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Modular Irregularity Strength on Some Corona Products of Graphs

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Abstract. We consider a finite graph with vertex set $V(G)$ and edge set $E(G)$. Let G be a graph of order n and f be an edge k-labeling that is a mapping from the set of edged of G to the set of numbers from $1, 2, ..., k$. The labeling f is called modular irregular labeling of the graph G if there exist a bijection $w_f : V(G) \to \mathbb{Z}_n$ defined by $w_f(u) = \sum_{uv \in E(u)} f(uv) \pmod{n}$ where $\mathbb{Z}_n$ is a group of integer modulo n. The modular weight of a vertex $u \in V(G)$ is the value of $w_f(u)$. The modular irregularity strength of G, denoted by $\mathrm{ms}(G)$, is defined as the smallest integer k such that G admits a modular irregular labeling with k as its maximum label. In this research, we focus on the corona product of a circulant graph with a null graph of order p, we have results for the corona product of G and H where G is a d-regular graph containing a perfect matching and H is a graph of order 3 by determining the modular irregularity strength $\mathrm{ms}(G \odot C_3) = n + 1$ and $\mathrm{ms}(G \odot (P_2 \cup P_1)) = \frac{3n}{2}$. Lastly, we find the modular irregularity strength $\mathrm{ms}(G \odot P_5) = \lceil \frac{5n+1}{3} \rceil$.

Key words and Phrases: modular irregular labeling, modular irregularity strength, circulant graph, regular graph, corona product.

1. INTRODUCTION

Graph labeling is an assignment of numbers to vertices or edges (or both) [1]. Many types of graph labeling have been studied, which can be found in a graph labeling survey by Gallian [2]. In this research, we consider a finite graph G that has a set of vertices $V(G)$ and a set of edges $E(G)$. Let G be a graph of order n. A graph G is called a regular graph if every vertex of G has the same degree and an irregular graph if every two distinct vertices of G has different degrees. It is easy to see that having an irregular simple graph is impossible. Motivated by an irregular

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graph, Chartrand et al. [3] defined an edge labeling $f : E(G) \to \{1, 2, \dots, k\}, k \in \mathbb{N}$, such that the weights for every two distinct vertices are different where the weight of vertex $u \in V(G)$ is defined by $w_f(u) = \sum_{uv \in E(u)} f(uv)$, where $E(u)$ is the set of edges in $E(G)$ incident with u as an irregular labeling of G. The irregularity strength of G, denoted by $\mathrm{s}(G)$, is defined as the smallest integer k such that G admits an irregular labeling with k as its maximum label. If there is no possible irregular labeling for G, then the irregularity strength of G is defined to be infinity, that is, $\mathrm{s}(G) = \infty$. One clear example of a graph without irregular labeling is a graph with isolated edges. Chartrand et al. give the lower bound of this invariant for any connected graph of order greater than 3 as follows.

Theorem 1.1. *([3]) Let G be a connected graph of order greater than 3 having n_i vertices of degree i, then*

$$\mathrm{s}(G) \geq \max_{1 \leq i \leq \Delta(G)} \left\{ \frac{n_i + i - 1}{i} \right\} .$$

In 2020, Bača, Muthugurupackiam, Kathiresan, and Ramya introduced modular irregular labeling [4]. Let G be a graph of order n and f is an edge k-labeling, $f : E(G) \to \{1, 2, \dots, k\}, k \in \mathbb{N}$. The labeling f is called modular irregular labeling of the graph G if there exist a bijection $w_f : V(G) \to \mathbb{Z}_n$ defined by $w_f(u) = \sum_{uv \in E(u)} f(uv) \pmod n$ where $\mathbb{Z}_n$ is a group of integer modulo n. The modular weight of a vertex $u \in V(G)$ is the value of $w_f(u)$. The modular irregularity strength of G, denoted by $\mathrm{ms}(G)$, is defined as the smallest integer k such that G admits a modular irregular labeling with k as its maximum label. Similarly to the strength of the irregularity, if G has no modular irregular labeling, the modular irregularity strength of G is defined as infinity, that is, $\mathrm{ms}(G) = \infty$. By the definition of modular irregular labeling, it is easy to see that any modular irregular labeling of G is also an irregular labeling for G. Hence, we have the following.

Theorem 1.2. *([4]) Let G be a graph without isolated edges. Then,* $\mathrm{s}(G) \leq \mathrm{ms}(G)$.

Not all graphs have modular irregular labeling. Characterization such that a graph does not have modular irregular labeling is also shown in [4]. In Theorem 1.3, the characterization is given.

Theorem 1.3. *([4]) If G is a graph of order $n, n \equiv 2 \pmod 4$, then G has no modular irregular labeling, that is,* $\mathrm{ms}(G) = \infty$.

Several classes of graphs have been shown to have modular irregularity labeling. The modular irregularity strength of several classes of graphs, namely cycle, star, triangular path, and gear graphs have been determined in [4]. Then, as an extension of the previous result of the cycle graph, the modular irregularity strength for the corona product of a cycle graph with a null graph has been determined in [5].

Recently, Barack *et al.* [6] determined the modular irregularity strength for the corona product of a regular graph containing a 1-factor with a null graph of odd order or with a path graph of order 3.

Theorem 1.4. *([6]) Let G be a regular graph of order n containing a 1-factor and $\overline{K_p}$ be a null graph of odd order p. Then*

$$\mathrm{ms}(G \odot \overline{K_p}) = pn.$$

Theorem 1.5. *([6]) Let G be a regular graph of order n containing a 1-factor, and P_3 be a path graph of order 3. Then*

$$\mathrm{ms}(G \odot P_3) = n + 1.$$

There are more results on modular irregularity strength of family graphs, such as path graph, cycle graph, star graph, double-star graph, friendship graph, generalized book graph, rose graph, daisy graph, sunflower graph, and firecracker graph [4, 7, 8, 9, 10]. There are also results on the modular irregularity strength of some dense graphs, such as complete graph and complete bipartite graph [11]. For disconnected graph, the modular irregularity strength of some disjoint copies of graphs has been obtained, such as cycle graphs, sun graphs, middle graph of cycle graphs and friendship graphs [12, 13].

Let n be an integer and let $s_i,\ i = 1, 2, \ldots, k$ be a strictly increasing sequence of positive integers, with $s_i \leq \left\lfloor \frac{n}{2} \right\rfloor,\ \forall i = 1, 2, \ldots, k$. The graph G of order n with the vertex set $V(G) = \{0,\ 1, \ldots, n-1\}$ and the edge set $E(G) = \{(u, v), v = u \pm s_i \pmod{n}, u \in V(G), 1 \leq i \leq k\}$ is called the circulant graph, denoted by $C_n(s_1, s_2, \ldots, s_k)$. The circulant graph $C_n(1)$ gives the cycle graph C_n, and the circulant graph $C_n\left(1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right)$ gives the complete graph K_n [14].

As the cycle graph is a particular case of a circulant graph, in this research we extend the previous result from $\mathrm{ms}\left(C_n \odot \overline{K_p}\right)$ to $\mathrm{ms}\left(C_n(1,2) \odot \overline{K_p}\right)$. As a result of Theorem 1.4, Barack, Sugeng, Semaničová-Feňovčíková, and Bača have found $\mathrm{ms}\left(G \odot \overline{K_3}\right)$. We now determine $\mathrm{ms}(G \odot H)$ where G is a regular graph containing a perfect matching G and $H \cong C_3$ or $H \cong P_2 \cup P_1$. These results will complete $\mathrm{ms}(G \odot H)$ for G is a regular graph containing perfect matching G and H is a graph of order 3. We also extend the previous result of $\mathrm{ms}(G \odot P_3)$ with $\mathrm{ms}(G \odot P_5)$.

2. MAIN RESULTS

Theorem 2.1. *Let $C_n(1, 2)$ be a circulant graph of order $n > 4$ and let $\overline{K_p}$ be a null graph of order p. If $(p + 1)n \not\equiv 2 \pmod{4}$, then $\mathrm{ms}\left(C_n(1, 2) \odot \overline{K_p}\right) = pn$.*

Proof. Let

$$V\left(C_n(1, 2) \odot \overline{K_p}\right) = \{a_i, b_{i,j} : 1 \leq i \leq n; 1 \leq j \leq p\}$$
$$E\left(C_n(1, 2) \odot \overline{K_p}\right) = \{a_i a_{i+1}, a_i a_{i+2}, a_i b_{i,j} : 1 \leq i \leq n; 1 \leq j \leq p\}$$

be the vertex set and edge set of $C_n(1, 2) \odot \overline{K_p}$, where a_i are the vertices of the circulant graph $C_n(1, 2)$ and $b_{i,j}$ are the pendant vertices. It is evident that $\left|V\left(C_n(1, 2) \odot \overline{K_p}\right)\right| = (p+1)n$. Therefore, according to Theorem 1.3, if $(p+1)n \equiv 2 \pmod{4}$, $\mathrm{ms}\left(C_n(1, 2) \odot \overline{K_p}\right) = \infty$. Let $a_{-1} = a_{n-1}$, $a_0 = a_n$, $a_{n+1} = a_1$, and

$a_{n+2} = a_2$. For the case where $(p+1)n \not\equiv 2 \pmod 4$, the proof will be divided into 2 cases, for odd p and even p. For odd p, define the edge labeling φ as follows.

$$\varphi(a_i b_{i,j}) = \begin{cases} jn - n + i, & 1 \le i \le n; j \text{ odd}, 1 \le j \le p, \\ jn + 1 - i, & 1 \le i \le n; j \text{ even}, 2 \le j \le p-1. \end{cases}$$

Then, for the labeling of the edge $a_i a_{i+1}$ and $a_i a_{i+2}$, we further divide into 3 subcases.

(1) For $pn - \frac{p-1}{2} \equiv 0 \pmod 4$ or $pn - \frac{p-1}{2} \equiv 2 \pmod 4$,

$$\varphi(a_i a_{i+1}) = \left\lfloor \frac{pn - \frac{p-1}{2}}{4} \right\rfloor, 1 \le i \le n,$$

$$\varphi(a_i a_{i+2}) = \left\lceil \frac{pn - \frac{p-1}{2}}{4} \right\rceil, 1 \le i \le n.$$

(2) For $pn - \frac{p-1}{2} \equiv 1 \pmod 4$,

$$\varphi(a_i a_{i+1}) = \begin{cases} \left\lceil \frac{pn - \frac{p-1}{2}}{4} \right\rceil, & i \text{ odd}, 1 \le i \le n, \\ \left\lfloor \frac{pn - \frac{p-1}{2}}{4} \right\rfloor, & i \text{ even}, 2 \le i \le n, \end{cases}$$

$$\varphi(a_i a_{i+2}) = \left\lfloor \frac{pn - \frac{p-1}{2}}{4} \right\rfloor, \ 1 \le i \le n.$$

(3) For $pn - \frac{p-1}{2} \equiv 3 \pmod 4$,

$$\varphi(a_i a_{i+1}) = \begin{cases} \left\lfloor \frac{pn - \frac{p-1}{2}}{4} \right\rfloor, & i \text{ odd}, 1 \le i \le n, \\ \left\lceil \frac{pn - \frac{p-1}{2}}{4} \right\rceil, & i \text{ even}, 2 \le i \le n, \end{cases}$$

$$\varphi(a_i a_{i+2}) = \left\lceil \frac{pn - \frac{p-1}{2}}{4} \right\rceil, \ 1 \le i \le n.$$

Observe that in each case we obtain that φ is a pn-labeling.

Since the vertex $b_{i,j}$ is a pendant vertex, the weight of $b_{i,j}$ is the same as the edge label that incident with it, that is:

$$w_\varphi(b_{i,j}) = \varphi(a_i b_{i,j}) = \begin{cases} jn - n + i, & 1 \le i \le n; j \text{ odd}, 1 \le j \le p, \\ jn + 1 - i, & 1 \le i \le n; j \text{ even}, 2 \le j \le p-1. \end{cases}$$

The edges that are incident with the vertex a_i in $C_n(1,2) \odot \overline{K_p}$ are the edges that are incident with a_i in $C_n(1,2)$ and the edges $a_i b_{i,j}, j = 1, 2, \ldots, p$. Hence, the modular weight of vertex a_i under the labeling φ is:

$$\begin{aligned} w_\varphi(a_i) &= \varphi(a_i a_{i+1}) + \varphi(a_{i-1} a_i) + \varphi(a_i a_{i+2}) + \varphi(a_{i-2} a_i) + \sum_{j=1}^{p} \varphi(a_i b_{i,j}) \\ &= pn + i + \frac{n(p-1)(p+1)}{2} \equiv pn + i \pmod{(p+1)n}. \end{aligned}$$

We find that the vertex weights are all distinct, and the set of all modular weights of vertex $v \in V\left(C_n(1,2) \odot \overline{K_p}\right)$:

$$\begin{aligned}
\left\{w_{\varphi}(v) : v \in V\left(C_n(1,2) \odot \overline{K_p}\right)\right\} &= \{w_{\varphi}(a_i) : 1 \le i \le n\} \\
&\quad \cup \{w_{\varphi}(b_{i,j}) : 1 \le i \le n, 1 \le j \le p\} \\
&= \{0, pn+1, pn+2, \dots, (p+1)n-1\} \\
&\quad \cup \{1, 2, \dots, pn\} \\
&= \{0, 1, 2, \ \dots, (p+1)n-1\}.
\end{aligned}$$

Hence, φ is a modular irregular labeling for $C_n(1,2) \odot \overline{K_p}$. and we can conclude that:

$$\operatorname{ms}\left(C_n(1,2) \odot \overline{K_p}\right) \le pn. \tag{1}$$

For even p, define an edge labeling ϕ as follows.

$$\phi(a_i b_{i,j}) = \begin{cases} jn - n + i, & 1 \le i \le n; j \text{ odd}, \ 1 \le j \le p-1, \\ jn + 1 - i, & 1 \le i \le n; j \text{ even}, \ 2 \le j \le p. \end{cases}$$

Then, for the labeling of the edge $a_i a_{i+1}$ and $a_i a_{i+2}$, we divide furthermore into 3 subcases.

(1) For $n \equiv 0 \pmod 4$,

$$\phi(a_i a_{i+1}) = \begin{cases} i, & 1 \le i \le \frac{n}{2}, \\ 2\left\lfloor \frac{n+1-i}{2} \right\rfloor + 1, & \frac{n}{2} < i \le n, \end{cases}$$

$$\phi(a_i a_{i+2}) = \begin{cases} pn, & i \equiv 1, 2 \pmod 4, \ 1 \le i \le n, \\ \frac{p(n-1)}{2} - 1, & i \equiv 0, 3 \pmod 4, \ 1 \le i \le n. \end{cases}$$

Observe that for this subcase we obtain that ϕ is a pn-labeling.

Since the vertex $b_{i,j}$ is a pendant vertex, then the weight of $b_{i,j}$ is the same as the label of the edge that incident with it, that is:

$$w_{\phi}(b_{i,j}) = \phi(a_i b_{i,j}) = \begin{cases} jn - n + i, & 1 \le i \le n; j \text{ odd}, \ 1 \le j \le p-1, \\ jn + 1 - i, & 1 \le i \le n; j \text{ even}, \ 2 \le j \le p. \end{cases}$$

The edges that incident with the vertex a_i in $C_n(1,2) \odot \overline{K_p}$ are the edges that are incident with a_i in $C_n(1,2)$ and the edges $a_i b_{i,j}, \ j = 1, 2, \dots, p$. Hence the modular weight of vertex a_i under the labeling ϕ is:

$$w_{\phi}(a_i) = \phi(a_i a_{i+1}) + \phi(a_{i-1} a_i) + \phi(a_i a_{i+2}) + \phi(a_{i-2} a_i) + \sum_{j=1}^{p} \phi(a_i b_{i,j})$$

$$w_{\phi}(a_i) = \begin{cases} pn + 1, & i = 1, \\ pn - 2 + 2i, & 2 \le i \le \frac{n}{2} + 1, \\ (p+1)n - 1, & i = \frac{n}{2} + 2, \\ pn + 1 + 2(n - 1 + i), & \frac{n}{2} + 3 \le i \le n. \end{cases}$$

We have that the vertex weights are all distinct, and the set of all modular weights of $v \in V\left(C_n(1,2) \odot \overline{K_p}\right)$:

$$\begin{aligned}
\left\{w_\phi\left(v\right) : v \in V\left(C_n(1,2) \odot \overline{K_p}\right)\right\} &= \{w_\phi(a_i) : 1 \le i \le n\} \\
&\quad \cup \{w_\phi(b_{i,j}) : 1 \le i \le n, 1 \le j \le p\} \\
&= \{0, pn+1, pn+2, \ldots, (p+1)n-1\} \\
&\quad \cup \{1, 2, \ldots, pn\} \\
&= \{0, 1, 2, \ \ldots, (p+1)n-1\}.
\end{aligned}$$

Hence, for this subcase, ϕ is a modular irregular labeling for $C_n(1,2) \odot \overline{K_p}$, and we can conclude that:

$$\operatorname{ms}\left(C_n(1,2) \odot \overline{K_p}\right) \le pn. \tag{2}$$

(2) For $n \equiv 1 \pmod 4$,

$$\phi(a_i a_{i+1}) = \begin{cases} \frac{1}{4}\left((p+2)n - (p-2)\right) + (i-1), & 1 \le i \le \frac{n+1}{2}, \\ \frac{1}{4}\left((p+2)n - (p-2)\right) + 2\left\lfloor \frac{n+1-i}{2} \right\rfloor, & \frac{n+1}{2} < i \le n, \end{cases}$$

$\phi(a_i a_{i+2}) = pn, \ 1 \le i \le n.$

Observe that for this subcase, we obtain that ϕ is a pn-labeling.

Since the vertex $b_{i,j}$ is a pendant vertex, then the weight of $b_{i,j}$ is the same as the label of the edge that incident with it, that is:

$$w_\phi(b_{i,j}) = \phi(a_i b_{i,j}) = \begin{cases} jn - n + i, & 1 \le i \le n; j \text{ odd}, \ 1 \le j \le p-1, \\ jn + 1 - i, & 1 \le i \le n; j \text{ even}, \ 2 \le j \le p. \end{cases}$$

The edges that incident with the vertex a_i in $C_n(1,2) \odot \overline{K_p}$ are the edges that are incident with a_i in $C_n(1,2)$ and the edges $a_i b_{i,j}, \ j = 1, 2, \ldots, p$. Hence, the modular weight of vertex a_i under the labeling ϕ is:

$$w_\phi(a_i) = \phi(a_i a_{i+1}) + \phi(a_{i-1} a_i) + \phi(a_i a_{i+2}) + \phi(a_{i-2} a_i) + \sum_{j=1}^{p} \phi(a_i b_{i,j})$$

$$w_\phi(a_i) = \begin{cases} pn + 1, & i = 1, \\ pn - 2 + 2i, & 2 \le i \le \frac{n+1}{2}, \\ \frac{p}{2}(p+1)n + 2(p+1)n, & i = \frac{n+1}{2} + 1, \\ pn + 2n + 3 - 2i \ , & \frac{n+1}{2} + 2 \le i \le n. \end{cases}$$

We have the vertex weights are all distinct, and the set of all modular weights of $v \in V\left(C_n(1,2) \odot \overline{K_p}\right)$:

$$\begin{aligned}
\left\{w_\phi\left(v\right) : v \in V\left(C_n(1,2) \odot \overline{K_p}\right)\right\} &= \{w_\phi(a_i) : 1 \le i \le n\} \\
&\quad \cup \{w_\phi(b_{i,j}) : 1 \le i \le n, 1 \le j \le p\} \\
&= \{0, pn+1, pn+2, \ldots, (p+1)n-1\} \\
&\quad \cup \{1, 2, \ldots, pn\} \\
&= \{0, 1, 2, \ldots, (p+1)n-1\}.
\end{aligned}$$

Hence, for this subcase, ϕ is a modular irregular labeling for $C_n(1,2) \odot \overline{K_p}$, and we can conclude that:

$$\text{ms}\left(C_n(1,2) \odot \overline{K_p}\right) \leq pn. \tag{3}$$

(3) For $n \equiv 3 \pmod 4$,

$$\phi(a_i a_{i+1}) = \begin{cases} i, & 1 \leq i \leq \frac{n+1}{2}, \\ 2\left\lceil \frac{n+1-i}{2} \right\rceil + 1, & \frac{n+1}{2} < i \leq n, \end{cases}$$

$$\phi(a_i a_{i+2}) = \frac{p}{2}\left(\frac{3\,(n-1)}{2} + 1\right) - 1, \ 1 \leq i \leq n.$$

Observe that for this subcase, we find that ϕ is a pn-labeling.

Since the vertex $b_{i,j}$ is a pendant vertex, the weight of $b_{i,j}$ is the same as the label of the edge that incident with it, that is:

$$w_\phi(b_{i,j}) = \phi(a_i b_{i,j}) = \begin{cases} jn - n + i, & 1 \leq i \leq n; j \text{ odd}, \ 1 \leq j \leq p-1, \\ jn + 1 - i, & 1 \leq i \leq n; j \text{ even}, \ 2 \leq j \leq p. \end{cases}$$

The edges that are incident with the vertex a_i in $C_n(1,2) \odot \overline{K_p}$ are the edges that are incident with a_i in $C_n(1,2)$ and the edges $a_i b_{i,j}$, $j = 1, 2, \ldots, p$. Hence, the modular weight of vertex a_i under the labeling ϕ is:

$$w_\phi(a_i) = \phi(a_i a_{i+1}) + \phi(a_{i-1} a_i) + \phi(a_i a_{i+2}) + \phi(a_{i-2} a_i) + \sum_{j=1}^{p} \phi(a_i b_{i,j})$$

$$w_\phi(a_i) = \begin{cases} pn + 2, & i = 1, \\ pn - 3 + 2i, & 2 \leq i \leq \frac{n+1}{2}, \\ \frac{p}{2}(p+1)n + (p+1)n, & \frac{n+1}{2} + 1, \\ pn + 2n + 4 - 2i\ , & \frac{n+1}{2} + 2 \leq i \leq n. \end{cases}$$

We have the vertex weights are all distinct, and the set of all modular weights of $v \in V\left(C_n(1,2) \odot \overline{K_p}\right)$:

$$\begin{aligned} \left\{w_\phi(v) : v \in V\left(C_n(1,2) \odot \overline{K_p}\right)\right\} &= \{w_\phi(a_i) : 1 \leq i \leq n\} \\ &\quad \cup \{w_\phi(b_{i,j}) : 1 \leq i \leq n, 1 \leq j \leq p\} \\ &= \{0, pn+1, pn+2, \ldots, (p+1)n - 1\} \\ &\quad \cup \{1, 2, \ldots, pn\} \\ &= \{0, 1, 2, \ \ldots, (p+1)n - 1\}. \end{aligned}$$

Hence, for this subcase, ϕ is a modular irregular labeling for $C_n(1,2) \odot \overline{K_p}$, and we can conclude that:

$$\text{ms}\left(C_n(1,2) \odot \overline{K_p}\right) \leq pn. \tag{4}$$

The modular irregularity strength of $C_n(1,2) \odot \overline{K_p}$ is at least pn because $C_n(1,2) \odot \overline{K_p}$ has pn pendant vertices, and each pendant vertex must have a different weight, that is:

$$\text{ms}\left(C_n(1,2) \odot \overline{K_p}\right) \geq pn. \tag{5}$$

From (1),(2),(3),(4),and (5), we can conclude that:

$$\operatorname{ms}\left(C_n(1,2) \odot \overline{K_p}\right) = pn.$$

□

As the circulant graph $C_n(1,2)$ is also a regular graph containing a perfect matching whenever n is even, for the case where n is even and p is odd, it is a corollary of Theorem 1.4. The example of Theorem 2.1 for modular irregular labeling in some corona product of circulant graph and complement of a complete graph is shown in Figure 1 below.

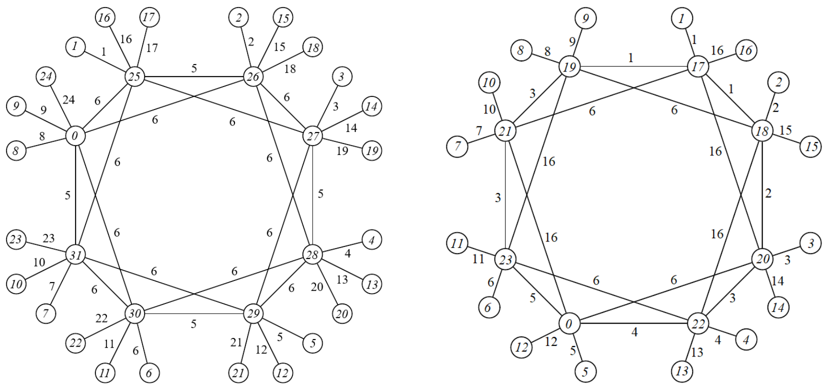

FIGURE 1. The modular irregular labeling of $C_8(1,2) \odot \overline{K_3}$ with $\operatorname{ms}\left(C_8(1,2) \odot \overline{K_3}\right) = 24$ and the modular irregular labeling of $C_8(1,2) \odot \overline{K_2}$ with $\operatorname{ms}\left(C_8(1,2) \odot \overline{K_2}\right) = 16$.

Theorem 2.2. *Let G be a d-regular graph of order n with $d > 1$ containing a perfect matching and C_3 be a cycle graph of order 3, then*

$$\operatorname{ms}(G \odot C_3) = n + 1.$$

Proof. Let G be a d-regular graph of order $n, d > 1$ containing a perfect matching $M(G)$. Let $a_i,\ i = 1, 2, \dots, n$ be the vertices of G and $b_{i,j}, i = 1, 2, \dots, n,\ j = 1, 2,\ 3$ be the vertices of an i-th copy of C_3, adjacent to a_i. For simplicity, let $b_{i,4} = b_{i,1}$.

We define an edge labeling τ as follows.

$$\tau(b_{i,j}b_{i,j+1}) = \begin{cases} 1, & 1 \le i \le n,\ j = 1, \\ n+1, & 1 \le i \le n,\ j = 2, \\ i, & 1 \le i \le n,\ j = 3, \end{cases}$$

$$\tau(a_i b_{i,j}) = \begin{cases} 1, & 1 \le i \le n,\ j = 1, \\ n+1-i, & 1 \le i \le n,\ j = 2, \\ n+1, & 1 \le i \le n,\ j = 3, \end{cases}$$

$$\tau(e) = \begin{cases} \left\lfloor \frac{2n}{d} \right\rfloor, & e \in E(G) \setminus M(G), \\ 2n - (d-1)\left\lfloor \frac{2n}{d} \right\rfloor, & e \in M(G). \end{cases}$$

Observe that the maximum value of the labeling τ is $n+1$. Hence, we see that τ is a $(n+1)$-labeling.

The weight of the vertex $b_{i,j},\ 1 \leq i \leq n, 1 \leq j \leq 3$ can be calculated as follows

$$\begin{aligned} w_\tau(b_{i,1}) &= \tau(a_i b_{i,1}) + \tau(b_{i,1} b_{i,2}) + \tau(b_{i,3} b_{i,1}) = 1 + 1 + i = i + 2, \\ w_\tau(b_{i,2}) &= \tau(a_i b_{i,2}) + \tau(b_{i,2} b_{i,3}) + \tau(b_{i,1} b_{i,2}) \\ &= (n + 1 - i) + (n+1) + 1 = 2n + 3 - i, \\ w_\tau(b_{i,3}) &= \tau(a_i b_{i,3}) + \tau(b_{i,3} b_{i,1}) + \tau(b_{i,2} b_{i,3}) \\ &= (n+1) + i + (n+1) = 2n + 2 + i, \end{aligned}$$

Hence, we get the set of modular weights of vertex $b_{i,j},\ 1 \leq i \leq n, 1 \leq j \leq 3$, which is

$$\{w_\tau(b_{i,j}) :\ 1 \leq i \leq n, 1 \leq j \leq 3\} = \{3,\ 4,\ 5, \ldots, 3n+2\}. \tag{6}$$

The edges that are incident with the vertex a_i in $G \odot C_3$ are the edges that are incident with a_i in G and the edges $a_i b_{i,j},\ 1 \leq j \leq 3$.. Therefore, the modular weight of the vertex a_i under the labeling τ is:

$$w_\tau(a_i) = \sum_{j=1}^{3} \tau(a_i b_{i,j}) + \sum_{e \in E_G(a_i)} \tau(e).$$

We get

$$\begin{aligned} w_\tau(a_i) &= 1 + (n+1-i) + (n+1) + (d-1)\left\lfloor \frac{2n}{d} \right\rfloor + 2n - (d-1)\left\lfloor \frac{2n}{d} \right\rfloor \\ &= 4n + 3 - i \end{aligned}$$

And the set of modular weight of the vertex $a_i, 1 \leq i \leq n$, is

$$\begin{aligned} \{w_\tau(a_i) :\ 1 \leq i \leq n\} &= \{4n+2,\ 4n+1,\ 4n,\ \ldots,\ 3n+3\} \\ &= \{0, 1,\ 2,\ 3n+3, 3n+4, \ldots, 4n-1\}. \end{aligned} \tag{7}$$

From (6) and (7), we have the set of all modular weights of $v \in V(G \odot C_3)$:

$$\begin{aligned} \{w_\tau(v) : v \in V(G \odot C_3)\} =&\ \{w_\tau(a_i) : 1 \leq i \leq n\} \cup \{w_\tau(b_{i,j}) : 1 \leq i \leq n, 1 \leq j \leq 3\} \\ &= \{0, 1, 2, 3n+3, 3n+4, \ldots, 4n-1\} \cup \{3, 4, 5, \ldots, 3n+2\} \\ &= \{0, 1, 2, \ldots, 4n-1\}. \end{aligned}$$

Therefore, τ is a modular irregular labeling of $G \odot C_3$, and we can conclude that

$$\mathrm{ms}(G \odot C_3) \leq n + 1. \tag{8}$$

Using Theorem 1.1 and Theorem 1.2 gives us

$$\mathrm{ms}(G \odot C_3) \geq \mathrm{s}(G \odot C_3) \geq \max\left\{\frac{3n-1}{3} + 1, \frac{n-1}{r+3} + 1\right\}$$

$$\mathrm{ms}(G \odot C_3) \geq n + \frac{2}{3}.$$

Since modular irregularity strength of a graph must be an integer, we have

$$\mathrm{ms}(G \odot C_3) \geq n + 1. \tag{9}$$

From (8) and (9), we can conclude that

$$\mathrm{ms}(G \odot C_3) = n + 1.$$

□

Theorem 2.3. *Let G be a regular graph of order n containing a perfect matching and $P_2 \cup P_1$ be a disjoint union of path graph of order 2 with path graph of order 1. Then*

$$\mathrm{ms}\left(G \odot (P_2 \cup P_1\ )\right) = \frac{3n}{2}.$$

Proof. Let G be a d-regular graph of order n containing a perfect matching $M(G)$. Let $a_i, i = 1, 2, \ldots, n$ be the vertices of G, $b_i, i = 1, 2, \ldots, n$ be the vertex of an i-th copy of P_1 adjacent to a_i, and $c_{i,j}, i = 1, 2, \ldots, n, j = 1, 2$ be the vertices of an i-th copy of P_2 adjacent to an a_i. For simplicity, let $H = P_2 \cup P_1$.

We define an edge labeling λ as follows.

$$\lambda(c_{i,1}c_{i,2}) = \frac{n}{2} + i,\ 1 \leq i \leq n,$$

$$\lambda(a_i c_{i,j}) = \begin{cases} \frac{n}{2}, & 1 \leq i \leq n, j = 1, \\ \frac{3n}{2}, & 1 \leq i \leq n, j = 2, \end{cases}$$

$$\lambda(a_i b_i) = i,\ 1 \leq i \leq n,$$

$$\lambda(e) = \begin{cases} \left\lfloor \frac{n}{d} \right\rfloor, & e \in E(G) \setminus M(G), \\ n - (d-1)\left\lfloor \frac{n}{d} \right\rfloor, & e \in M(G). \end{cases}$$

It is clear that the maximum value of the labeling λ is $\frac{3n}{2}$, so λ is a $\frac{3n}{2}$-labeling. The edge incident with the vertex $b_i, i = 1, 2, \ldots, n$ is only the edge $a_i b_i$, hence the weight of the vertex $b_i, i = 1, 2, \ldots, n$ is

$$w_\lambda\,(b_i) = \lambda(a_i b_i) = i,\ 1 \leq i \leq n,$$

and the set of modular weight in modulo $4n$ is:

$$\{w_\lambda(b_i) : 1 \leq i \leq n\} = \{1, 2, \ldots, n\}.$$

The edges incident with the vertex $c_{i,j},\ i = 1, 2, \ldots, n, j = 1, 2$ are the edge $a_i c_{i,j}$ and $c_{i,1}c_{i,2}$, therefore the weight of the vertex $c_{i,j}, i = 1, 2, \ldots, n, j = 1, 2$ is

$$\begin{aligned} w_\lambda(c_{i,1}) = &\quad \lambda(a_i c_{i,1}) + \lambda(c_{i,1}c_{i,2}) = \tfrac{n}{2} + \tfrac{n}{2} + i = n + i, \quad 1 \leq i \leq n, \\ w_\lambda(c_{i,2}) = &\quad \lambda(a_i c_{i,2}) + \lambda(c_{i,1}c_{i,2}) = \tfrac{3n}{2} + \tfrac{n}{2} + i = 2n + i, \quad 1 \leq i \leq n. \end{aligned}$$

Hence, the set of modular weight in modulo $4n$ is:

$$\{w_\lambda(c_{i,j}) : 1 \leq i \leq n, 1 \leq j \leq 2\} = \{n + 1, n + 2, \ldots, 3n\}.$$

Lastly, the edges that incident with the vertex $a_i, i = 1, 2, \ldots, n$ are the edges $a_i c_{i,j}, j = 1, 2$, the edge $a_i b_i$, and the edges that incident with a_i in G, hence the weight of the vertex $a_i, i = 1, 2, \ldots, n$ is

$$\begin{aligned} w_\lambda(a_i) &= \lambda(a_i c_{i,1}) + \lambda(a_i c_{i,2}) + \lambda(a_i b_i) + \sum_{e \in E_G(a_i)} \lambda(e) \\ &= \frac{n}{2} + \frac{3n}{2} + i + (d-1)\left\lfloor \frac{n}{d} \right\rfloor + n - (d-1)\left\lfloor \frac{n}{d} \right\rfloor = 3n + i \end{aligned}$$

Hence, the set of modular weight in modulo $4n$ is:

$$\{w_\lambda(a_i) : 1 \leq i \leq n\} = \{0, 3n+1, 3n+2, \ldots, 4n-1\}.$$

So, we have that the set of modular weights of all the vertices of $G \odot H$ is

$$\begin{aligned} \{w_\lambda(v) : v \in G \odot H\} &= \{w_\lambda(a_i) : 1 \leq i \leq n\} \cup \{w_\lambda(b_i) : 1 \leq i \leq n\} \\ &\quad \cup \{w_\lambda(c_{i,j}) : 1 \leq i \leq n,\ 1 \leq j \leq 2\} \\ &= \{0, 3n+1, 3n+2, \ldots, 4n-1\} \cup \{1, 2, \ldots, n\} \\ &\quad \cup \{n+1, n+2, \ldots, 3n\} \\ &= \{0, 1, 2, 3, \ldots, 4n-1\} \end{aligned}$$

Therefore, λ is a modular irregular labeling of $G \odot H$, and we can conclude that

$$\mathrm{ms}(G \odot H) \leq \frac{3n}{2}. \tag{10}$$

To minimize the greatest label used, the greater weights must be in the vertices with the greatest degree. Because $G \odot H$ has 1 as its minimum degree, the weight 0 must be obtained at least by $4n$. Suppose that the weights of $3n+1$ up to $4n$ are located in the vertices of G, because there are n vertices with $deg(v) = \Delta(G \odot H)$. Then, the weight of the vertices from H must consist of 1 to $3n$. Let $\mathrm{ms}(G \odot H) = s$. Since we want to minimize s, the weight $n+1$ to $3n$ must be located in the vertices of degree 2, so we get

$$\begin{aligned} 2s &\geq 3n \\ s &\geq \frac{3n}{2}. \end{aligned}$$

Hence, we get the lower-bound

$$\mathrm{ms}(G \odot H) \geq \frac{3n}{2}. \tag{11}$$

By (10) and (11), we can conclude that

$$\mathrm{ms}(G \odot H) = \mathrm{ms}(G \odot (P_2 \cup P_1)) = \frac{3n}{2}.$$

□

Theorem 2.4. *Let G be a regular graph of order n containing perfect matching, and P_5 be a path graph of order 5. Then*

$$\mathrm{ms}(G \odot P_5) = \left\lceil \frac{5n+1}{3} \right\rceil.$$

Proof. Let G be a d-regular graph of order n that contains a perfect matching $M(G)$. Let $a_i, i = 1, 2, \ldots, n$ be the vertices of G and $b_{i,j}$ be the vertices of an i-th copy of P_5, adjacent to an a_i.

We define an edge labeling ϕ as follows.

$$\phi(b_{i,j}b_{i,j+1}) = \begin{cases} 1, & 1 \le i \le n, j = 1, \\ \left\lfloor \frac{5n+1}{3} \right\rfloor, & n \equiv 0 \pmod 6, 1 \le i \le n, j = 2, 3, \\ \left\lceil \frac{5n+1}{3} \right\rceil, & n \equiv 2, 4 \pmod 6, 1 \le i \le n, j = 2, 3, \\ n+1, & 1 \le i \le n, j = 4, \end{cases}$$

$$\phi(a_i b_{i,j}) = \begin{cases} i, & 1 \le i \le n, j = 1, \\ \left\lfloor \frac{n}{3} \right\rfloor + i, & 1 \le i \le n, j = 2, \\ \left\lceil \frac{2n+1}{3} \right\rceil + i, & n \equiv 0, 4 \pmod 6, 1 \le i \le n, j = 3, \\ \left\lfloor \frac{2n+1}{3} \right\rfloor + i & n \equiv 2 \pmod 6, 1 \le i \le n, j = 3, \\ \left\lfloor \frac{n}{3} \right\rfloor + n + 1 - i, & 1 \le i \le n, j = 4, \\ n + 1 - i & 1 \le i \le n, j = 5, \end{cases}$$

$$\phi(e) = \begin{cases} \left\lfloor \frac{\frac{5n}{3} - 2}{d} \right\rfloor, & n \equiv 0 \pmod 6, e \in E(G) \setminus M(G), \\ \frac{5n}{3} - 2 - (d-1) \left\lfloor \frac{\frac{5n}{3} - 2}{d} \right\rfloor, & n \equiv 0 \pmod 6, e \in M(G), \\ \left\lfloor \frac{5n+2}{3d} \right\rfloor, & n \equiv 2 \pmod 6, e \in E(G) \setminus M(G), \\ \frac{5n+2}{3} - (d-1) \left\lfloor \frac{5n+2}{3d} \right\rfloor, & n \equiv 2 \pmod 6, e \in M(G), \\ \left\lfloor \frac{5n-2}{3d} \right\rfloor, & n \equiv 4 \pmod 6, e \in E(G) \setminus M(G), \\ \frac{5n-2}{3} - (d-1) \left\lfloor \frac{5n-2}{3d} \right\rfloor, & n \equiv 4 \pmod 6, e \in M(G). \end{cases}$$

Observe that the maximum value of the labeling ϕ is $\left\lceil \frac{5n+1}{3} \right\rceil$. Hence, we obtain that ϕ is a $\left\lceil \frac{5n+1}{3} \right\rceil$-labeling.

The weight of vertex $b_{i,j}$, $1 \le i \le n, 1 \le j \le 5$ can be calculated as follows For $j = 1$.

$$w_\phi(b_{i,1}) = \phi(a_i b_{i,1}) + \phi(b_{i,1} b_{i,2}).$$

For $j = 2, 3,$ or 4.

$$w_\phi(b_{i,j}) = \phi(a_i b_{i,j}) + \phi(b_{i,j} b_{i,j+1}) + \phi(b_{i,j-1} b_{i,j}).$$

For $j = 5$.

$$w_\phi(b_{i,5}) = \phi(a_i b_{i,5}) + \phi(b_{i,4} b_{i,5}).$$

Hence, we get

$$w_\phi(b_{i,j}) = \begin{cases} i + 1, & 1 \le i \le n, j = 1, \\ 2n + 1 + i, & 1 \le i \le n, j = 2, \\ 4n + 1 + i, & 1 \le i \le n, j = 3, \\ 4n + 2 - i, & 1 \le i \le n, j = 4, \\ 2n + 2 - i, & 1 \le i \le n, j = 5. \end{cases}$$

and the set of modular weight of vertex $b_{i,j}, 1 \le i \le n, 1 \le j \le 5$, is

$$\{w_\phi(b_{i,j}) : 1 \le i \le n, 1 \le j \le 5\} = \{2, 3, 4, \ldots, 5n + 1\}.$$

The edges that are incident with the vertex a_i in $G \odot P_5$ are the edges that are incident with a_i in G and the edges $a_ib_{i,j}, 1 \leq j \leq 5.$. Therefore, the modular weight of the vertex a_i under the labeling ϕ is:

$$w_\phi(a_i) = \sum_{j=1}^{5} \phi(a_ib_{i,j}) + \sum_{e \in E_G(a_i)} \phi(e).$$

Hence, we get

$$w_\phi(a_i) = 5n + 1 + i,\ 1 \leq i \leq n.$$

The set of modular weights of the vertex $a_i, 1 \leq i \leq n$, is

$$\begin{aligned}\{w_\phi(a_i) :\ 1 \leq i \leq n\} &= \{5n+2, 5n+3, \ldots, 6n, 6n+1\} \\ &= \{0, 1, 5n+2, 5n+3, \ldots, 6n-1\}.\end{aligned}$$

We have the set of all the modular weights of $v \in V(G \odot P_5)$:

$$\begin{aligned}\{w_\phi(v) : v \in V(G \odot P_5)\} &= \{w_\phi(a_i) :\ 1 \leq i \leq n\} \\ &\quad \cup \{w_\phi(b_{i,j}) :\ 1 \leq i \leq n, 1 \leq j \leq 5\} \\ &= \{0, 1, 5n+2, 5n+3, \ldots, 6n-1\} \cup \{2, 3, \ldots, 5n+1\} \\ &= \{0, 1, 2, \ldots, 6n-1\}.\end{aligned}$$

Therefore, ϕ is a modular irregular labeling of $G \odot P_5$, and we can conclude that

$$\mathrm{ms}(G \odot P_5) \leq \left\lceil \frac{5n+1}{3} \right\rceil. \tag{12}$$

Using Theorem 1.1 and Theorem 1.3 gives us

$$\mathrm{ms}(G \odot P_5) \geq \mathrm{s}(G \odot P_5) \geq \max\left\{\frac{2n-1}{2} + 1, \frac{3n-1}{3} + 1, \frac{n-1}{r+5} + 1\right\}$$

$$\mathrm{ms}(G \odot P_5) \geq n + \frac{2}{3}.$$

Since $\mathrm{ms}(G \odot P_5)$ must be an integer, we have $\mathrm{ms}(G \odot P_5) \geq n + 1$.

Suppose that there exists an $n+1$-modular irregular labeling ϕ of $G \odot P_5$. The weight of vertices of degree 2 or 3 is in the range $2 \leq w_\phi(v) \leq 2n+2$ or $3 \leq w_\phi(v) \leq 3n$, respectively. Observe that only $3n+2$ weights are available for these vertices, but the number of vertices of degree 2 or 3 is $5n$. So, there must be some vertices that have the same weight. Therefore, $\mathrm{ms}(G \odot P_5) > n + 1$.

To increase the lower bound, we use a different approach. We can safely assume that the highest weight must be in the vertices to the greatest extent to minimize the highest label used k. Because $G \odot P_5$ has 2 as its minimum degree, the weights 0 and 1 must be obtained at least by $6n$ and $6n+1$, respectively. Suppose that the weights of $5n+2$ up to $6n+1$ are located in the vertices of G, because there are n vertices with $deg(v) = \Delta(G \odot P_5)$. Then, the vertices of P_5 must "cover" the weights from 2 to $5n+1$. Let $\mathrm{ms}(G \odot P_5) = s$. Since we want to minimize s, we get that

$$3s \geq 5n + 1$$

$$s \geq \frac{5n+1}{3}.$$

Hence, we get the new lower-bound

$$\mathrm{ms}(G \odot P_5) \geq \left\lceil \frac{5n+1}{3} \right\rceil. \tag{13}$$

From (12) and (13) we can conclude that

$$\mathrm{ms}(G \odot P_5) = \left\lceil \frac{5n+1}{3} \right\rceil.$$

□

3. CONCLUDING REMARKS

In this research, we determined the modular irregularity strength for $C_n(1,2) \odot \overline{K_p}$ that is, $\mathrm{ms}\left(C_n(1,2) \odot \overline{K_p}\right) = pn$, which we proved in Theorem 2.1 by constructing the modular irregular labeling for $C_n(1,2) \odot \overline{K_p}$ and show the lower bound for its modular irregularity strength. We completed the previous result on the modular irregularity strength for the corona product $G \odot H$, where G is a d-regular graph containing a perfect matching G and H is a graph of order 3, as follows.

$$\mathrm{ms}(G \odot H) = \begin{cases} 3n, & \text{if } H \cong \overline{K_3}, \\ \frac{3n}{2}, & \text{if } H \cong P_2 \cup P_1, \\ n+1, & \text{if } H \cong P_3 \text{ or } H \cong C_3 \text{ and } d > 1. \end{cases}$$

Lastly, we also determined the modular irregularity strength for the corona product of a regular graph containing a 1-factor G with a path graph of order 5, $\mathrm{ms}(G \odot P_5) = \left\lceil \frac{5n+1}{3} \right\rceil$. Further research can be conducted to determine the modular irregularity strength for the corona product of the circulant graph and the complement of complete graph for other parameters of the circulant graph, e. g. $C_n(1,k) \odot \overline{K_p}$ or the modular irregularity strength for the corona product of the graph G with path P_m for $m \neq 3$ or $m \neq 5$.

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