J. Indones. Math. Soc.
Special Edition (2011), pp. 133–135.

HOW PROVABLY GRACEFUL ARE THE TREES?

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The Ringel-Kotzig-Rosa conjecture that all trees are graceful, originating in Ringel [3] and Rosa [4] is nearing 50 years without being resolved. Much evidence has been produced in support of the conjecture (see Gallian [2]), the majority of which falls under the "Here is another family of graceful trees" or "Trees of this type are graceful, and as the order of these trees increases the number of ways to gracefully label them increases" categories. But what can be said about all trees? Do they all possess a property that is close to being graceful? Is there a numerical measure by which we can say "All trees are at least this graceful"? Two open problems of this nature follow.

A tree T of order $|V(T)| = n$ and size $|E(T)| = m = n - 1$ is graceful if there is a bijection $f : V(T) \to \{0, 1, 2, \ldots, n-1\}$ such that the corresponding induced function $f : E(T) \to \{1, 2, \ldots, n-1\}$ defined by $f(uv) = |f(u) - f(v)|$ is also a bijection. The tree $T1$ is gracefully labeled in Fig. 1(a).

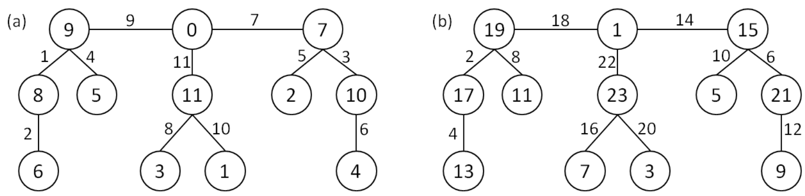

Figure 1. Graceful and simply sequential labelings of $T1$.

PROBLEM 1: Simply Sequential Trees.

For the tree $T1$ in Fig. 1, we have $n + m = 12 + 11 = 23$, and the labeling in Fig. 1(b) illustrates a bijection $f : V(T1) \bigcup E(T1) \to \{1, 2, \ldots, 23\}$ where again $f(uv) = |f(u) - f(v)|$. A graph G of order $|V(G)| = n$ and size $|E(G)| = m$ is

2000 Mathematics Subject Classification: 05C78.
Received: 09-08-2011, revised: 09-09-2011, accepted: 04-12-2011.

k-sequential if there is a bijection $f : V(G) \bigcup E(G) \to \{k, k+1, \ldots, n+m+k-1\}$ with $f(uv) = |f(u) - f(v)|$ for all $uv \in E(G)$. A 1-sequential graph is called *simply sequential*. Simply sequential graphs and k-sequential graphs were introduced in Bange, Barkauskas and Slater [1] and Slater [5].

The simply sequential labeling f^* in Fig 1(b) is obtained from the labeling $f : V(T1) \to \{0, 1, 2, \ldots, 11\}$ in Fig 1(a) by letting $f^*(v) = 2f(v) + 1$. This illustrates the following theorem.

Theorem 1 ([1])**.** *A tree T is graceful if and only if T is simply sequential via a function f^* such that $f^*(v)$ is odd for each vertex $v \in V(T)$.*

For example, labeling the vertices of path P_4 as $(0, 3, 1, 2)$ produces edge labels $(3, 2, 1)$, and $(1, 7, 3, 5)$ yields edge labels $(6, 4, 2)$. Thus, tree T is graceful implies T is simply sequential, and the truth of the R-K-R conjecture would imply the following conjecture.

Conjecture 2 (Slater [1])**.** *All trees are simply sequential.*

Note that labeling the vertices of P_4 as $(7, 6, 2, 5)$ yields edge labels $(1, 4, 3)$, and we have another simply sequential labeling of P_4. The star $K_{1,3}$ has eleven essentially different simply sequential labelings (see [1]), only two of which have all odd vertex labels. It might well be possible to show that all trees are simply sequential, providing strong support for the R-K-R conjecture.

PROBLEM 2: Matching $f(E(T))$ and $f(V(T)) - \{0\}$.

Trivially we note that each tree has one more vertex than edges. If $f : V(T) \to \{0, 1, \ldots, n-1\}$ is a graceful labeling with $f(v) = 0$, then f is an injection with $f(E(T)) = f(V(T) - v)$. That is, except for zero on vertex v, the set of vertex labels is the same as the set of edge labels. Note that no graceful labeling $f(V(T2)) \to \{0, 1, \ldots 5\}$ for the tree $T2$ in Fig. 2 has $f(v) = 0$. However, the illustrated injection $f(V(T2)) \to \{0, 1, \ldots, 6\}$ has $f(E(T2)) = f(V(T2) - v) = \{1, 3, 4, 5, 6\}$.

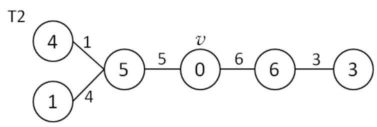

FIGURE 2. $f(E(T2)) = f(V(T2) - v)$.

The following result, interesting in its own right, was used as a lemma in Slater [6] to show that all countably infinite trees are k-graceful for all $k \geq 1$.

Theorem 3 ([6])**.** *Let T be any finite tree with $v \in V(T)$. There is a one-to-one function $f : V(T) \to \mathbb{N} - \{1, 2, \ldots, k-1\} = \{0, k, k+1, k+2, \ldots\}$ such that $f(v) = 0$ and $f(E(T)) = f(V(T) - v)$.*

Let T_n denote a tree of order n with $v \in V(T_n)$. Let $\mathcal{F}(T_n; v) = \{f : V(T) \rightarrow \{0, 1, 2, \ldots\} |$ f is 1-1, $f(v) = 0$, and $f(E(T)) = f(V(T) - v)\}$. Note that the above theorem states that $\mathcal{F}(T_n; v)$ is nonempty for all T_n and all $v \in V(T_n)$. Note also that $\mathcal{F}(T_n; v)$ contains any graceful labeling f with $f(v) = 0$. How large a label $f(w)$ is required for any $w \in V(T_n)$ for an $f \in \mathcal{F}(T_n; v)$?

Let $LG(T_n; v) = MIN_{f \in \mathcal{F}(T_n;v)} MAX\{f(w) : w \in V(T_n)\}$. For example, $LG(T2; v) = 6$.

We can state the R-K-R conjecture as follows.

R-K-R Conjecture: For any tree T_n there is at least one $v \in V(T_n)$ such that $LG(T_n; v) = n - 1$.

Many problems involving $LG(T_n; v)$ suggest themselves, including the following:

PROBLEM 2a: For which trees T_n is $LG(T_n; v) = n - 1$ for all $v \in T_n$? That is, which trees have graceful labelings with $f(v) = 0$ for any choice of v? (We can call such trees “charming”.)

PROBLEM 2b: For a given tree T_n, what is $MAX\{LG(T_n; v) | v \in T_n\}$?

PROBLEM 2c: What is the largest value of $LG(T_n; v)$ over all trees T_n of order n and all $v \in T_n$?

In the spirit of asking “How close to graceful can we prove all trees to be?”, there is the following problem.

PROBLEM 2d: For how small a function $h(n)$ can we show that, for all trees T_n of order n, we have $MIN\{LG(T_n; v) : v \in V(T_n)\} \leq h(n)$?

The R-K-R conjecture is that we have $h(n) = n - 1$ for Problem 2d.

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