\mu^{\ddagger}_{\beta}$ of μ is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A, then the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy UP-subalgebra (resp. fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) of A for all $\alpha \in [0, \dagger]$.*

Proof. It follows from Theorems 3.5 and 2.33 (resp. Theorems 3.7 and 2.35, Theorems 3.9 and 2.37, Theorems 3.11 and 2.39). □

4. Conclusions

In the present paper, we have introduced the concepts of fuzzy α-translations of a fuzzy set of type I and of fuzzy β-translations of a fuzzy set of type II in UP-algebras and investigated some of its essential properties. Finally, we have important relationships between two types of fuzzy translations of a fuzzy set. We think this work would enhance the scope for further study in this field of fuzzy sets. It is our hope that this work would serve as a foundation for the further study in this field of fuzzy sets in UP-algebras.

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