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Explicit Formulas for the Anti-Trace of 3-by-3 Matrix Powers via Generating Functions

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Abstract. We study the anti-trace—the sum of anti-diagonal entries—of powers of 3-by-3 matrices. Unlike the ordinary trace, the anti-trace has no spectral characterization, so closed forms for the anti-trace of matrix powers are valuable. Starting from the Cayley–Hamilton recurrence for matrix powers and solving a trivariate generating function, we derive explicit closed-form expressions whose coefficients are signed multinomial combinations of the sums of principal minors and their anti-principal counterparts. The formulas are purely algebraic and remain valid over any commutative ring. As a structural application, we analyze block anti-diagonal matrices of size 3m-by-3m.

Key words and Phrases: anti-trace, trace, Cayley–Hamilton, generating function, block anti-diagonal matrix.

1. INTRODUCTION

The trace of a matrix is the sum of its diagonal entries and appears throughout mathematics and its applications. In particular, traces of matrix powers arise in network analysis, number theory, dynamical systems, and control [1]. For instance, in a connected simple graph the total number of triangles equals $Tr(A^3)/6$, where A is the adjacency matrix [2]; for integer matrices one has the Euler congruence

$$Tr(A^{p^r}) \equiv Tr(A^{p^{r-1}}) \pmod{p^r}$$

for all primes p and $r \in \mathbb{N}$; and in dynamical systems and control, traces of matrix polynomials enter Lefschetz numbers and Lyapunov equations [3, 4]. Because $Tr(A)$

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equals the sum of eigenvalues, $Tr(A^n)$ is spectrally transparent: it is the sum of eigenvalues raised to the nth power.

By contrast, the *anti-trace* $Tr_a(A)$—the sum of the anti-diagonal entries—has no analogous spectral identity. In the absence of such a formula, computing $Tr_a(A^n)$ by first forming A^n is expensive and obscures structure. Recent work addresses special cases for traces (e.g., tridiagonal or persymmetric matrices [5, 6, 7]) and gives closed forms for 2×2 powers in terms of trace and determinant [8, 9]. For anti-trace, [10] obtained a 2×2 formula involving trace, anti-trace, and determinant. To the best of our knowledge, explicit closed forms for $Tr_a(A^n)$ beyond 2×2 have not been written down in a way that is both exact and ring-valid.

To clarify the contribution of this work relative to the existing literature, we emphasize that earlier studies on traces or anti-traces of matrix powers often rely on eigenvalue-based methods or recursive computational techniques over the real or complex numbers. In contrast, the present paper provides genuine explicit closed-form formulas with explicitly described multinomial coefficients, derived via a generating-function framework combined with the Cayley–Hamilton theorem. The approach avoids spectral arguments entirely and remains valid over arbitrary commutative rings.

Organization. Section 2 recalls basic notions (trace/anti-trace and generating functions). Section 3 develops the generating-function method and presents the closed forms. The same section records the block–anti–diagonal consequences.

2. PRELIMINARIES

We assume familiarity with standard linear algebra. This section records the notation and basic tools used later. Throughout, R denotes a commutative ring with identity, and $M_n(R)$ the ring of $n \times n$ matrices over R. For readability we sometimes specialize to $R = \mathbb{R}$ in examples, but every algebraic identity below holds over R.

2.1. Trace and anti-trace.

For $A = [a_{ij}] \in M_n(R)$, the *trace* is

$$Tr(A) = \sum_{i=1}^{n} a_{ii}.$$

It is linear and cyclic: $Tr(\alpha A + \beta B) = \alpha\, Tr(A) + \beta\, Tr(B)$, and for any conformable matrices

$$Tr(X_1 X_2 \cdots X_r) = Tr(X_2 \cdots X_r X_1). \tag{1}$$

When R is a field— or $R \subseteq F$ embeds in a field F—then over an algebraic closure of F, $Tr(A)$ equals the sum of the eigenvalues of A counted with algebraic multiplicity; see [11].

The *anti-trace* is the sum of the anti-diagonal entries,

$$Tr_a(A) = a_{1n} + a_{2,n-1} + \cdots + a_{n1}.$$

Let $J \in M_n(R)$ be the exchange (reversal) permutation matrix with ones on the anti-diagonal and zeros elsewhere. Then $J^2 = I$ and $J^\top = J$, and

$$Tr_a(A) = Tr(JA) = Tr(AJ). \tag{2}$$

For $n = 3$, $J = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$, so $Tr_a(A) = a_{13} + a_{22} + a_{31}$, consistent with [10].

2.2. **Generating functions and multinomials.**

Given a univariate sequence $(f_n)_{n\geq 0}$, its ordinary generating function (OGF) is $F(x) = \sum_{n\geq 0} f_n x^n$. For a three-term linear recurrence $f_{n+3} = af_{n+2} + bf_{n+1} + cf_n$ with fixed a, b, c, the OGF satisfies

$$(1 - ax - bx^2 - cx^3)\, F(x) = \text{a polynomial determined by } f_0, f_1, f_2.$$

We will also use trivariate OGFs of the form

$$F(x, y, z) = \sum_{i,j,k\geq 0} \mu_{i,j,k}\, x^i y^j z^k,$$

together with the multinomial coefficients

$$\binom{r}{i, j, k} = \frac{r!}{i!\, j!\, k!}, \qquad i, j, k \geq 0,\ i + j + k = r.$$

All generating functions are formal power series (no convergence issues). These tools let us encode and solve the coefficient recurrences appearing later.

3. MAIN RESULTS

The goal of this section is to obtain an explicit closed formula for the anti–trace of powers of 3×3 matrices over R via generating functions, and to record structural consequences for block–anti–diagonal matrices.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in M_3(R),$$

and let A_{jk} denote the submatrix obtained by deleting the jth row and kth column. Write S_k for the sum of the $k \times k$ principal minors:

$$S_1 = a_{11}+a_{22}+a_{33} = Tr(A), \qquad S_2 = \det(A_{11})+\det(A_{22})+\det(A_{33}), \qquad S_3 = \det(A).$$

Likewise, define the *anti–principal minors*

$$Q_1 = a_{31}+a_{22}+a_{13} = Tr_a(A), \qquad Q_2 = \det(A_{31})+\det(A_{22})+\det(A_{13}), \qquad Q_3 = \det(A).$$

Note that $S_3 = Q_3$. Now, by the characteristic equation of A,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0,$$

and by Cayley–Hamilton,

$$A^3 - S_1A^2 + S_2A - S_3I = 0.$$

Hence, for all $n \in \mathbb{N}$,

$$A^{n+3} = S_1A^{n+2} - S_2A^{n+1} + S_3A^n. \tag{3}$$

Applying the linear functional Tr_a to (3) yields the third–order recurrence

$$Tr_a(A^{n+3}) = S_1\, Tr_a(A^{n+2}) - S_2\, Tr_a(A^{n+1}) + S_3\, Tr_a(A^n), \qquad n \geq 0. \tag{4}$$

3.1. **Anti–trace formula via generating functions.**

Using (4) we compute the first ten values of $Tr_a(A^n)$:

TABLE 1. $Tr_a(A^n)$ from the recurrence (4)

n	$Tr_a(A^n)$
1	$Q_1(1)$
2	$Q_1(S_1) + Q_2(1) - S_2$
3	$Q_1(S_1^2 - S_2) + Q_2(S_1) - S_2(S_1) + S_3$
4	$Q_1(S_1^3 - 2S_2(S_1) + S_3) + Q_2(S_1^2 - S_2) - S_2(S_1^2) + S_2^2 + S_3(S_1)$
5	$Q_1(S_1^4 - 3S_2(S_1^2) + S_2^2 + 2S_3(S_1)) + Q_2(S_1^3 - 2S_2(S_1) + S_3) - S_2(S_1^3)$ $+2S_2^2(S_1) - 2S_2S_3 + S_3(S_1^2)$
6	$Q_1(S_1^5 - 4S_2(S_1^3) + 3S_2^2(S_1) + 3S_3(S_1^2) - 2S_2S_3) + Q_2(S_1^4 - 3S_2(S_1^2) + S_2^2 + 2S_3(S_1))$ $-S_2(S_1^4) + 3S_2^2(S_1^2) + S_3(S_1^3) - S_2^3 - 4S_2S_3(S_1) + S_3^2$
7	$Q_1(S_1^6 - 5S_2(S_1^4) + 6S_2^2(S_1^2) - S_2^3 + 4S_3(S_1^3) - 6S_2S_3(S_1) + S_3^2)$ $+Q_2(S_1^5 - 4S_2(S_1^3) + 3S_2^2(S_1) + 3S_3(S_1^2) - 2S_2S_3)$ $-S_2(S_1^5) + S_3(S_1^4) + 4S_2^2S_1^3 - 6S_2S_3(S_1^2) - 3S_2^3(S_1) + 2S_3^2(S_1) + 3S_2^2S_3$
8	$Q_1(S_1^7 - 6S_2(S_1^5) + 10S_2^2(S_1^3) + 5S_3(S_1^4) - 12S_2S_3(S_1^2) - 4S_2^3(S_1) + 3S_3^2(S_1) + 3S_2^2S_3)$ $+Q_2(S_1^6 - 5S_2(S_1^4) + 4S_3(S_1^3) + 6S_2^2(S_1^2) - 6S_2S_3(S_1) - S_2^3 + S_3^2)$ $-S_2(S_1^6) + S_3(S_1^5) + 5S_2^2(S_1^4) - 8S_2S_3(S_1^3) - 6S_2^3(S_1^2) + 3S_3^2(S_1^2) + 9S_2^2S_3(S_1) + S_2^4 - 3S_2S_3^2$
9	$Q_1(S_1^8 - 7S_2(S_1^6) + 6S_3(S_1^5) + 15S_2^2(S_1^4) - 20S_2S_3(S_1^3) - 10S_2^3(S_1^2) + 6S_3^2(S_1^2)$ $+12S_2^2S_3(S_1) + S_2^4 - 3S_2S_3^2)$ $+Q_2(S_1^7 - 6S_2(S_1^5) + 5S_3(S_1^4) + 10S_2^2(S_1^3) - 12S_2S_3(S_1^2) - 4S_2^3(S_1) + 3S_3^2(S_1) + 3S_2^2S_3)$ $-S_2(S_1^7) + S_3(S_1^6) + 6S_2^2(S_1^5) - 10S_2S_3(S_1^4) - 10S_2^3(S_1^3) + 4S_3^2(S_1^3)$ $+18S_2^2S_3(S_1^2) + 4S_2^4(S_1) - 9S_2S_3^2(S_1) - 4S_2^3S_3 + S_3^3$
10	$Q_1(S_1^9 - 8S_2(S_1^7) + 7S_3(S_1^6) + 21S_2^2(S_1^5) - 30S_2S_3(S_1^4) - 20S_2^3(S_1^3) + 10S_3^2(S_1^3)$ $+30S_2^2S_3(S_1^2) + 5S_2^4(S_1) - 12S_2S_3^2(S_1) - 4S_2^3S_3 + S_3^3)$ $+Q_2(S_1^8 - 7S_1^6S_2 + 6S_1^5S_3 + 15S_1^4S_2^2 - 20S_2S_3(S_1^3) - 10S_2^3(S_1^2) + 6S_1^2S_3^2 + 12S_1S_2^2S_3 + S_2^4 - 3S_2S_3^2)$ $-S_2(S_1^8) + S_3(S_1^7) + 7S_2^2(S_1^6) - 12S_1^5S_2S_3 - 15S_1^4S_2^3 + 5S_3^2(S_1^4) + 30S_2^2S_3(S_1^3) + 10S_1^2S_2^4$ $-18S_2S_3^2(S_1^2) - 16S_2^3S_3(S_1) + 3S_1S_3^3 - S_2^5 + 6S_2^2S_3^2$

The pattern suggests

$$Tr_a(A^n) = Q_1\left(\sum_{i+2j+3k=n-1} \alpha_{i,j,k} S_1^i S_2^j S_3^k\right) + Q_2\left(\sum_{i+2j+3k=n-2} \beta_{i,j,k} S_1^i S_2^j S_3^k\right)$$
$$+ \sum_{i+2j+3k=n} \gamma_{i,j,k} S_1^i S_2^j S_3^k, \tag{5}$$

with $\alpha_{i,j,k}, \beta_{i,j,k}, \gamma_{i,j,k}$ depending only on (i,j,k).

- **Term** $\alpha_{i,j,k}$
  Using equation (4), the coefficients of $S_1^i S_2^j S_3^k$ on both sides implies

$$\alpha_{i,j,k} = \alpha_{i-1,j,k} - \alpha_{i,j-1,k} + \alpha_{i,j,k-1}$$

where $i, j, k \geq 1$. Define a generating function $F_1(x, y, z)$ as follows

$$F_1(x, y, z) = \sum_{i,j,k \geq 0} \alpha_{i,j,k} x^i y^j z^k$$

Accordingly, we obtain

$$F_1(x, y, z) - xF_1(x, y, z) + yF_1(x, y, z) - zF_1(x, y, z) = x - y + z \tag{6}$$

The above equation is true since for $i+j+k \geq 2$ the function $\alpha_{i,j,k}$ satisfies

$$\alpha_{i,j,k} x^i y^j z^k - \alpha_{i-1,j,k} x^{i-1} y^j z^k x + \alpha_{i,j-1,k} x^i y^{j-1} z^k y - \alpha_{i,j,k-1} x^i y^j z^{k-1} z = 0$$

Therefore, by equation (6) it implies

$$\begin{aligned}
F_1(x, y, z) &= \frac{x - y + z}{1 - x + y - z} \\
&= (x - y + z) \left(1 + \sum_{t=1}^{\infty} (x - y + z)^t \right) \\
&= (x - y + z) \left(1 + \sum_{t=1}^{\infty} \sum_{k_1+k_2+k_3=t} \binom{t}{k_1, k_2, k_3} x^{k_1} (-y)^{k_2} z^{k_3} \right) \\
&= (x - y + z) \left(1 + \sum_{t=1}^{\infty} \sum_{k_1+k_2+k_3=t} (-1)^{k_2} \binom{t}{k_1, k_2, k_3} x^{k_1} y^{k_2} z^{k_3} \right)
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
\alpha_{i,j,k} &= (-1)^j \binom{i+j+k-1}{i-1, j, k} - (-1)^{j-1} \binom{i+j+k-1}{i, j-1, k} + (-1)^j \binom{i+j+k-1}{i, j, k-1} \\
&= (-1)^j \left(\binom{i+j+k-1}{i-1, j, k} + \binom{i+j+k-1}{i, j-1, k} + \binom{i+j+k-1}{i, j, k-1} \right) \\
&= (-1)^j \left(\frac{(i+j+k-1)!}{(i-1)!(j-1)!(k-1)!} \left(\frac{1}{jk} + \frac{1}{ik} + \frac{1}{ij} \right) \right) \\
&= (-1)^j \left(\frac{(i+j+k-1)!}{(i-1)!(j-1)!(k-1)!} \left(\frac{i+j+k}{ijk} \right) \right) \\
&= (-1)^j \frac{(i+j+k)!}{i!j!k!} \\
&= (-1)^j \binom{i+j+k}{i, j, k}
\end{aligned}$$

As a conclusion, the function $\alpha_{i,j,k}$ can be written as

$$\alpha_{i,j,k} = (-1)^j \binom{i+j+k}{i, j, k} \tag{7}$$

- **Term** $\beta_{i,j,k}$
  Without loss of generality, the same technique is applied in order to find

$\beta_{i,j,k}$. It yields a similar expression as $\alpha_{i,j,k}$.

$$\beta_{i,j,k} = (-1)^j \binom{i+j+k}{i,j,k} \tag{8}$$

- **Term** $\gamma_{i,j,k}$
Interestingly, using the same method for finding $\gamma_{i,j,k}$ differs slightly from the other cases. By comparing the coefficient of $S_1^i S_2^j S_3^k$ on both sides and using equation (4), this implies for $i, j, k \geq 1$.

$$\gamma_{i,j,k} = \gamma_{i-1,j,k} - \gamma_{i,j-1,k} + \gamma_{i,j,k-1}$$

Define the following generating function

$$F_2(x,y,z) = \sum_{i,j,k\geq 0} \gamma_{i,j,k} x^i y^j z^k$$

Consequently, it results

$$F_2(x,y,z) - xF_2(x,y,z) + yF_2(x,y,z) - zF_2(x,y,z) = -y + z \tag{9}$$

The above equation is true since for $i+j+k \geq 2$ the function $\gamma_{i,j,k}$ satisfies

$$\gamma_{i,j,k}x^i y^j z^k - \gamma_{i-1,j,k}x^{i-1}y^j z^k x + \gamma_{i,j-1,k}x^i y^{j-1} z^k y - \gamma_{i,j,k-1}x^i y^j z^{k-1} z = 0$$

Therefore, by equation (9) it implies

$$\begin{aligned}
F_2(x,y,z) &= \frac{-y+z}{1-x+y-z} \\
&= (-y+z)\left(1 + \sum_{t=1}^{\infty}(x-y+z)^t\right) \\
&= (-y+z)\left(1 + \sum_{t=1}^{\infty} \sum_{k_1+k_2+k_3=t} \binom{t}{k_1,k_2,k_3} x^{k_1}(-y)^{k_2} z^{k_3}\right) \\
&= (-y+z)\left(1 + \sum_{t=1}^{\infty} \sum_{k_1+k_2+k_3=t} (-1)^{k_2}\binom{t}{k_1,k_2,k_3} x^{k_1} y^{k_2} z^{k_3}\right)
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
\gamma_{i,j,k} &= -(-1)^{j-1}\binom{i+j+k-1}{i,j-1,k} + (-1)^j \binom{i+j+k-1}{i,j,k-1} \\
&= (-1)^j \left(\binom{i+j+k-1}{i,j-1,k} + \binom{i+j+k-1}{i,j,k-1}\right) \\
&= (-1)^j \left(\frac{(i+j+k-1)!}{(i-1)!(j-1)!(k-1)!}\left(\frac{1}{ik} + \frac{1}{ij}\right)\right) \\
&= (-1)^j \left(\frac{(i+j+k-1)!}{(i-1)!(j-1)!(k-1)!}\left(\frac{j+k}{ijk}\right)\right) \\
&= (-1)^j \frac{(i+j+k-1)!}{i!j!k!}(j+k)
\end{aligned}$$

$$= (-1)^j \frac{(j+k)}{(i+j+k)} \binom{i+j+k}{i,j,k}$$

As a conclusion, the function $\gamma_{i,j,k}$ can be written as

$$\gamma_{i,j,k} = (-1)^j \frac{(j+k)}{(i+j+k)} \binom{i+j+k}{i,j,k} \tag{10}$$

For example, if $\gamma_{1,2,1} = 9$, then the coefficient of $S_1 S_2^2 S_3$ is 9 in $Tr_a(A^8)$ and this is true based on Table 1.

To sum up, by substituting equations (7), (8), and (10) into equation (5), hence the explicit formula for anti-trace is obtained and stated in the following theorem.

Theorem 3.1. *For a 3×3 matrix A and positive integer n,*

$$\begin{aligned} Tr_a(A^n) = Q_1 &\left(\sum_{i+2j+3k=n-1} (-1)^j \binom{i+j+k}{i,j,k} S_1^i S_2^j S_3^k \right) \\ &+ Q_2 \left(\sum_{i+2j+3k=n-2} (-1)^j \binom{i+j+k}{i,j,k} S_1^i S_2^j S_3^k \right) \\ &+ \sum_{i+2j+3k=n} (-1)^j \frac{(j+k)}{(i+j+k)} \binom{i+j+k}{i,j,k} S_1^i S_2^j S_3^k. \end{aligned}$$

□

Example 3.2. *Let*

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then

$$S_1 = 6, \quad S_2 = \det(A_{11}) + \det(A_{22}) + \det(A_{33}) = 11, \quad S_3 = \det(A) = 6,$$

and the anti–principal sums are

$$Q_1 = a_{31} + a_{22} + a_{13} = 4, \qquad Q_2 = \det(A_{31}) + \det(A_{22}) + \det(A_{13}) = -1.$$

Since A is upper triangular with $(1,3)$ entry 2, one checks directly that

$$(A^n)_{13} = 3^n - 1, \qquad (A^n)_{22} = 2^n, \qquad (A^n)_{31} = 0,$$

hence

$$Tr_a(A^n) = 2^n + 3^n - 1 \quad \textit{for all } n \geq 1.$$

We now compare with the theorem.

- *For $n = 4$, the table/recurrence gives*

$$\begin{aligned} Tr_a(A^4) &= Q_1(S_1^3 - 2S_2S_1 + S_3) + Q_2(S_1^2 - S_2) - S_2S_1^2 + S_2^2 + S_3S_1 \\ &= 4(216 - 2 \cdot 11 \cdot 6 + 6) + (-1)(36 - 11) - 11 \cdot 36 + 11^2 + 6 \cdot 6 \\ &= 96, \end{aligned}$$

which matches $2^4+3^4-1=16+81-1=96$.

- *For* $n=5$,

$$\begin{aligned}
Tr_a(A^5) &= Q_1(S_1^4-3S_2S_1^2+S_2^2+2S_3S_1)+Q_2(S_1^3-2S_2S_1+S_3)\\
&\quad - S_2S_1^3+2S_2^2S_1-2S_2S_3+S_3S_1^2\\
&= 4(1296-3\cdot 11\cdot 36+121+2\cdot 6\cdot 6)+(-1)(216-2\cdot 11\cdot 6+6)\\
&\quad - 11\cdot 216+2\cdot 121\cdot 6-2\cdot 11\cdot 6+6\cdot 36\\
&= 274,
\end{aligned}$$

which matches $2^5+3^5-1=32+243-1=274$.

3.2. More results on block–anti–diagonal matrices.

We investigate the anti–trace of powers of block–anti–diagonal $3m\times 3m$ matrices with 3×3 blocks.

Theorem 3.3 (Even number of blocks)**.** *Let m be even and*

$$A=\begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & A_m\\ \mathbf{0} & \mathbf{0} & \cdots & A_{m-1} & \mathbf{0}\\ \vdots & \vdots & \iddots & \vdots & \vdots\\ \mathbf{0} & A_2 & \cdots & \mathbf{0} & \mathbf{0}\\ A_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\end{pmatrix}, \qquad A_k\in M_3(R).$$

Then for all $n\geq 1$:

(i) $Tr_a(A^{2n})=0$.

(ii) $Tr_a(A^{2n+1})=\displaystyle\sum_{i=1}^{m} Tr_a\big((A_{m+1-i}A_i)^n\,A_{m+1-i}\big)$.

Proof. A direct block multiplication shows

$$A^2=\operatorname{diag}\big(A_mA_1,\ A_{m-1}A_2,\ \ldots,\ A_1A_m\big).$$

Hence A^{2n} is block–diagonal for every n. When m is even, the global anti–diagonal of the $m\times m$ block grid passes between diagonal blocks,

$$A^{2n}=\begin{pmatrix} (A_mA_1)^n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & (A_{m-1}A_2)^n & \cdots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & (A_2A_{m-1})^n & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & (A_1A_m)^n\end{pmatrix}$$

Then every entry on the (full) anti–diagonal of A^{2n} is zero; thus $Tr_a(A^{2n})=0$.

For odd powers, we have

$$A^{2n+1} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & (A_m A_1)^n A_m \\ \mathbf{0} & \mathbf{0} & \cdots & (A_{m-1}A_2)^n A_{m-1} & \mathbf{0} \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ \mathbf{0} & (A_2 A_{m-1})^n A_2 & \cdots & \mathbf{0} & \mathbf{0} \\ (A_1 A_m)^n A_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

As we can see, A^{2n+1} has nonzero blocks exactly on the block anti–diagonal:

$$(A^{2n+1})_{i,m+1-i} = (A_{m+1-i}A_i)^n \, A_{m+1-i}.$$

Along the full matrix anti–diagonal, one picks out the anti–diagonals of these blocks yield $Tr_a(A^{2n+1}) = \sum_{i=1}^{m} Tr_a\big((A_{m+1-i}A_i)^n \, A_{m+1-i}\big)$ □

Example 3.4 (Even number of blocks)**.** *Take $m = 2$ and*

$$A = \begin{pmatrix} \mathbf{0} & A_2 \\ A_1 & \mathbf{0} \end{pmatrix}, \qquad A_1 = \mathrm{diag}(1, 2, 3), \quad A_2 = \mathrm{diag}(4, 5, 6).$$

Then

$$A^2 = \mathrm{diag}(A_2A_1, \ A_1A_2) = \mathrm{diag}(\mathrm{diag}(4, 10, 18), \ \mathrm{diag}(4, 10, 18)).$$

Because m is even, the global anti–diagonal of A^{2n} passes through off–diagonal blocks only; hence $Tr_a(A^{2n}) = 0$. In particular, $Tr_a(A^2) = 0$.

For odd powers,

$$A^3 = \begin{pmatrix} A_2A_1A_2 & \mathbf{0} \\ \mathbf{0} & A_1A_2A_1 \end{pmatrix} \implies Tr_a(A^3) = Tr_a(A_2A_1A_2) + Tr_a(A_1A_2A_1).$$

Since the anti–trace of a diagonal 3×3 matrix $\mathrm{diag}(d_1, d_2, d_3)$ equals d_2, we get

$$Tr_a(A_2A_1A_2) = 50, \qquad Tr_a(A_1A_2A_1) = 20, \qquad Tr_a(A^3) = 70,$$

exactly as predicted by Theorem 3.3(ii) with $n = 1$.

Theorem 3.5 (Odd number of blocks)**.** *Let m be odd and $c = \frac{m+1}{2}$. With A as above, for all $n \geq 1$:*

(i) $Tr_a(A^{2n}) = Tr_a\big([A_c]^{2n}\big)$.

(ii) $Tr_a(A^{2n+1}) = Tr_a\big([A_c]^{2n+1}\big) \ + \sum_{\substack{i=1 \\ i\neq c}}^{m} Tr_a\big((A_{m+1-i}A_i)^n \, A_{m+1-i}\big)$.

Proof. As before, A^{2n} is obtained via direct block multiplication,

$$A^{2n} = \begin{pmatrix} (A_mA_1)^n & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (A_{m-1}A_2)^n & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \cdots & \cdots & \iddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & [A_c]^{2n} & \vdots & \vdots & \vdots \\ \vdots & \vdots & \iddots & \cdots & \cdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & (A_2A_{m-1})^n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & (A_1A_m)^n \end{pmatrix}$$

When m is odd, the global anti–diagonal meets only the central diagonal block, so $Tr_a(A^{2n})$ reduces to the anti–trace of $[A_c]^{2n}$.

For odd powers, we have

$$A^{2n+1} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & (A_mA_1)^nA_m \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & (A_{m-1}A_2)^nA_{m-1} & \mathbf{0} \\ \vdots & \vdots & \cdots & \cdots & \iddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & [A_c]^{2n+1} & \vdots & \vdots & \vdots \\ \vdots & \vdots & \iddots & \cdots & \cdots & \vdots & \vdots \\ \mathbf{0} & (A_2A_{m-1})^nA_2 & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ (A_1A_m)^nA_1 & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

So, $A^{2n+1} = A^{2n}A$ has nonzero blocks on the block anti–diagonal with

$$(A^{2n+1})_{i,m+1-i} = \begin{cases} (A_{m+1-i}A_i)^n\, A_{m+1-i}, & i \neq c, \\ [A_c]^{2n+1}, & i = c. \end{cases}$$

Taking the sum of the anti–traces of these blocks along the full anti–diagonal gives

$$Tr_a(A^{2n+1}) = Tr_a\big([A_c]^{2n+1}\big) \;+\; \sum_{\substack{i=1 \\ i\neq c}}^{m} Tr_a\big((A_{m+1-i}A_i)^n\, A_{m+1-i}\big)$$

□

Example 3.6 (Odd number of blocks)**.** *Let $m = 3$ with central index $c = 2$ and*

$$A = \begin{pmatrix} \mathbf{0} & \mathbf{0} & A_3 \\ \mathbf{0} & A_2 & \mathbf{0} \\ A_1 & \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad A_1 = \mathrm{diag}(1,2,3),\ A_2 = \mathrm{diag}(4,5,6),\ A_3 = \mathrm{diag}(7,8,9).$$

Then

$$A^2 = \mathrm{diag}(A_3A_1,\ A_2^2,\ A_1A_3)$$

so the global anti–diagonal meets only the middle block. Therefore

$$Tr_a(A^2) = Tr_a(A_2^2) = Tr_a(\mathrm{diag}(16,25,36)) = 25,$$

confirming Theorem 3.5(i) for $n = 1$.
For odd powers,

$$A^3 = \begin{pmatrix} A_3A_1A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_2^3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_1A_3A_1 \end{pmatrix},$$

and hence, using $Tr_a(\operatorname{diag}(d_1, d_2, d_3)) = d_2$,

$$\begin{aligned} Tr_a(A^3) &= Tr_a(A_2^3) \; + \; Tr_a(A_3A_1A_3) \; + \; Tr_a(A_1A_3A_1) \\ &= 125 \; + \; 128 \; + \; 32 = 285, \end{aligned}$$

which matches Theorem 3.5(ii) with $n = 1$.

3.3. **Computational Remarks.**

For a single numerical instance (A, n), computing A^n by exponentiation by squaring and then taking Tr_a is typically faster than evaluating the closed form. A 3×3 matrix power by squaring uses $\Theta(\log n)$ matrix multiplications; each 3×3 multiplication costs a fixed number of scalar operations (e.g., 27 multiplies and 18 additions in the naive scheme), so the overall cost is $\Theta(\log n)$.

By contrast, the closed form

$$\begin{aligned} Tr_a(A^n) = Q_1 \left(\sum_{i+2j+3k=n-1} \alpha_{i,j,k} S_1^i S_2^j S_3^k \right) + Q_2 \left(\sum_{i+2j+3k=n-2} \beta_{i,j,k} S_1^i S_2^j S_3^k \right) \\ + \sum_{i+2j+3k=n} \gamma_{i,j,k} S_1^i S_2^j S_3^k, \end{aligned}$$

has $\Theta(n^2)$ terms. Indeed, the number of triples $(i, j, k) \in \mathbb{N}^3$ with $i + 2j + 3k = n$ is

$$N(n) = \sum_{k=0}^{\lfloor n/3 \rfloor} \left(\left\lfloor \frac{n - 3k}{2} \right\rfloor + 1 \right) = \frac{n^2}{12} + O(n),$$

so a direct evaluation of the multinomial sums scales quadratically in n (after a one-time extraction of S_1, S_2, S_3, Q_1, Q_2 from A). This explains the observed timings: for a single large n with floating-point entries, the direct method is usually faster.

However, the closed form remains computationally useful in exact and modular arithmetic. Over general commutative rings (e.g., $\mathbb{Z}$, $\mathbb{Z}/p^r\mathbb{Z}$, or polynomial rings), the recurrence/closed form permits exact computations and congruence arguments; forming A^n numerically would either be impossible or unreliable. The structural block anti-diagonal identities in Subsection 3.2 also reduce large problems to 3×3 blocks, which a direct power need not reveal.

In summary, for a single floating-point instance (A, n) the direct method is often the fastest ($\Theta(\log n)$), while the closed form (and its induced recurrence) is preferable for symbolic/modular settings, when many values $Tr_a(A^n)$ are required, or when exploiting block structure.

4. CONCLUSION

We developed an exact, generating–function approach for computing the anti–trace $Tr_a(A^n)$ of powers of 3×3 matrices over R, yielding closed formulas whose coefficients are multinomial combinations of principal and anti–principal minors. The key point is that anti–trace admits an algebraic treatment via the Cayley–Hamilton recurrence; no spectral information is required. In the block setting, the resulting identities are driven purely by placement along the block anti–diagonal, and they therefore remain valid over any commutative coefficient ring. Practically, the formulas avoid forming A^n and the block rules let one read off anti–traces from constituent 3×3 blocks. Natural next steps include extending the coefficient description beyond 3×3, studying congruence refinements, and exploring additional structured classes.

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Author Contributions. Jeriko Gormantara: Conceptualization, methodology, writing-original draft, writing-review & editing. Devi Fitri Ferdania: Investigation, validation. Fajar Yuliawan: Formal analysis. Hanni Garminia: Supervision. All authors discussed the results and contributed to the final manuscript.

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REFERENCES

[1] C. Brezinski, P. Fika, and M. Mitrouli, "Estimations of the trace of powers of positive self-adjoint operators by extrapolation of the moments," *Electronic Transactions on Numerical Analysis*, vol. 39, pp. 144–155, 2012. https://etna.ricam.oeaw.ac.at/vol.39.2012/pp144-155.dir/pp144-155.pdf.

[2] H. Avron, "Counting triangles in large graphs using randomized matrix trace estimation," in *Workshop on Large-Scale Data Mining: Theory and Applications (LDMTA)*, pp. 1–8, 2010. https://www.semanticscholar.org/paper/Counting-Triangles-in-Large-Graphs-using-Randomized-Avron/24716ee2bf34934e8eb70a7aca4ffa38b544ca81.

[3] A. V. Zarelua, "On congruences for the traces of powers of some matrices," *Proceedings of the Steklov Institute of Mathematics*, vol. 263, no. 1, pp. 78–98, 2008. https://doi.org/10.1134/S008154380804007X.

[4] B. N. Datta and K. Datta, "An algorithm for computing powers of a Hessenberg matrix and its applications," *Linear Algebra and its Applications*, vol. 14, no. 3, pp. 273–284, 1976. https://doi.org/10.1016/0024-3795(76)90072-0.

[5] M. T. Chu, "Symbolic calculation of the trace of the power of a tridiagonal matrix," *Computing*, vol. 35, no. 3, pp. 257–268, 1985. https://doi.org/10.1007/BF02240193.

[6] J. Gutiérrez-Gutiérrez, "Positive integer powers of certain tridiagonal matrices," *Applied Mathematics and Computation*, vol. 202, no. 1, pp. 133–140, 2008. https://doi.org/10.1016/j.amc.2008.01.022.

[7] J. Gutiérrez-Gutiérrez, "Powers of real persymmetric anti-tridiagonal matrices with constant anti-diagonals," *Applied Mathematics and Computation*, vol. 206, no. 2, pp. 919–924, 2008. https://doi.org/10.1016/j.amc.2008.10.003.

[8] J. Pahade and M. Jha, "Trace of positive integer power of real 2×2 matrices," *Advances in Linear Algebra & Matrix Theory*, vol. 5, no. 4, pp. 150–156, 2015. http://dx.doi.org/10.4236/alamt.2015.54015.

[9] K. Patil, "On the trace of powers of square matrices," *Hacettepe Journal of Mathematics and Statistics*, vol. 50, no. 1, pp. 14–23, 2021. https://doi.org/10.15672/hujms.558995.

[10] A. K. Amir, J. Gormantara, and M. Jusmawati, "A method for calculating the sum of diagonal and anti diagonal of power matrices without explicitly calculating its matrices," *Journal of Physics: Conference Series*, vol. 1277, no. 1, p. 012038, 2019. https://doi.org/10.1088/1742-6596/1277/1/012038.

[11] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge: Cambridge University Press, 2 ed., 2012. https://doi.org/10.1017/CBO9781139020411.