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WHEN IS AN ABELIAN WEAKLY CLEAN RING CLEAN?

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Abstract. We answer the title question in the affirmative by showing that any abelian weakly clean ring for which 2 belongs to its Jacobson radical (in particular, if 2 is nilpotent) has to be clean. Some constructive examples, one of which illustrates that this is no longer true removing the condition on the 2, are given as well.

Key words: weakly clean rings, clean rings, nilpotents, Jacobson radical.

Abstrak. Kami menjawab pertanyaan yang tertera dalam judul secara afirmatif dengan menunjukkan bahwa sebarang *abelian weakly clean ring* dimana 2 merupakan anggota dari *Jacobson Radical*nya (secara khusus, jika 2 adalah nilpoten) haruslah juga *clean*. Beberapa contoh konstruktif, yang mengilustrasikan bahwa hal ini tidak lagi benar dengan menghapus kondisi pada 2 juga disajikan pula.

Kata kunci: *weakly clean rings, clean rings, nilpotents, Jacobson radical*

1. Introduction and Backgrounds

Throughout this note, let R be an associative unital ring with identity element 1, with unit group $U(R)$, with Jacobson radical $J(R)$, with center $C(R)$, with set of idempotents $Id(R)$ and with set of nilpotents $Nil(R)$. If $Nil(R)$ is an ideal, we just write $N(R)$ for simpleness. All other notations are standard and the terminology is classical. The most important concepts used in the sequel are recollected below. For instance, a ring R is said to be *abelian* if all its idempotents are central, that is, $Id(R) \subseteq C(R)$. Unfortunately, in abelian rings $Nil(R)$ need not be always an ideal.

In [10] the following fundamental notion was defined.

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Definition 1. A ring R is called *clean* if each $r \in R$ can be expressed as $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$.

Likewise, in [10] was pointed out that R is clean if, and only if, $R/J(R)$ is clean and all idempotents lift modulo $J(R)$.

The "clean" concept was generalized there to the following one:

Definition 2. A ring R is said to be *an exchange ring* if, for every $a \in R$, there exists an idempotent $e \in Ra$ such that $1 - e \in R(1 - a)$.

In [10] was obtained that R is an exchange ring if, and only if, $R/J(R)$ is an exchange ring and all idempotents lift modulo $J(R)$.

Besides, it was established there that Definitions 1 and 2 are tantamount for abelian rings. However, there is an exchange ring that is not clean.

On the other hand, in [1] was introduced the notion of *weakly clean* rings but only in the commutative aspect as follows:

Definition 3. A ring R is called *weakly clean* if each $r \in R$ can be expressed as either $r = u + e$ or $r = u - e$, where $u \in U(R)$ and $e \in Id(R)$.

Certainly, the last definition can be stated in general for arbitrary rings. Evidently, all clean rings are weakly clean, but the converse does not hold even in the commutative version (see, e.g., [1]). However, every weakly clean ring of characteristic 2 is clean, and vice versa. Our motivation to write the present paper is to strengthen this self-evident observation by requiring that 2 lies in $J(R)$ that supersedes the condition $2 = 0$.

Moreover, the next necessary and sufficient condition for an abelian ring to be weakly clean, slightly reformulated for the left hand-side, is needed for our further applicable purposes (see [5] or [12]):

Criterion. *The abelian ring R is weakly clean if, and only if, for any $x \in R$ there exists $e \in Id(R)$ such that $e \in Rx$ and either $1 - e \in R(1 - x)$ or $1 - e \in R(1 + x)$.*

Mimicking [7], recall that a ring is *nil clean* if each its element can be written as the sum of a nilpotent and an idempotent, whereas imitating [6] a ring is *weakly nil clean* if each its element can be written as the sum or the difference of a nilpotent and an idempotent. In [6] it was proved that R is a commutative weakly nil clean ring and $2 \in N(R)$ if, and only if, R is nil clean. It was also demonstrated that there is a clean ring of characteristic 2 which is not weakly nil clean.

So, the purpose of the current article is to demonstrate that an analogous result can be viewed for weakly clean and clean rings. We will also improve in some way the above Criterion in the sense of Definition 2.1 from [3]. This shall be done in the next section.

2. Main Results

We first begin with the promised above extension of Theorem 2.1 from [5], especially of the implication ($\Rightarrow$), in the sense of Definition 2.1 in [3].

Theorem 2.1. *Suppose R is an abelian ring. Then R is weakly clean if, and only if, for each $x \in R$ there exist $e \in Id(R)$ and $a, b \in R$ such that $ab = ba$, $e = ax = xa$ and either $1 - e = b(1 - x) = (1 - x)b$ or $1 - e = b(1 + x) = (1 + x)b$.*

Proof. The sufficiency follows directly from [5], so that we will concentrate on the necessity. To that aim, given $x \in R$, then we may write $x = u + f$ or $x = u - f$ with $u \in U(R)$ and $f \in Id(R)$. Clearly, in the first case, $ux = u(u+f) = u^2 + uf = u^2 + fu = (u + f)u = xu$ and by the same reason $u^{-1}x = xu^{-1}$. Similarly, $ux = u(u - f) = u^2 - uf = u^2 - fu = (u - f)u = xu$ and by the same token $u^{-1}x = xu^{-1}$ in the second case.

Now, setting $e = u(1 - f)u^{-1}$, we observe that $e = e^2$. Furthermore, when $x = u + f$, we have $(x - e)u = (u + f - u(1 - f)u^{-1})u = u^2 + fu - u(1 - f) = u^2 + 2uf - u = x^2 - x$ and hence $e - x = (x - x^2)u^{-1}$. Thus $e = x + (x - x^2)u^{-1} = x(1+(1-x)u^{-1}) = (1+(1-x)u^{-1})x = (1+(1-u-f)u^{-1})x = (1-f)u^{-1}x = u^{-1}ex$, because by what we have shown above $u^{-1}x = xu^{-1}$. Likewise, $1 - e = 1 - x - (x - x^2)u^{-1} = (1 - xu^{-1})(1 - x) = (1 - x)(1 - xu^{-1})$.

Next, when $x = u - f$, we have $(x + e)u = (u - f + u(1 - f)u^{-1})u = u^2 - fu + u - uf = u^2 - 2uf + u = x^2 + x$ whence $x + e = (x^2 + x)u^{-1}$. So, $e = (x^2 + x)u^{-1} - x = x((x + 1)u^{-1} - 1) = ((x + 1)u^{-1} - 1)x$ since by what we have demonstrated above $u^{-1}x = xu^{-1}$. Also, $1 - e = 1 + x - (x^2 + x)u^{-1} = (1 + x)(1 - xu^{-1}) = (1 - xu^{-1})(1 + x)$. That is why, one sees that in both cases we can put $a = u^{-1}e$ and $b = 1 - xu^{-1}$. Therefore, it is routinely checked that $ab = ba$, because $xu^{-1} = u^{-1}x$ and e is central, as asserted. ∎

Remark. Actually, in both Theorem 2.1 above and Theorem 2.2 from [3] the element $e = u(1 - f)u^{-1}$ just equals to $e = 1 - f$ because u and f commute. However, the calculations there show that even more general setting could be true.

Recall that two left (respectively, right) ideals I and J of a ring R are called *comaximal* if their sum generate all of R, that is, $I + J = R$ (see [8] for the commutative variant). The next statement is well-known (see, e.g., [9] and [4]); however for completeness of the exposition and reader's convenience we rework the proof from [9] to obtain the following:

Proposition 2.2. *An abelian ring R is clean if, and only if, for any two left (respectively, right) comaximal ideals I and J of R, there exists an idempotent $e \in I$ with the property that $1 - e \in J$.*

Proof. "$\Rightarrow$". If $I + J = R$, there is $a \in I$ such that $1 - a \in J$. Write $1 - a = u + e$, where $u \in U(R)$ and $e \in Id(R)$. Then $u^{-1}(1 - a) = u^{-1}u + u^{-1}e = 1 + u^{-1}e$, whence by multiplying both sides with $1-e$, we deduce that $u^{-1}(1-a)(1-e) = 1-e$. Thus, since $1 - e \in Id(R)$, we have that $u^{-1}(1 - e)(1 - a) = 1 - e$, and hence $1 - e \in J$. Moreover, $a = 1 - u - e$, so that multiplying both sides with e, we derive that $ae = -ue$, and consequently $-u^{-1}ae = e$, i.e., $-u^{-1}ea = e \in I$, as required.

"$\Leftarrow$". If $a \in R$, one may decompose $Ra + R(1-a) = R$, so there is an idempotent $e \in Ra$ with $1-e \in R(1-a)$. Write $e = ra$ and $1-e = s(1-a)$, where $r, s \in R$. But

$$[re - s(1-e)][ea + (1-e)(a-1)] = rea - s(1-e)(a-1) =$$

$$= rae - s(a-1)(1-e) = e + (1-e) = 1.$$

Moreover, one sees that

$$ea+(1-e)(a-1)+(1-e) = ea+(1-e)(a-1+1) = ea+(1-e)a = ea+a-ea = a.$$

By what we have established above, $ea+(1-e)(a-1)$ is a unit, while $1-e$ is an idempotent, so that one can conclude that R is by definition clean, as claimed. ∎

So, we come to our first basic result.

Theorem 2.3. *Suppose that R is an abelian ring with $2 \in Nil(R)$. Then R is weakly clean if, and only if, R is clean.*

Proof. The direction "if" being trivial true in general (even when $2 \notin Nil(R)$), we concentrate on the "and only if" one. To that goal, let I and J be two arbitrary comaximal left ideals of R, say $R = I + J$. Thus, there is $a \in I$ such that $1-a \in J$. Write either $1-a = u+e$ or $1-a = u-e$, where $u \in U(R)$ and $e \in Id(R)$. Furthermore, in the first case, $u^{-1}(1-a) = 1 + u^{-1}e$ whence $u^{-1}(1-a)(1-e) = u^{-1}(1-e)(1-a) = 1-e \in J$ because $1-e \in Id(R)$. Moreover, $a = 1-u-e$, so that $ae = -ue$. Hence, $-u^{-1}ae = -u^{-1}ea = e \in I$, as desired.

In the second situation, $1-a = u-e$ which implies that $u^{-1}(1-a) = 1-u^{-1}e$ and therefore $u^{-1}(1-a)(1-e) = u^{-1}(1-e)(1-a) = 1-e \in J$ since $1-e \in Id(R)$. Also, $a = 1-u+e$ yields that $ae = 2e - ue = (2-u)e$. But 2 being a central nilpotent obviously commutes with u and thus $2-u \in U(R)$ because it is well known that the sum/difference of a nilpotent and a unit which commute one to other, is again a unit. In fact, since $2u = u2$, we have that $u^{-1}2u = u^{-1}u2 = 1 \cdot 2 = 2$ and hence by multiplying both sides from the right by u^{-1}, we obtain $u^{-1}2 = 2u^{-1}$. Furthermore, write $2-u = u(u^{-1}2-1)$. Since as already have seen u^{-1} and 2 commute, the element $u^{-1}2$ is immediately a nilpotent and thus $u^{-1}2-1$ is a unit. Finally, $2-u$ must be a unit, too, as asserted. Consequently, one observes that $(2-u)^{-1}ae = (2-u)^{-1}ea = e \in I$, as wanted.

Finally, we apply Proposition 2.2 to get the assertion. ∎

It is well known that $N(R) \subseteq J(R)$ whenever R is a commutative ring. However, this inclusion is not valid provided R is abelian. In fact, the following two constructions are fulfilled:

Example 1: There exists an indecomposable ring with zero Jacobson radical containing nontrivial nilpotent elements.

Inside $M_2(\mathbb{Z})$, the ring of all 2×2 matrices over the ring $\mathbb{Z}$ of integers, let $R = \mathbb{Z} \cdot 1 + M_2(2\mathbb{Z})$. Observe that there are no nonzero idempotents in $M_2(2\mathbb{Z})$, from which it follows that 1 and 0 are the only idempotents in R. So, R should be indecomposable, and hence abelian. For any odd prime p, the map $M_2(2\mathbb{Z}) \to M_2(\mathbb{Z}) \to M_2(\mathbb{Z}/p\mathbb{Z})$ is surjective since all that is needed is to check this on the entries of the matrices and it is obvious that the map $2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ is surjective because p is odd, so $R \cap M_2(p\mathbb{Z})$ is a maximal ideal of R. The intersection of these is 0, so it must be that $J(R)$ is zero. However, there are plenty of nilpotents in R. ∎

Example 2: There exists an abelian ring R such that $Nil(R)$ is not contained in $J(R) = \{0\}$.

Suppose that $A = \mathbb{Z}[X]$ is the ring of polynomials over the ring of integers $\mathbb{Z}$. Consider the ring of all 2×2 matrices $R = M_2(\mathbb{Z}[X])$ over the polynomial ring A. Setting

$$S = \{(a, b; c, d) \in R \mid a(0) = d(0), c(0) = 0\},$$

where $a, b, c, d \in A$. Writing $a(X) = a_0 + a_1X + \cdots + a_nX^n$, $b(X) = b_0 + b_1X + \cdots + b_nX^n$, $c(X) = c_0 + c_1X + \cdots + c_nX^n$ and $d(X) = d_0 + d_1X + \cdots + d_nX^n$, it must be that $a_0 = d_0$ and $c_0 = 0$. Thus $c(X) = c_1X + \cdots + c_nX^n$ and hereafter we leave to the reader to verify that the subring S of R works to obtain the desired example. ∎

Nevertheless, all central nilpotents of any ring must lie in the Jacobson radical. Since 2 is a central element, if it is a nilpotent, it will follow that 2 lies in $J(R)$. So, it is rather natural to ask of whether or not Theorem 2.3 can be refined to this new condition on the Jacobson radical. We will do that in what follows, but we foremost need the following parallel claim (see Theorem 2.5 below) to that for clean and exchange rings quoted above - unfortunately we prove it only for abelian rings.

Recall that the idempotents of a ring R can be *lifted modulo the ideal* L if, given $x \in R$ with $x - x^2 \in L$, there exists $e \in Id(R)$ such that $e - x \in L$. Replacing x by $-x$ this condition is equivalent to the following: if $x + x^2 \in L$, there exists $e \in Id(R)$ such that $e + x \in L$.

The following technicality is necessary for our presentation and is of independent interest as well.

Lemma 2.4. *For each ring R the following equality holds:*

$$U(R) = U(R) + J(R).$$

Proof. The left hand-side is apparently contained in the right hand-side. To show the converse, given $x \in U(R)+J(R)$, we write that $x = u+j$, where $u \in U(R)$ and $j \in J(R)$. Since $x = u(1+u^{-1}j)$ and $u^{-1}j \in J$, it is enough to show only that $1+j$ lies in $U(R)$. But this follows immediately from the characterization of $J(R)$ in the form

$$J(R) = \{x \in R \mid 1 - rxs \in U(R), \forall r, s \in R\},$$

which gives the desired inclusion $1 + J(R) \subseteq U(R)$. And thus, $x \in U(R)$, as required. ∎

Theorem 2.5. *Suppose R is an abelian ring with $2 \in J(R)$. Then R is weakly clean if, and only if, $R/J(R)$ is weakly clean and all idempotents in R lift modulo $J(R)$.*

Proof. Let R be weakly clean. Since homomorphic (in particular, epimorphic) images of weakly clean rings are obviously weakly clean, it immediately follows that $R/J(R)$ is weakly clean as well. To treat the second part, utilizing the Criterion from Section 1 or Theorem 2.1, for any $x \in R$ there is $e \in Id(R)$ such that $e \in Rx$ and either $1-e \in R(1-x)$ or $1-e \in R(1+x)$. We thus deduce that either $e - x = e(1-x) - (1-e)x \in Rx(1-x) - R(1-x)x = R(x-x^2)$ or $e + x = e(1+x) + (1-e)x \in Rx(1+x) + R(1+x)x = R(x+x^2)$ and, moreover, $e - x = e(1-x) - (1-e)x \in Rx(1-x) - R(1+x)x = R(x-x^2) - R(x+x^2)$ or $e + x = e(1+x) + (1-e)x \in Rx(1+x) + R(1-x)x = R(x+x^2) + R(x-x^2)$. But one sees that $x - x^2 = (x+x^2) - 2x^2$ and hence $x - x^2 \in J(R)$ if and only if $x + x^2 \in J(R)$ because $2x^2 \in J(R)$. Therefore, in either case, we deduce that $e - x \in J(R)$ or $e + x \in J(R)$, as required. So, in both cases, the conditions for lifting are satisfied and we are done.

Conversely, let us assume that $R/J(R)$ is weakly clean and all idempotents of R can be lifted modulo $J(R)$. Given $x \in R$ and write $\overline{x} = x + J(R)$ where we denote $\overline{R} = R/J(R)$. Thus there exists $\overline{a} \in \overline{R}\overline{x}$ such that $\overline{a}^2 = \overline{a}$, and $\overline{c} \in \overline{R}$ such that $\overline{1} - \overline{a} = \overline{c}(\overline{1} + \overline{x})$. We may without loss of generality assume that $a \in Rx$. Choose $\overline{f} = \overline{a}$ with $f^2 = f$. Hence, it is easily checked that $u = 1 - f + a = 1 - (f - a) \in U(R)$, and so define $e = u^{-1}fu = u^{-1}fa$. In fact, $a - f \in J(R)$ and hence, by what we have observed above, $u \in 1 + J(R) \subseteq U(R)$, as claimed. Therefore, $e^2 = e \in Rx$. Since $\overline{e} = \overline{f} = \overline{a}$, we deduce that $1 - e - c(1+x) \in J(R)$. Hence, the usage of Lemma 2.4 insures that $e + c(1+x) \in U(R)$, and it is then a routine exercise to obtain the equality $Re + R(1+x) = R$. So, we may write $1 = te + s(1+x)$, where $t, s \in R$. Setting $g = e + (1-e)te$, it is plainly verified that $g^2 = g \in Rx$ and now $1 - g = (1-e)s(1+x) \in R(1+x)$, as required to apply the Criterion from the introductory section to get the result. ∎

Notice that the restriction on 2 to lie in $J(R)$ was not used in the proof of sufficiency.

So, we are now ready to extend Theorem 2.3 to the following statement:

Theorem 2.6. *Suppose that R is an abelian ring with $2 \in J(R)$. Then R is weakly clean if, and only if, R is clean.*

Proof. One direction being elementary, we consider the central one. So, given R is weakly clean, we apply Theorem 2.5 to obtain that $R/J(R)$ is weakly clean and all idempotents of R are lifted modulo $J(R)$. Since 2 is in $J(R)$, the quotient ring $R/J(R)$ is clean having characteristic 2. We furthermore employ [10] to infer that R is clean, as expected. ∎

Remark. It is worthwhile notice that in [1] was constructed a concrete example of a commutative weakly clean ring in which $2 \in U(R)$ that is not clean; for instance, this is the ring $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$, consisting of the joint elements of $\mathbb{Z}_{(3)}$ and $\mathbb{Z}_{(5)}$, where $\mathbb{Z}_{(3)}$ is the ring of integers localized at 3 and $\mathbb{Z}_{(5)}$ is the ring of integers localized at 5.

Moreover, there is a simple direct confirmation of the validity of the last theorem like this: Letting $x \in R$, we write $x = u + e$ with $u \in U(R)$ and $e \in Id(R)$, so we are done. In the remaining case when $x = u - e$, we simply write $x = (u - 2e) + e$, where a direct check by Lemma 2.4 shows that $u - 2e \in U(R)$ since $2 \in J(R)$, and thus we are finished after all, because x is a clean element in both cases. We emphasize also that by a combination of this observation and [10], the above Theorem 2.5 may be successfully obtained. Although the existence of that proof, our basic aim was to demonstrate a more general approach by obtaining some other strong results.

Likewise, concrete examples of weakly clean rings with $2 \in J(R)$ are weakly clean rings whose units are unipotents; in fact, $-1 = 1 + n$, where n is a nilpotent, so that $2 = n$ is a nilpotent.

The following extends Theorem 1.9 (2) of [1] to abelian rings.

Corollary 2.7. *Let R be an abelian ring with $2 \in J(R)$. Then $R[[X]]$ is weakly clean if, and only if, R is weakly clean.*

Proof. Since R is a homomorphic image of $R[[X]]$, the necessity follows directly.

As for the sufficiency, because of the equality $J(R[[X]]) = J(R) + XR[[X]]$, we observe that $R[[X]]/J(R[[X]]) \cong R/J(R)$ is weakly clean. Since all idempotents of $R[[X]]$, that are precisely the idempotents in R, can be lifted modulo $J(R)$ by using Theorem 2.5, and hence modulo $J(R[[X]]) \supseteq J(R)$, Theorem 2.5 applies to show that $R[[X]]$ must be weakly clean, as stated. ∎

Remark. Actually, Corollary 2.7 can be deduced by Theorem 2.6 and the corresponding result for clean rings.

3. Open Problems

In closing we pose the following questions of some interest and importance:

Problem 1. Whether Theorem 2.5 remains valid for an arbitrary not necessarily abelian ring without the limitation $2 \in J(R)$?

If yes, the next question will have a positive answer too.

Problem 2. Does it follow that Theorem 2.6 (in particular, Theorem 2.3) remains true for an arbitrary not necessarily abelian ring?

Problem 3. If R is weakly clean and $e \in Id(R)$ (or e is a full idempotent, that is, $ReR = R$), is it true that eRe is weakly clean?

Problem 4. Characterize uniquely weakly clean rings?

For a nice characterization of uniquely clean rings the interested reader may see [11] and [2].

Conjecture 5. Every ring R is clean if, and only if, it is an exchange weakly clean ring?

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