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PERFECT 3-COLORINGS OF GP(5,2), GP(6,2), AND GP(7,2) GRAPHS

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Abstract. In this paper we enumerate the parameter matrices of all perfect 3-colorings of GP(5,2), GP(6,2), and GP(7,2) graphs.

Key words and Phrases: perfect coloring, equitable partition, generalized Petersen graph.

Abstrak. Pada paper ini ditentukan matrik parameter dari 3-pewarnaan sempurna dari graf GP(5,2), GP(6,2), dan GP(7,2).

Kata kunci:pewarnaan sempurna, partisi *equitable*, generalisasi graf Petersen

1. INTRODUCTION

The concept of a perfect m-coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another term for this concept in the literature as "equitable partition" (see Godsil [8]). The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of nontrivial perfect codes in Johnson graphs. Therefore, some effort has been done on enumerating the parameter matrices of some Johnson graphs, including $J(6,3)$, $J(7,3)$, $J(8,3)$, $J(8,4)$, and $J(v,3)$ (v odd) (see Avgustinovich and Mogilnykh [2, 3] and Gavrilyuk and Goryainov [7]).

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Fon-Der-Flass enumerated the parameter matrices of n-dimensional hypercube Q_n for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional cube with a given parameter matrix (see Fon-Der-Flass [4, 5, 6]). Also, there are more results obtained by researchers on perfect colorings of graphs (see [1, 10, 11]).

In this article we enumerate the parameter matrices of all perfect 3-colorings of $GP(5,2)$, $GP(6,2)$, and $GP(7,2)$ graphs.

2. Definition and Concepts

In this section, we give some basic definitions and concepts.

Definition 2.1. *The generalized Petersen graph $GP(n,k)$ has vertices, respectively, edges given by*

$$V(GP(n,k)) = \{a_i, b_i : 0 \le i \le n-1\},$$
$$E(GP(n,k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} : 0 \le i \le n-1\},$$

where the subscripts are expressed as integers modulo n (≥ 5), and k (≥ 1) is the "skip". Moreover, The distinct eigenvalues of GP(5,2) are 3, 1, and -2.

Definition 2.2. *For a graph G and an integer m, a mapping $T : V(G) \to \{1, \cdots, m\}$ is called a perfect m-coloring with matrix $A = (a_{ij})_{i,j \in \{1,\cdots,m\}}$, if it is surjective, and for all i, j, for every vertex of color i, the number of its neighbors of color j is equal to a_{ij}. The matrix A is called the parameter matrix of a perfect coloring. In the case $m = 3$, we call the first color white, the second color black, and the third color red.*

The next theorem is useful to enumerate parameter matrices.

Theorem 2.3 (Godsil and Gordon[9]). *If T is a perfect coloring of a graph G in m colors, then any eigenvalue of T is an eigenvalue of G.*

3. MAIN RESULTS

In this section, we first present some results concerning necessary conditions for the existence of perfect 3-colorings of the graph $GP(n,k)$ with a given parameter matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

The simplest necessary condition for the existence of a perfect 3-colorings of a cubic connected graph with the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is,

$$a + b + c = d + e + f = g + h + i = 3.$$

Also, it is clear that we cannot have $b = c = 0$, $d = f = 0$, or $g = h = 0$, since the graph is connected. In addition, $b = 0 \Leftrightarrow d = 0$, $c = 0 \Leftrightarrow g = 0$, and $f = 0 \Leftrightarrow h = 0$.

By using the conditions above and an easy computation, it can be seen that there are only 109 matrices that can be a parameter matrix corresponding to a perfect 3-coloring in a cubic connected graph.

In this paper, we consider all perfect 3-colorings, up to renaming the colors; i.e, we identify the perfect 3-coloring with the matrices

$$\begin{bmatrix} a & c & b \\ g & i & h \\ d & f & e \end{bmatrix}, \begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix},$$

$$\begin{bmatrix} e & f & d \\ h & i & g \\ b & c & a \end{bmatrix}, \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}, \begin{bmatrix} i & g & h \\ c & a & b \\ f & d & e \end{bmatrix},$$

obtained by switching the colors with the original coloring.

Also, by using this condition and some computation, it is clear that we should consider 22 out of 109 matrices. These matrices are listed below.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

The next proposition gives a formula for calculating the number of white, black and red vertices, in a perfect 3-coloring.

Proposition 3.1. *Let T be a perfect 3-coloring of a graph G with the matrix* $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

(1) *If $b, c, f \neq 0$, then*

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

(2) *If $b = 0$, then*

$$|W| = \frac{|V(G)|}{\frac{c}{g} + 1 + \frac{ch}{fg}}, |B| = \frac{|V(G)|}{\frac{f}{h} + 1 + \frac{fg}{ch}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

(3) *If $c = 0$, then*

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{bf}{dh}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{dh}{bf}}.$$

(4) *If $f = 0$, then*

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{cd}{bg}}, |R| = \frac{|V(G)|}{\frac{g}{c} + 1 + \frac{bg}{cd}}.$$

PROOF. **(1):** Consider the 3-partite graph obtained by removing the edges uv such that u and v are the same color. By counting the number of edges between parts, we can easily obtain $|W|b = |B|d$, $|W|c = |R|g$, and $|B|f = |R|h$. Now, we can conclude the desired result from $|W| + |B| + |R| = |V(G)|$.
The proof of (2), (3), (4) is similar to (1). □

The next theorem can be useful to find the eigenvalues of a parameter matrix.

Theorem 3.2. *If* $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ *be a parameter matrix of a k-regular graph, then the eigenvalues of A are*

$$\lambda_{1,2} = \frac{\operatorname{tr} A - k}{2} \pm \sqrt{\left(\frac{\operatorname{tr} A - k}{2}\right)^2 - \frac{\det A}{k}} \quad , \quad \lambda_3 = k.$$

PROOF. By using the condition $a+b+c = d+e+f = g+h+i = k$, it is clear that one of the eigenvalues is k. Therefore $\det A = k\lambda_1\lambda_2$. From $\lambda_2 = \operatorname{tr} A - \lambda_1 - k$, we get

$$\det A = k\lambda_1(\operatorname{tr} A - \lambda_1 - k) = -k\lambda_1^2 + k(\operatorname{tr} A - k)\lambda_1.$$

By solving the equation $\lambda^2 + (k - \operatorname{tr} A)\lambda + \frac{\det A}{k} = 0$, we obtain

$$\lambda_{1,2} = \frac{\operatorname{tr} A - k}{2} \pm \sqrt{\left(\frac{\operatorname{tr} A - k}{2}\right)^2 . - \frac{\det A}{k}}$$

□

The parameter matrices of GP(5,2) are enumerated in the next theorem.

Theorem 3.3. *The graph $GP(5,2)$ has perfect 3-colorings only with the matrices* $\begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

Proof. As the graph $GP(5,2)$ has 10 vertices, by using Proposition 3.1, it can be seen that the graph $GP(5,2)$ can have perfect 3-coloring only with the matrices below.

$$\begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

By using Theorem 2.3 and 3.2, it can be seen that there are no perfect 3-colorings with the matrices $\begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.
Now, consider the mapping $T : Gp(5,2) \to \{1,2,3\}$ by $T(a_0) = 1$, $T(a_1) = T(a_4) = T(b_0) = 2$, and $T(a_2) = T(a_3) = T(b_1) = T(b_2) = T(b_3) = T(b_4) = 3$. It is easliy seen that this mapping is a perfect 3-coloring with the matrix $\begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.
Next, consider the mapping $T : Gp(5,2) \to \{1,2,3\}$ by $T(a_0) = T(b_0) = 1$, $T(a_1) = T(a_4) = T(b_2) = T(b_3) = 2$, and $T(a_2) = T(a_3) = T(b_1) = T(b_4) = 3$. It can be easily seen that this mapping is a perfect 3-coloring with the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. □

Now, we present a useful lemma in order to enumerate the parameter matrices of $GP(n,2)$, especially $GP(6,2)$ and $GP(7,2)$.

Lemma 3.4. *There are no perfect 3-colorings of $GP(n,2)$ with the matrices*

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Proof. We give the proof only for the matrix $\begin{bmatrix}1 & 2 & 0\\ 2 & 0 & 1\\ 0 & 1 & 2\end{bmatrix}$; the other cases are similar. Suppose, contrary to our claim, there is a perfect 3-coloring of $GP(n,2)$ with the matrix $\begin{bmatrix}1 & 2 & 0\\ 2 & 0 & 1\\ 0 & 1 & 2\end{bmatrix}$, say T. Let $T(a_0)=1$. First, take $T(b_0)=1$. Hence, $T(a_1)=T(b_2)=2$ and, in consequence, $T(a_2)=1$. Then $T(a_3)=1$ and $T(b_1)=3$. From $T(a_3)=T(a_2)=1$, we conclude that $T(b_2)=T(b_3)=T(a_4)=2$. As $T(b_2)=2$ and $T(b_0)=T(a_2)=1$, we get $T(b_4)=3$ which is a contradiction of $T(a_4)=T(b_2)=2$. Now, suppose that $T(b_0)=2$ and, by symmetry, $T(a_1)=1$. Hence, $T(b_1)=T(a_2)=2$. From $T(a_2)=T(b_0)=2$, we conclude that $T(b_2)=1$ and, in consequence, $T(a_3)=T(a_4)=T(b_3)=3$. As $T(a_2)=T(b_0)=2$ and $T(b_2)=1$, we get $T(b_4)=1$ which is a contradiction of $T(a_4)=3$. □

Now, we finish this paper by enumerating the parameter matrices of $GP(6,2)$ and $GP(7,2)$.

Theorem 3.5. *The graph $GP(6,2)$ has perfect 3-colorings only with the matrices* $\begin{bmatrix}0 & 1 & 2\\ 1 & 1 & 1\\ 2 & 1 & 0\end{bmatrix}$ *and* $\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$.

Proof. As the graph $GP(6,2)$ has 12 vertices, by using Proposition 3.1, it can be seen that the graph $GP(6,2)$ can have perfect 3-coloring only with the matrices below.

$$\begin{bmatrix}1 & 2 & 0\\ 2 & 0 & 1\\ 0 & 1 & 2\end{bmatrix}, \begin{bmatrix}2 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 2\end{bmatrix}, \begin{bmatrix}0 & 1 & 2\\ 1 & 1 & 1\\ 2 & 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}.$$

By using Lemma 3.4, it can be seen that there are no perfect 3-colorings with the matrices $\begin{bmatrix}1 & 2 & 0\\ 2 & 0 & 1\\ 0 & 1 & 2\end{bmatrix}$ and $\begin{bmatrix}2 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 2\end{bmatrix}$.
Now, consider the mapping $T: Gp(6,2)\to\{1,2,3\}$ by $T(a_2)=T(a_5)=T(b_2)=T(b_5)=1$, $T(a_1)=T(a_4)=T(b_1)=T(b_4)=2$, and $T(a_0)=T(a_3)=T(b_0)=T(b_3)=3$. It is easliy seen that this mapping is a perfect 3-coloring with the matrix $\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$.
Next, consider the mapping $T: Gp(6,2)\to\{1,2,3\}$ by $T(a_0)=T(a_3)=T(b_2)=T(b_4)=1$, $T(a_1)=T(a_4)=T(b_1)=T(b_4)=2$, and $T(a_2)=T(a_4)=T(b_0)=T(b_3)=3$. It can be easily seen that this mapping is a perfect 3-coloring with the matrix $\begin{bmatrix}0 & 1 & 2\\ 1 & 1 & 1\\ 2 & 1 & 0\end{bmatrix}$. □

Theorem 3.6. *The graph $GP(7,2)$ has no perfect 3-colorings.*

PROOF. As the graph $GP(7,2)$ has 14 vertices, by using Proposition 3.1, it can be seen that the graph $GP(7,2)$ can have perfect 3-coloring only with the matrices below.

$$\begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

By using Lemma 3.4, it can be seen that there are no perfect 3-colorings with the above matrices which is our claim. □

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