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An Existence and Uniqueness of Solutions for a Stochastic Differential Equations with ψ-Caputo Fractional Derivative

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Abstract. The present work investigates the existence and uniqueness of solutions for a fractional stochastic differential equations with ψ-Caputo fractional derivatives. We establish important theoretical conclusion and examine the energy growth bounds and asymptotic behavior of the stochastic solution using the Banach's fixed point theorem. Additionally, we consider our analysis into single-valued function derived from multivalued mappings. We provide examples to validate in our methodology.

Key words and Phrases: ψ-Caputo fractional stochastic differential equation, Single valued map, Energy-growth bound, Asymptotic behavior.

1. INTRODUCTION

Fractional calculus extends standard differentiation and integration to noninteger orders, which creates a wider mathematical framework. This enables for the simulation of process with memory and mutations, which are common in many science and engineering systems. Additional studies support both the theory and applications of fractional calculus [1, 2, 3, 4, 5, 6, 7]. It remains an effective framework for understanding and solving complex problems. Fractional derivatives such as Caputo fractional derivative, Riemann-Liouville fractional derivative and ψ-Fractional derivatives have specific benefits for modeling complex operations.

Almeida [8] expanded the standard definition for Caputo fractional derivatives by introducing a flexible formulation that includes more general integral operators and different types of kernels. The ψ-Caputo derivatives enriches a function $\psi(t)$, resulting in more flexibility in modeling complex systems with variable scaling attributes (refer [9]). The ψ-Caputo derivative uses a kernel function ψ to model different systems behaviors, making it a useful tool for describing complex phenomena. Sharma et al. [10] discussed the existence and controllability results for

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fractional integro-differential stochastic delayed system with impulsive effects. A function that may generate numerous outputs from the single input is known as the multi-valued function. A single-valued function is the specific case of the multi-valued function. It is defined that there is exactly one output for every input. A single-valued function is a part multi-valued functions that it has a single element in their output set.

In [11], Hariharan et al. investigated the existence and uniqueness of Hilfer-Kkatugampola fractional differential equations with almost sectorial operators. Zhou and Zhang in [12] combined the probability density function and the Laplace transform to construct a framework of moderate solutions. Moreover, Peng and Jia [13] investigated the existence and uniqueness of the solutions for fractional differential equation whose derivative has an impact on an arbitrary function. In [14] Yang discussed the existence uniqueness of mild solution for a class of ψ-Caputo fractional stochastic evolution equation with varying-time delay driven by fractional Brownian motion. In [15], authors investigated on ψ-fractional stochastic system with Rosenblatt process. In [16], authors were studied long term behaviours of the mild solution to a Hilfer time-fractional stochastic differential equation. Sharma et al. [17] investigated the existence of the mild solution and approximate controllability of a class of Caputo conformable fractional neutral-type stochastic system with the instantaneous impulsive effects and nonlocal conditions.

In [18], Sharma et al. discussed the existence and controllability results for fractional Sobolev-type stochastic system involving delays and impulses. El-Sayed and Ibrahim [19] demonstrated the Cauchy problem of the multi-valued fractional differential equation. Nandhaprasadh and Udhayakumar [20] examined the existence of mild solutions for fractional stochastic evolution equations with infinite delay on an infinite interval. Hammami and Horrigue [21] investigated the existence and uniqueness of solutions to the ψ-Riemann-Liouville fractional stochastic differential equation is determined using Banach's fixed point theorem. Omaba et al. [22] used Banach fixed point theorem to prove existence and uniqueness of mild solution.

Motivated by the above articles, the following system is investigated by the fractional stochastic differential inclusion expressed in the Caputo sense in the presence of stochastic noise,

$$\begin{cases} {}^{C}D^{\gamma}_{\psi}U(t) \in g[t,U(t)]dt + \sum_{i=1}^{m} \rho^{i}[t,U(t)]dW_i(t) + F(t,U(t)), \\ U(0) = u_0, \end{cases} \tag{1}$$

here we take $f(t,U(t)) \in F(t,U(t))$ is a single valued function. Then the above inclusion, we consider the equation defined by

$$\begin{cases} {}^{C}D^{\gamma}_{\psi}U(t) = g[t,U(t)]dt + \sum_{i=1}^{m} \rho^{i}[t,U(t)]dW_i(t) + f(t,U(t)), \\ U(0) = u_0, \end{cases} \tag{2}$$

where $t \in [0,\vartheta],\ 0 < \gamma < 1,\ g : [0,\vartheta] \times \mathbb{R} \to \mathbb{R},\ \rho : [0,\vartheta] \times \mathbb{R} \to \mathbb{R}^m$ are the two well-defined maps, $F : [0,\vartheta] \times \mathbb{B} \to 2^{\mathbb{B}} - \{\tau\}$ is a set valued and $f : [0,\vartheta] \times \mathbb{B} \to \mathbb{B}$ is a single valued map and $\mathbb{B}$ is a Banach space.

The main contribution of this work is constructing and analysing the generalized fractional-order system with the ψ-Caputo derivative. Using single-valued nonlinear functions and Banach's fixed point theorem, we proved the existence and uniqueness of the solutions. Along with the asymptotic behavior of these solutions, we are verified their stability under the suitable conditions. An energy growth bound was derived to characterize the boundedness of the system. An example is presented to illustrate the significance of the existence and uniqueness results.

This work is organized as follows: Section 2 to recalls some fundamental concepts, results and properties related to fractional calculus and stochastic differential equations. Section 3 constructs the existence and uniqueness of the solution to the non-linear stochastic ψ-Caputo fractional differential equation. Section 4 demonstrates the upper growth bound. Section 5 establishes an exponential growth bound on the energy solution at time $t \in [0,\vartheta]$. Finally, Section 6 provides an example for verifying our methodology.

2. PRELIMINARIES

In this section, we provide some basic concepts of stochastic differential equations and $\psi-$fractional operator. Let us take $P = [0,\vartheta]$, where $0 < \gamma < 1,\ 0 < \delta < 1$ and let ψ be a non$-$negative function on P that is continuous and whose derivative $\psi^{'}$ is also non$-$negative. Simply, a given $\rho \in P$, we represent the function ψ_p defined by

$$\psi_p(t) = \psi(t) - \psi(p), \quad \forall\ t \in P.$$

Definition 2.1. [23],[24] *The ψ-Caputo fractional derivative of the function $f \in C^n(P,\mathbb{R}), \gamma > 0$, then the order γ is defined by*

$$^{C}D^{\gamma,\psi}(f)(t) = I^{n-\gamma,\psi}{}_{\psi}\vartheta_x^n f(x). \tag{3}$$

where, ${}_{\psi}\vartheta_x = (\frac{1}{\psi'(x)}\frac{d}{dx})$ and $n = [\gamma] + 1$.

The fractional integral of the function f with respect to a function ψ is defined by

$$I^{\gamma,\psi} f(x) = \frac{1}{\Gamma(\gamma)} \int_0^x \psi'(t)\psi_0^{\gamma-1}(x) f(t)dt, \tag{4}$$

provided the integral exists.

Lemma 2.2. [25] *Let $0 < p$ and $\gamma > p, f \in C\gamma, \psi[P,\mathbb{R}]$, then we have,*

$$^{C}D^{p,\psi} I^{\gamma,\psi} = I^{\gamma-1,\psi} f(t), \qquad \forall\ t \in P. \tag{5}$$

In particular, any non-negative integer $n \leq [\gamma] + 1$,

$$D^n I^{\gamma,\psi} f(t) = I^{\gamma-n,\psi} f(t), \qquad \forall\ t \in P. \tag{6}$$

Lemma 2.3. [26],[27] *If $\gamma > 0$ and $\delta > 0$, then we obtain*

1. $I^{\gamma,\psi}\left(\psi_0^{\delta-1}\right) = \frac{\Gamma(\delta)}{\Gamma(\delta+\gamma)}\psi_0^{\delta+\gamma-1}$.
2. ${}^{C}D^{\gamma,\psi}\left(\psi_0^{\delta-1}\right) = \frac{\Gamma(\delta)}{\Gamma(\delta-\gamma)}\psi_0^{\delta-\gamma-1}$.

Lemma 2.4. [28], [29] *If $0 < \gamma < 1$ and $f \in C^n[P, \mathbb{R}]$, then we obtain*

$$I^{\gamma,\psi}\left({}^{c}D^{\gamma,\psi}f\right)(t) = f(t) - \sum_{n=0}^{r-1}\frac{f^{(n)}(a^+)}{n!}(\psi_0(t))^n.$$

In particular for $\gamma \in (0,1)$, we get

$$I^{\gamma,\psi}\left({}^{C}D^{\gamma,\psi}f\right)(t) = f(t) - f(0), \qquad \forall\ t \in P. \tag{7}$$

Definition 2.5. [30] *A one dimensional Brownian motion $\{W(t), \mathbb{R}^+ \to \mathbb{R}\}$ is a real valued stochastic process that satisfies:*

I. $W(0) = 0$.
II. *Independent increment: When $q \leq t$, the increments $W(t) - W(q)$ and $W(t-q)$ are independent.*
III. *Normal increment : The difference $W(q+t) - W(q)$ follows a normal distribution $O(0,t)$ for all $q, t \geq 0$, where $\mathcal{O}(\lambda, \sigma^2)$ denotes a normal distribution with the mean λ and the variance σ^2.*
IV. *Brownian motion has almost continuous, then the function $t \to W(t)$ is continuous.*

Lemma 2.6. [21], [30] *(Itô isometry) Let $H : P \times \Lambda \to \mathbb{R}$ be a stochastic process(means that F_t^W- measurable for each $t \in P$), we have*

$$\mathbb{E}\left[\left(\int_0^{\vartheta} X_t\ dW_t\right)^2\right] = \mathbb{E}\left[\int_0^{\vartheta} X_t^2 dt\right]. \tag{8}$$

Definition 2.7. [21], [30] *We delegate a group of square-integrable functions on Λ to $H(\Lambda)$, with the following norm defined as*

$$\|x\|_H^2 = \int_{\Lambda} \mathbb{E}|x(z)|^2\ dz. \tag{9}$$

Theorem 2.8. [31] *(Banach's Fixed Point Theorem) Let T be a complete metric space and let $M : T \to T$ be a contraction on T. Then M has unique fixed point $y \in T$ such that $M(y^*) = y^*$.*

Lemma 2.9. [32] *(Gronwall Inequality) Let $K > 0$ and $x, y : P \to [0, \infty]$ be two continuous function satisfies*

$$y(t) \leq K + \int_0^t x(z)y(z)dz, \qquad t \in P.$$

Then, the regular Gronwall inequality is given by,

$$y(z) \leq K \exp\left(\int_0^t y(z)dz\right).$$

3. MAIN RESULT

Around this article, referring the filtered probability space which satisfies the standard requirements and $M = 1, 2, 3, 4, \cdots, m$ by $(\Lambda, \eta, \eta_t, \mathbb{P})$. Then the following notations must be used.

- $W(t) = \begin{pmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_m(t) \end{pmatrix}$ is a m-dimensional Brownian motion.
- $g : P \times \mathbb{R} \to \mathbb{R}$ and $\rho^i : P \times \mathbb{R} \to \mathbb{R}^m, for\ i \in M$, are well-defined maps.

For easier use, to indicate the components of ρ, throughout the text using the superscript $\rho^i_j, i, j \in M$. The following stochastic differential equation is taken into consideration.

$$\begin{cases} {}^C D^\gamma_\psi U(t) = g[t, U(t)]dt + \sum_{i=1}^m \rho^i[t, U(t)]dW_i(t) + f(t, U(t)), \\ U(0) = u_0. \end{cases}$$

where $t \in P, 0 < \gamma < 1$. Using the Lemma (2.4), the problem can change in its equivalent integral form:

$$\begin{aligned} U(t) =& u_0 + \frac{1}{\Gamma(\gamma)} \int_0^t \psi_0^{\gamma-1}(z)\psi'(z) \Bigg[g(z, U(z))dz + \sum_{i=1}^m \rho^i(z, U(z))dW_i(z) \\ & + f(z, U(z)) \Bigg] dz. \end{aligned} \tag{10}$$

The existence and uniqueness of solutions to the nonlinear stochastic ψ-Caputo fractional differential equation will be demonstrated in this section. For this, we assume that all $\tau \in g$ or $\rho^i_j, (t, u) \in P \times \mathbb{R}$ and $i, j \in M$.

(1) Lipschitz continuity: For any $t \in [0, \vartheta]$ and $u \in \mathbb{R}$, there exist a positive constant J, we have

$$\psi'(t)|\tau(t,u) - \tau(t,v)|^2 \leq J(|u-v|^2). \tag{11}$$

(2) Linear growth: For any $t \in [0, \vartheta]$ and $u \in \mathbb{R}$, there exist a positive constant G, we have

$$\psi'(t)|\tau(t,u)|^2 \leq G(1 + |u|^2). \tag{12}$$

Define an operator $A : H(\Lambda) \to H(\Lambda)$ by:

$$\begin{aligned} A(U(t)) =& u_0 + \frac{1}{\Gamma(\gamma)} \int_0^t \psi_0^{\gamma-1}(z)\psi'(z) \Bigg[g(z, U(z))dz + \sum_{i=1}^m \rho^i(z, U(z))dW_i(z) \\ &+ f(z, U(z)) \Bigg] dz. \end{aligned} \tag{13}$$

To prove the solutions of the existence and uniqueness, we first prove the following lemma.

Lemma 3.1. *Let U and V be the random field solutions, which satisfies*

$$\mathbb{E}\|A(U) - A(V)\|^2 \leq \frac{3(m^2+2)H\psi^{2\gamma-1}(\vartheta)}{(2\gamma-1)\Gamma^2(\gamma)} \|U - V\|^2.$$

Proof. By using the conditions such that, algebraic inequality $\Bigg(\sum_{i=1}^q r_i \Bigg)^2 \leq q \sum_{i=1}^q r_i^2$, the It$\widehat{o}$ isometry, the Lipschitz continuity and the linear growth we get the following evaluation

$$\begin{aligned} \mathbb{E}(|A(U(t)) - A(V(t))|^2) \leq& \frac{3\psi_0^{2\gamma-1}(t)}{(2\gamma-1)\Gamma(\gamma)} \\ & \Bigg[\mathbb{E}\Bigg(\int_0^t \psi'(z) \Big| g(z, U(z)) - g(z, V(z)) \Big|^2 dz \Bigg) \\ & + \mathbb{E}\Bigg(\int_0^t \psi'(z) \Big| \sum_{i=1}^m \rho^i(z, U(z)) - \sum_{i=1}^m \rho^i(z, V(z)) \Big|^2 dW_i(z) \Bigg) \\ & + \mathbb{E}\Bigg(\int_0^t \psi'(z) \Big| f(z, U(z)) - f(z, V(z)) \Big|^2 dz \Bigg) \Bigg] \\ \leq& \frac{3H\psi_0^{2\gamma-1}(t)}{(2\gamma-1)\Gamma^2(\gamma)} [2 + m^2] \int_0^t \mathbb{E}(|U(z) - V(z)|^2) dz \\ \leq& \frac{3(m^2+1)H\psi_0^{2\gamma-1}(\vartheta)}{(2\gamma-1)\Gamma^2(\gamma)} \|U - V\|^2. \end{aligned}$$

This completes the proof. $\square$

The proof of the following theorem is completed by applying Theorem (2.8) together with the lemma (3.1).

Theorem 3.2. *Given L be a constant in inequality* (11) *which satisfies*

$$L < \frac{3(2\gamma-1)\Gamma^2(\gamma)}{\psi_0^{2\gamma-1}(\vartheta)(m^2+2)}.$$

Thus, the equation (14) has unique solution.

4. ENERGY GROWTH-BOUND

The upper growth moment bound was proved in this section. It is essential to demonstrate the succeeding lemma.

Lemma 4.1. *Suppose that U is the random field solution which resolves the problem and fulfills*

$$\sup_{t\in P} \mathbb{E}|U(z)| < \infty. \tag{14}$$

Given that the Linear growth (12) *holds, we obtain:*

$$\mathbb{E}(|A(U(t))|^2) \leq c_2 + 2c_1\left(\int_0^\vartheta \mathbb{E}|U(z)|^2 dz\right),$$

where, $c_0 = 4\mathbb{E}(u_0^2)\vartheta, \qquad c_1 = \frac{4(m^2+2)G^2\psi_0^{2\gamma-1}}{(2\gamma-1)\Gamma^2(\gamma)} \qquad$ *and* $\qquad c_2 = c_0 + \vartheta c_1.$

Proof. By using the following conditions such that,the linear growth condition algebraic inequality $\left(\sum_{i=1}^q r_i\right)^2 \leq q\sum_{i=1}^q r_i^2$, and the Cauchy-Schwartz inequality, the It$\hat{o}$ isometry equation (6), we get

$$\begin{aligned}
\mathbb{E}(|A(U(t))|^2) \leq 4\mathbb{E}(u_0^2)\vartheta + \frac{4\psi_0^{2\gamma-1}(t)}{(2\gamma-1)\Gamma^2(\gamma)}\Bigg[\mathbb{E}\left(\int_0^t \psi'(z)(|g(z,U(z))|^2)dz\right) \\
+\mathbb{E}\left(\int_0^t \psi_0'\Big(|\sum_{i=0}^m \rho^i(z,U(z))|^2\Big)dz\right) \\
+\mathbb{E}\left(\int_0^t \psi'(z)\Big(|f(z,U(z))|^2\Big)dz\right)\Bigg] \\
\leq 4\mathbb{E}(u_0^2)\vartheta + \frac{4\psi_0^{2\gamma-1}(t)}{(2\gamma-1)\Gamma^2(\gamma)}\mathbb{E}\left(\int_0^t G(1+|U(z)|^2)dz\right) \\
+\frac{4\psi_0^{2\gamma-1}(t)m^2}{(2\gamma-1)\Gamma^2(\gamma)}\mathbb{E}\left(\int_0^t G(1+|U(z)|^2)dz\right)
\end{aligned}$$

$$+\frac{4\psi_0^{2\gamma-1}(t)m^2}{(2\gamma-1)\Gamma^2(\gamma)}\mathbb{E}\left(\int_0^t G(1+|U(z)|^2)dz\right)$$
$$\leq 4\mathbb{E}(u_0^2)\vartheta+\frac{4(m^2+2)\psi_0^{2\gamma-1}(t)G^2}{(2\gamma-1)\Gamma^2(\gamma)}\mathbb{E}\left(\int_0^t G(1+|U(z)|^2)dz\right).$$

Take,

$$c_0=4\mathbb{E}(u_0^2)\vartheta \quad and \quad c_1=\frac{4(m^2+2)\psi_0^{2\gamma-1}(t)G^2}{(2\gamma-1)\Gamma^2(\gamma)}.$$

Which follows that

$$\begin{aligned}\mathbb{E}(|A(U(t))|^2) &\leq c_0+c_1\left(\int_0^\vartheta (1+2\mathbb{E}|U(z)|^2)dz\right)\\ &\leq \underbrace{(c_0+c_1\vartheta)}_{c_2}+2c_1\int_0^\vartheta \mathbb{E}|U(z)|^2dz\\ &\leq c_2+2c_1\int_0^\vartheta \mathbb{E}|U(z)|^2dz.\end{aligned}$$

Therefore, we conclude it by

$$\|A(U)\|^2\leq c_2+2c_1\|U\|^2.$$

Now, we can simply demonstrate an upper growth bound for a random solution u_0 is bounded.

Then the following theorem, we combine the Gronwall inequality with Lemma (4.1). $\square$

Theorem 4.2. *Given that the condition of linear growth stated in* (12)*, we get*

$$\mathbb{E}(|A(U(t))|^2)\leq c_2e^{2c_1t}.$$

5. ASYMPTOTIC BEHAVIOR

From the previous result, the energy solution is satisfies an exponential growth bound at a given time $t\in[0,\vartheta]$. Therefore, behavior of an energy solution and finding a growth of time is very large. Then the growth rate of the solution has the finite upper bound.

Corollary 5.1. *Let the condition of Theorem* (4.2) *and* $0<\gamma<1$ *are exist, thus we can get*

$$\lim_{t\to+\infty} sup\frac{log\Big(\mathbb{E}(|A(U(t))|^2)\Big)}{t}\leq 2c_1.$$

Proof. Recall from Theorem (4.2) that

$$\mathbb{E}(|A(U(t))|^2) \leq c_2 e^{2c_1 t}.$$

Taking *log* on both sides then the equation becomes,

$$log\Big(\mathbb{E}(|A(U(\vartheta))|^2)\Big) \leq log(c_2) + 2c_1 t.$$

Then, dividing both sides by 't' and taking lim sup, we get

$$\lim_{t\to+\infty} sup \frac{log\Big(\mathbb{E}(|A(U(\vartheta))|^2)\Big)}{t} \leq \lim_{t\to+\infty}\left(\frac{log(c_2)}{t} + 2c_1\right) \leq 2c_1.$$

Hence proved. □

6. EXAMPLES

In this section, we demonstrate the fractional stochastic differential equation and verifying that it is unique and bounded.

Example 6.1. *Given* $\gamma = \frac{3}{4}$, $\psi(t) = \sqrt{t+1}$, $m = 1$ *and* $P = [0, 1]$ *to define the following fractional stochastic differential equation:*

$$\begin{cases} {}^C D^{\gamma}_{\psi} U(t) = \cos(U(t))dt + t^2 U(t)t + U(t)dW(t), \\ U(0) = u_0 = 1. \end{cases} \tag{15}$$

Here we place $g[t, U(t)] = \cos(U(t))$, $f[t, U(t)] = t^2 U(t)$ and $\rho[t, U(t)] = U(t)$.
It can be observed that the maps g and ρ meet the conditions (11) and (12) with $G = J = \frac{e}{3}$.
Further,

$$\frac{3(2\gamma - 1)\Gamma^2(\gamma)}{\psi_0^{2\gamma-1}(\vartheta)(m^2+2)} = 1.1 > J.$$

By verifying Theorem (4.2) and the existence and uniqueness of the solution (15), we obtain

$$\mathbb{E}(|A(U(t))|^2) \leq 12.3 e^{8.3t}.$$

Additionally, solution achieves

$$\lim_{t\to+\infty} sup \frac{log\Big(\mathbb{E}(|A(U(\vartheta))|^2)\Big)}{t} \leq 8.3.$$

Therefore, the system (15) is unique and bounded.

Example 6.2. *Given* $\gamma = \frac{2}{3}$, $\psi(t) = e^t$, $m = 1$ *and* $P = [0, 1]$ *to define the following fractional stochastic differential equation:*

$$\begin{cases} {}^{C}D^{\gamma}_{\psi}U(t) = [\frac{3}{5}U(t) + \psi_0^{\frac{3}{5}}(t)]dt + \sqrt{t}U(t)dt + \frac{U(t)}{1+\psi_0(t)}dW(t), \\ U(0) = u_0 = 1. \end{cases} \tag{16}$$

Here we place $g[t, U(t)] = \frac{3}{5}U(t) + \psi_0^{\frac{3}{5}}(t),\ f[t, U(t)] = \sqrt{t}U(t)$ and $\rho[t, U(t)] = \frac{U(t)}{1+\psi_0(t)}$.
Then, it can be observed that the maps g and ρ meets the conditions (11) and (12) with $G = J = \frac{1}{3}$.
Further,

$$\frac{3(2\gamma - 1)\Gamma^2(\gamma)}{\psi_0^{2\gamma-1}(\vartheta)(m^2 + 2)} = 0.9 > J.$$

By verifying Theorem (4.2) and the existence and uniqueness of the solution (16), we obtain

$$\mathbb{E}(|A(U(t))|^2) \leq 6.1e^{2.6t}.$$

Additionally, solution achieves

$$\lim_{t\to+\infty} sup \frac{log\Big(\mathbb{E}(|A(U(\vartheta))|^2)\Big)}{t} \leq 2.6.$$

Therefore, the system (16) is unique and bounded.

7. CONCLUDING REMARKS

In this study, the ψ-Caputo fractional derivative involving the single valued function was discussed and a general analytical framework ensuring the existence and uniqueness of the solutions. The existence and uniqueness of solution was derived by using the Banach's fixed point theorem. The obtained energy growth-bound and asymptotic results describe the bounded and long-term behavior of the system. An example is provided to demonstrate our findings. This methodology provides a basis for future investigations of nonlinear and coupled systems. Practical applications will be improved by the development of the numerical techniques.

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Declarations.
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