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Antimagic Labeling of Graph Unions of Trees and 4-Cycles

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Abstract. Let $G(V, E)$ be a graph with $V(G)$ as the set of vertices and $E(G)$ as the set of edges. A labeling is a bijection $f : E(G) \to \{1, 2, \cdots, |E(G)|\}$. For each vertex v, let $\phi(v)$ be the sum of the labels on the edges incident to v. If all values $\phi(v)$ are distinct, the labeling is antimagic, and the graph is antimagic if such a labeling exists. This paper explores the concept of antimagic labeling in graph theory, with a particular focus on the union of various tree structures, including stars, single brooms, and double brooms. We apply extended Skolem sequences to prove that for a wide range of parameters, unions of multiple 3-paths with appropriately many 4-cycles yield antimagic graphs. Additionally, we analyze combinations of 4-cycles paired with different tree structures.

Key words and Phrases: antimagic labeling, trees, cycles, extended Skolem sequences.

1. INTRODUCTION

Graph labeling places symbols on the structural elements of a graph in order to encode combinatorial information. Depending on which elements receive labels, one distinguishes vertex labelings, edge labelings, and total labelings where both vertices and edges are assigned values [1]. The modern development of labeling theory goes back to the foundational work of Rosa [2] and subsequent contributions by Golomb on graph numbering [3].

In this article, we focus on antimagic edge labelings. Let $G = (V, E)$ be a finite graph with a set of vertices $V(G)$ and an edge set $E(G)$. Fix a bijection $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$. For each vertex $v \in V(G)$, set $\phi_f(v) = \sum_{e \in E(G)} f(e)$, the sum of the labels on the edges incident with v. We call f an *antimagic edge labeling* if $\phi_f(u) \neq \phi_f(v)$ for all distinct vertices $u, v \in V(G)$. A graph G is

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antimagic if it admits such a labeling. For brevity, tG denotes the disjoint union of t copies of G.

The formal notion of antimagic graphs itself was introduced by Hartsfield and Ringel [4]. Beyond their intrinsic combinatorial interest, antimagic labelings have found use in cryptography. They have been incorporated into block-cipher style constructions and secure communication schemes, illustrating how discrete labeling constraints can be leveraged for diffusion and uniqueness properties. For instance, Prihandini and Adawiyah designed a Hill-cipher variant informed by antimagic principles [5]; Gurjar and Krishnaa surveyed several antimagic labeled families with a view toward cryptographic frameworks [6] and Gondalia discussed implications for secure data transfer [7].

A long-standing conjecture due to Hartsfield and Ringel asserts that every tree other than K_2 is antimagic [4]. Although a full resolution is not yet available, substantial progress has been made for broad subfamilies. Kaplan, Lev, and Roditty showed that any nontrivial rooted tree in which every non-leaf vertex has at least two children is antimagic [8]. Liang, Wong, and Zhu showed that if a tree T has no vertices of degree two, then its subdivision T^*, obtained by inserting a new vertex on every edge, is antimagic [9]. For caterpillars, Lozano et al. established that a caterpillar of order n is $(\lfloor (n-1)/2 \rfloor - 2)$-antimagic [10]. For any caterpillar whose spine has order $p-2$, $p-1$, or p, with p prime, Wong and Zhu proved that it is 1-antimagic [11]. For caterpillars whose maximum degree is three, Deng and Li showed the graphs are antimagic [12].

Disconnected structures have also been investigated as natural extensions of the connected case. Shang et al. analyzed star forests and identified antimagic conditions when there is no K_2 component and at most one component is $K_{1,2}$ [13]. It was proved by Chang et al. that each double spider admits an antimagic labeling[14] and shown by Sierra et al. that any forest with no more than one vertex of degree two is antimagic [15]. Additional families and techniques appear in [16, 17, 18, 19, 20]; for a broad survey of labeling results, including antimagic labelings, see Gallian [21].

Together, these developments provide a robust platform for studying more elaborate constructions. In particular, they motivate the present work on disjoint unions that combine familiar tree classes with 4-cycles, and on constructive labeling frameworks that yield antimagic behavior across wide parameter ranges. The results below aim to widen the catalogue of antimagic graphs and to supply flexible methods that apply simultaneously to trees and small even cycles.

2. ANTIMAGIC LABELING OF GRAPH UNIONS OF MULTIPLE TYPE OF TREES

In this paper, we first investigate the antimagic labeling of unions of 3-paths, P_3. To establish the notation used in labeling P_3, it is relatively straightforward and follows naturally from the set of integers assigned to its edges. We provide an independent proof of the following result, which has also been previously established in [22].

Proposition 2.1. *The graph tP_3 is not antimagic for $t \geq 2$.*

Proof. Let v_i denote the degree-2 vertex in the i-th copy of tP_3, $i = 1, 2, \ldots, t$. Assume on the contrary that tP_3 has an antimagic labeling f. Without loss of generality, assume that the edges incident to v_1 are labeled with 1 and a_1. This means that the vertex sum of v_1 is a_1+1. If $a_1 < 2t$, then $a_1+1 \in \{2, 3, \ldots, 2t\} \setminus \{a_1\}$ which means that in the labeling ϕ, an edge incident to v_i $(i \geq 2)$ receives the label a_1+1. This implies that the vertex sum of the pendant vertex of this edge is a_1+1, a contradiction. Hence, assume that $a_1 = 2t$. This means that the remaining edges of tP_3 are labeled with $\{2, 3, \ldots, 2t-1\}$.

Without loss of generality, assume that the edges incident to v_2 are labeled with 2 and a_2. That is, the vertex sum of v_2 is a_2+2. Now $a_2 \neq 2t-1$, otherwise the vertex sum of v_2 is $2t+1$, which is also the vertex sum of v_1 in ϕ, a contradiction. Since $a_2 < 2t-1$, then $a_2+2 \in \{3, 4, \ldots, 2t-1\} \setminus \{a_2\}$. But this implies that, in ϕ, an edge incident to $v_j (j \geq 3)$ is labeled with a_2+2, which means that the vertex sum of the pendant vertex adjacent to v_j is a_2+2, a contradiction. This completes the proof. □

Let S_n be a star which is also a complete bipartite graph $K_{1,n}$ for any positive integer n.

Proposition 2.2. *If $G = tS_n$ where $n \geq 3$ and $t \geq 1$, then G is antimagic.*

Proof. We label the edges of the k-th S_n with the integers $\{k, t+k, 2t+k, \cdots, (n-1)t+k\}$ for $k = 1, 2, \cdots, t$ as shown in the Figure 1. The vertex sums of tS_n are given by $\phi(V(tS_n)) = \cup_{k=1}^{t} \left\{k, t+k, \ldots, (n-1)t+k, \frac{n(2k+(n-1)t)}{2}\right\}$. The vertex sums of G are all unique, which implies that G is antimagic. □

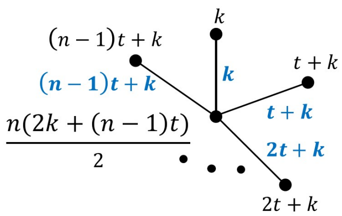

Figure 1. An antimagic labeling of the k-th S_n

Theorem 2.3 ([13, Theorem 2.1])**.** *If G is a union of stars containing no S_1 and at most one S_2, then G is antimagic.*

Proposition 2.2 is a specific case within the broader scope of Theorem 2.3 proved by [13]. However, the two results use different labeling methods. Figure 2 shows antimagic labeling for $4S_3$, Figure 2(a) applies the approach described in Proposition 2.2, while Figure 2(b) follows the method outlined in the proof of Theorem 2.3.

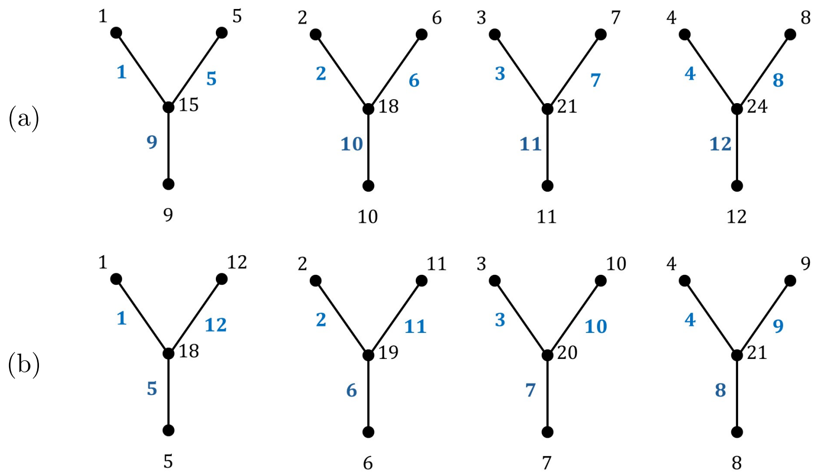

FIGURE 2. Examples of antimagic labeling of $4S_3$

Definition 2.4. *A linear forest is defined as a collection of disjoint paths, each containing more than one vertex.*

A P_k-free linear forest refers to a linear forest in which none of its components is a path of order k. In 2018, Shang [23] established the following theorem concerning linear forests, providing significant insights into their properties and structure.

Theorem 2.5 ([23, Theorem 2.8])**.** *P_2, P_3-free linear forests are antimagic.*

Let P be a linear forest and $P = \cup_{k=1}^{t} P_{k,n_k}$ where P_{k,n_k} is the k-th path of order $n_k \geq 4$ for all k. Let $v_{k,1}, v_{k,2}, \ldots, v_{k,n_k}$ be the vertices in sequential order of P_{k,n_k} for $k = 1, 2, \ldots, t$, where $v_{k,1}$ and v_{k,n_k} are the pendant vertices at each end of the path. Let $e_{k,i}$ be the edges between the vertices $v_{k,i}$ and $v_{k,i+1}$ for $i = 1, 2, \ldots, n_k - 1$. Let $f : E(P) \to \{1, 2, \ldots, n\}$, where $n = \sum_{k=1}^{t} n_k - t$. According to Theorem 2.5, f serves as an antimagic labeling of P. Moreover, the first and last edges of the path P_{k,n_k} are assigned labels from the smallest integers $2t$ in the set $\{1, 2, \ldots, n\}$, as determined by the equations provided below:

$$f(e_{k,1}) = 2k - 1 \quad \text{and} \quad f(e_{k,n_{k-1}}) = 2k \quad \text{for} \quad k = 1, 2, \ldots, t. \tag{1}$$

Definition 2.6. *A broom $B_{n,d}$ is a graph of n vertices, which has a path P with d vertices and $(n - d)$ pendant vertices, all of these being adjacent to either the origin*

u or the terminus v of the path P [24]*. A broom forest is a disjoint union of broom graphs. If $B^{(k)}_{n_k,d_k}$ denotes the k-th broom, then the single broom forest is defined as $B = \bigcup_{k=1}^{t} B^{(k)}_{n_k,d_k}$, where t is the number of brooms in the forest.*

Proposition 2.7. *A single broom forest $B = \cup_{k=1}^{t} B^{(k)}_{5,3}$ is antimagic for $t \geq 1$.*

Proof. Let $f : E(B) \to \{1, 2, \ldots, 4t\}$ be a function such that $f(e'_{k,1}) = k$, $f(e'_{k,2}) = 2t - k + 1$, $f(e_{k,1}) = 2t + 2k - 1$, and $f(e_{k,2}) = 2t + 2k$, for $k = 1, 2, \ldots, t$. Then, the vertex sums are given by $\phi(u_{k,1}) = k$, $\phi(u_{k,2}) = 2t - k + 1$, $\phi(v_{k,1}) = 4t + 2k$, $\phi(v_{k,2}) = 4t+4k-1$, and $\phi(v_{k,3}) = 2t+2k$. Since the vertex sums of B are distinct, B is antimagic. $\square$

Theorem 2.8. *A single broom forest $B = \cup_{k=1}^{t} B^{(k)}_{n'_k+2,n'_k}$ is antimagic for $t \geq 1$ and $n'_k \geq 6$.*

Proof. Suppose that a single broom forest B is constructed from a linear forest $P = \cup_{k=1}^{t} P_{k,n_k}$, where P is antimagic. Let $f : E(P) \to \{1, 2, \ldots, n - t\}$ be an antimagic labeling that satisfies (1). Now let $f_1 : E(B) \to \{1, 2, \ldots, n + t\}$ be a function such that $f_1(e'_{k,1}) = k$, $f_1(e'_{k,2}) = 2t - k + 1$ and $f_1(e_{k,i}) = f(e_{k,i}) + 2t$ for $i = 1, 2, \ldots, n_{k-1}$. Note that $f_1(e_{k,i}) \geq 4t + 1$ for $2 \leq i \leq n_k - 2$ and it is clear that

$$\phi(u_{k,1}) = k,\ \phi(u_{k,2}) = 2t - k + 1,\ \phi(v_{k,1}) = 4t + 2k,\ \phi(v_{k,n_k}) = 2t + 2k; \qquad (2)$$

$$\phi(v_{k,i}) = f_1(e_{k,i-1}) + f_1(e_{k,i}),\ i = 2, \ldots, n_k - 1. \qquad (3)$$

Note that the vertex sum in (2) is all distinct and the largest vertex sum is $6t$. Since P is antimagic, $\phi(v_{k,i}) \geq (2t + 1) + (4t + 1) = 6t + 2$ in (3) are all different. Hence, B is antimagic. $\square$

Figure 3 illustrates an example of an antimagic labeling of single broom forests.

Definition 2.9. *A double broom graph $B(r, s, s)$ is obtained from a path P_r with vertices $u_1, u_2, \ldots, u_r$ by attaching s pendant (leaf) vertices $v_1, \ldots, v_s$ to the left end vertex u_1 and attaching s pendant vertices $w_1, \ldots, w_s$ to the right end vertex u_r; equivalently, its vertex set is*

$$V\big(B(r, s, s)\big) = \{u_1, \ldots, u_r\} \cup \{v_1, \ldots, v_s\} \cup \{w_1, \ldots, w_s\}$$

and its edge set is

$$E\big(B(r, s, s)\big) = \{u_j u_{j+1} : 1 \leq j \leq r - 1\} \cup \{u_1 v_i : 1 \leq i \leq s\} \cup \{u_r w_i : 1 \leq i \leq s\}.$$

Hence, $B(r, s, s)$ has $r + 2s$ vertices and $r + 2s - 1$ edges [25]*. A double broom forest is a disjoint union of double brooms. If $B^{(k)}(r_k, s_k, s_k)$ denotes the k-th double broom, then the forest is defined as $D = \bigcup_{k=1}^{t} B^{(k)}(r_k, s_k, s_k)$, where t is the number of double brooms in the forest.*

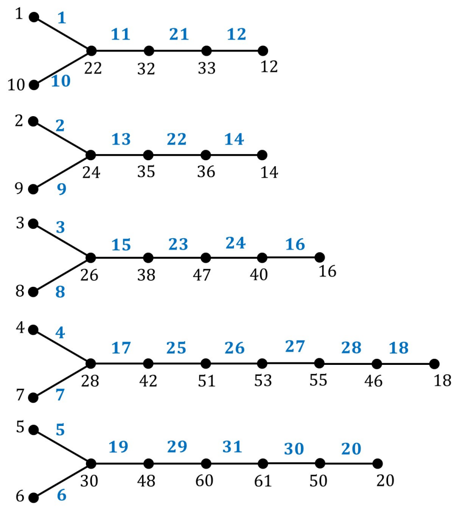

FIGURE 3. An antimagic labeling of $B = B^{(1)}_{6,4} \cup B^{(2)}_{6,4} \cup B^{(3)}_{7,5} \cup B^{(4)}_{9,7} \cup B^{(5)}_{8,6}$

An example of an antimagic labeling for double broom forests is shown in the Figure 4.

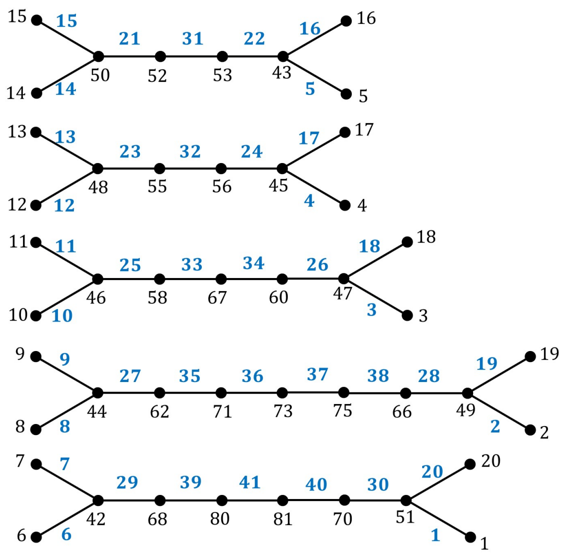

FIGURE 4. An antimagic labeling of $D = B^{(1)}(4,2,2) \cup B^{(2)}(4,2,2) \cup B^{(3)}(5,2,2) \cup B^{(4)}(7,2,2) \cup B^{(5)}(6,2,2)$

Proposition 2.10. *A double broom forest* $D = \cup_{k=1}^{t} B^{(k)}(2,2,2)$ *is antimagic for* $t \geq 1$.

Proof. Let $f : E(D) \to \{1,2,\ldots,5t\}$ be a function such that $f(e'_{k,1}) = 4k-3$, $f(e'_{k,2}) = 4k-1$, $f(e_{k,1}) = 4t+k$, $f(e^*_{k,1}) = 4k-2$, $f(e^*_{k,2}) = 4k$ for $k = 1,2\ldots,t$. Then the vertex sums of D given by $\phi(u_{k,1}) = 4k-3$, $\phi(u_{k,2}) = 4k-1$, $\phi(v_{k,1}) = 4t+9k-4$, $\phi(v_{k,2}) = 4t+9k-2$, $\phi(w_{k,1}) = 4k-2$ and $\phi(w_{k,2}) = 4k$ are all distinct. Therefore, D is antimagic. □

Proposition 2.11. *A double broom forest* $D = \cup_{k=1}^{t} B^{(k)}(3,2,2)$ *is antimagic for* $t \geq 1$.

Proof. Let $f : E(D) \to \{1,2,\ldots,6t\}$ be a function defined as $f(e'_{k,1}) = 2t+k$, $f(e'_{k,2}) = 3t+k$, $f(e_{k,1}) = 4t+2k-1$, $f(e_{k,2}) = 4t+2k$, $f(e^*_{k,1}) = 2t-2k+2$, $f(e^*_{k,2}) = 2t-2k+1$ for $k = 1,2\ldots,t$. Based on this labeling, the corresponding vertex sums in D are: $\phi(u_{k,1}) = 2t+k$, $\phi(u_{k,2}) = 3t+k$, $\phi(v_{k,1}) = 9t+4k-1$, $\phi(v_{k,2}) = 8t+4k-1$, $\phi(v_{k,3}) = 8t-2k+3$, $\phi(w_{k,1}) = 2t-2k+2$ and $\phi(w_{k,2}) = 2t-2k+1$. All of these vertex sums are pairwise distinct. Hence, D is antimagic. □

Theorem 2.12. *A double broom forest* $D = \cup_{k=1}^{t} B^{(k)}(n^*_k - 4, 2, 2)$ *is antimagic for* $t \geq 1$ *and* $n^*_k \geq 8$.

Proof. Suppose that the double broom forest D is constructed from a linear forest $P = \cup_{k=1}^{t} P_{k,n_k}$ where P is antimagic. Let $f : E(D) \to \{1,2,\ldots,n-t\}$ be an antimagic labeling that satisfies (1). Now, let $f_1 : E(D) \to \{1,2,\ldots,n+3t\}$ be a function defined as $f_1(e'_{k,1}) = 3t-2k+2$, $f_1(e'_{k,2}) = 3t-2k+1$, $f_1(e^*_{k,1}) = 3t+k$, $f_1(e^*_{k,2}) = t-k+1$ for $k = 1,2\ldots,t$; additionally, $f_1(e_{k,i}) = f(e_{k,i}) + 4t$ for $i = 1,2,\ldots,n_k-1$. Note that $f_1(e_{k,i}) \geq 6t+1$ for $2 \leq i \leq n_k - 2$. The vertex sums of D are given by

$$\phi(u_{k,1}) = 3t-2k+2,\ \phi(u_{k,2}) = 3t-2k+1,\ \phi(w_{k,1}) = 3t+k, \tag{4}$$

$$\phi(w_{k,2}) = t-k+1,\ \phi(v_{k,1}) = 10t-2k+2,\ \phi(v_{k,n_k}) = 8t+2k+1,$$

$$\phi(v_{k,i}) = f_1(e_{k,i-1}) + f_1(e_{k,i}),\ i = 2,\ldots,n_k-1. \tag{5}$$

We can see that the vertex sums in (4) are all distinct and the largest vertex sum is $10t+1$. Since P is antimagic, $\phi(v_{k,i}) \geq (4t+1) + (4t+2t+1) = 10t+2$ in (5) are all different. Hence, D is antimagic. □

The double broom $B(n,2,2)$ is a tree characterized by exactly two vertices of degree 3. This construction can be generalized by joining two disjoint copies of the star S_n through a single edge. The resulting tree, denoted $B(2,n-1,n-1)$, consists of $2n$ vertices and $2n-1$ edges. In this structure, the two central vertices, each initially serving as the center of an S_n, have degree n, while all remaining vertices have degree 1.

Proposition 2.13. $tB(2,n-1,n-1)$ *is antimagic for the integers* $n \geq 4$ *and* $t \geq 1$.

Proof. Since $B(2, n-1, n-1)$ consists of two copies of S_n, we first label the edge that joins the two copies S_n and then label the edges within each S_n. For the k-th copy of $B(2, n-1, n-1)$, we assign the edge connecting the two copies of S_n the label $(2n-2)t+k$, while the edges within the two copies of S_n are labeled with the integers $\{k, t+k, 2t+k, \cdots, (n-1)t+k\}$ and $\{nt+k, (n+1)t+k, \cdots, (2n-1)t+k\}$ for $k = 1, 2, \cdots, t$. As a result, all vertex sums in $tB(2, n-1, n-1)$ are unique, given by the set $\phi(V(tB(2, n-1, n-1))) = \bigcup_{k=1}^{t} \left\{k, t+k, \ldots, (n-1)t+k, \dfrac{n(2k+(n-1)t)}{2}\right\} \cup \left\{nt+k, (n+1)t+k, \ldots, (2n-1)t+k, \dfrac{n(3nt+2k-t)}{2}\right\}$. Therefore, $tB(2, n-1, n-1)$ is antimagic. □

Theorem 2.14. *$tB(2, m-1, n-1)$ is antimagic for the integers $m, n \geq 4$ and $t \geq 1$.*

Proof. For the k-th copy of $B(2, m-1, n-1)$, we first label S_m with integers $\{k, t+k, 2t+k, \cdots, (m-1)t+k\}$ and S_n with integers $\{mt+k, (m+1)t+k, \cdots, (m+n-1)t+k\}$, respectively. The edge connecting S_m and S_n is assigned with the label $(m+n)t+k$, for $k = 1, 2, \ldots, t$. It can be shown that all the vertex sums of $tB(2, m-1, n-1)$ are distinct, so we conclude that $tB(2, m-1, n-1)$ is antimagic. □

3. ANTIMAGIC LABELING OF GRAPH UNIONS OF MULTIPLE 4-CYCLES AND VARIOUS TYPES OF TREES

We investigate antimagic labelings of disjoint unions of 4-cycles C_4. A labeling of a cycle C_4 is written in the form (a, b, c, d), where a, b, c, and d are distinct and represent the edge labels encountered when traversing the cycle cyclically, either clockwise or counterclockwise. Throughout this section, we use the notation (a, b, c, d) to denote a labeled C_4, and (x, y) to denote the labeling of a path P_3. For example, the labeling $(1, 2, 4, 3)$ is illustrated in Figure 5.

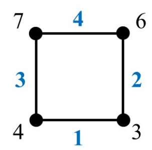

FIGURE 5. An antimagic labeling of C_4

Proposition 3.1. *If $G = tC_4$ where $t \geq 1$, then G is antimagic.*

Proof. We label the edges of k-th C_4 with the integers $(4k-3, 4k-2, 4k, 4k-1)$ for $k = 1, 2, \cdots, t$. The vertex sums of C_4 are given by $\phi(V(tC_4)) = \cup_{k=1}^{t}\{8k-5, 8k-4, 8k-2, 8k-1\}$. Since all the vertex sums of G are distinct, G is antimagic. □

Proposition 2.1 establishes that the graph tP_3 is not antimagic for $t \geq 2$. However, when considering the disjoint union $G = C_4 \cup P_3$, We observe that G does admit an antimagic labeling. In what follows, we present all possible antimagic labelings for the graph $G = C_4 \cup P_3$, as summarized in Table 1. Furthermore, Table 2 illustrates the different configurations in which distinct positive integers can be assigned to the edges of the graph $G = C_4 \cup 2P_3$, while preserving the antimagic property.

Table 1. All the antimagic labelings of $G = C_4 \cup P_3$

No	Labeling of C_4	Labeling of P_3
1	$(3,5,6,4)$	$(1,2)$
2	$(2,6,5,4)$	$(1,3)$
3	$(2,5,6,4)$	$(1,3)$
4	$(2,6,4,5)$	$(1,3)$
5	$(1,6,4,5)$	$(2,3)$

Table 2. All the antimagic labelings of $G = C_4 \cup 2P_3$

No	Labeling of C_4	Labeling of $2P_3$	No	Labeling of C_4	Labeling of $2P_3$
1	$(3,7,6,8)$	$(1,2),(4,5)$	13	$(2,5,7,8)$	$(1,4),(3,6)$
2	$(3,5,7,8)$	$(1,2),(4,6)$	14	$(3,4,7,6)$	$(1,5),(2,8)$
3	$(3,5,8,6)$	$(1,2),(4,7)$	15	$(2,6,7,8)$	$(1,5),(3,4)$
4	$(3,4,8,7)$	$(1,2),(5,6)$	16	$(2,6,8,7)$	$(1,5),(3,4)$
5	$(3,5,4,8)$	$(1,2),(6,7)$	17	$(2,7,6,8)$	$(1,5),(3,4)$
6	$(3,4,5,7)$	$(1,2),(6,8)$	18	$(4,5,8,7)$	$(1,6),(2,3)$
7	$(4,6,7,8)$	$(1,3),(2,5)$	19	$(2,7,4,8)$	$(1,6),(3,5)$
8	$(4,6,8,7)$	$(1,3),(2,5)$	20	$(4,5,6,8)$	$(1,7),(2,3)$
9	$(4,7,6,8)$	$(1,3),(2,5)$	21	$(4,5,8,6)$	$(1,7),(2,3)$
10	$(4,5,8,7)$	$(1,3),(2,6)$	22	$(4,6,5,8)$	$(1,7),(2,3)$
11	$(4,6,5,8)$	$(1,3),(2,7)$	23	$(1,7,5,8)$	$(2,3),(4,6)$
12	$(3,8,5,7)$	$(1,4),(2,6)$	24	$(1,6,7,8)$	$(2,4),(3,5)$

Proposition 3.2. *Let $G = P_3 \cup tC_4$ for the positive integer $t \geq 1$. Then G is antimagic.*

Proof. We label the only P_3 graph as $(1,2)$ and note that the vertex sums for the P_3 path are $\{1,2,3\}$. For the k-th C_4, the edges are labeled with $(4k-1, 4k, 4k+2, 4k+1)$ for $k = 1,2,\ldots,t$. The vertex sums of the 4-cycles are given by $\phi(V(tC_4)) = \cup_{k=1}^{t}\{8k-1, 8k, 8k+2, 8k+3\}$. Since all the vertex sums of G are distinct, G is antimagic. □

Proposition 3.3. *Let G be a graph consisting of unions of paths P_n or cycles C_4, where n is a positive integer with $n \geq 3$. If G is antimagic, then $G' = G \cup C_4$ is also antimagic.*

Proof. Suppose that $|E(G)| = m$ where m is a positive integer. Given that G is antimagic, there is a one-to-one correspondence f between the $E(G)$ and the label set $\{1, 2, \cdots, m\}$ such that $\phi(v) \neq \phi(u)$ holds for any two distinct vertices u and v. We label the edges of C_4 with integers $(m+1, m+2, m+4, m+3)$. In the graph G, the largest edge label (if it exists) is $m + (m-1) = 2m - 1$, which is less than $2m + 3$, the smallest edge label in C_4. Hence, $G' = G \cup C_4$ is antimagic. $\square$

Referring to Theorem 2.5 and Proposition 3.3, we can immediately derive the following results.

Proposition 3.4. *Let T be a P_2, P_3-free linear forest. Then $G = T \cup tC_4$ is antimagic for a positive integer $t \geq 1$.*

Definition 3.5. *An* extended Skolem sequence *of order n is a sequence $S = (s_1, s_2, \ldots, s_{2n+1})$ of length $2n+1$, consisting of integers, that satisfies the following conditions:*

(i) *For each integer $k \in \{1, 2, \ldots, n\}$, the sequence contains exactly two occurrences of k;*
(ii) *If the two positions $i < j$ satisfy $s_i = s_j = k$, then $j - i = k$;*
(iii) *There exists exactly one position i such that $s_i = 0$.*

An alternative representation of an extended Skolem sequence, as introduced in [26], is as a set of ordered pairs $\{(x_k, y_k) : 1 \leq k \leq n,\ y_k - x_k = k\}$, such that the union of all indices $\{x_k, y_k\}$ equals the entire index set $\{1, 2, \ldots, 2n+1\}$. Abrham and Kotzig [27] proposed a connection between extended Skolem sequences and a specific type of additive permutation, called σ-permutations. In their work, they demonstrated that each extended Skolem sequence corresponds uniquely to a σ-permutation.

The subsequent theorems will establish the application of extended Skolem sequences of order n in constructing antimagic labelings for the graph G.

Theorem 3.6. *Given $G = nP_3 \cup \left\lceil \dfrac{3n+5}{16} \right\rceil C_4$ with $n = 4q+1 \geq 9$, the graph G is antimagic.*

Proof. For any integer $n \geq 9$, we examine the extended Skolem sequence, where each element is a triple of the form $(k, x_k + n, y_k + n)$, with $k = 1, 2, \ldots, n$:

$$\begin{cases} (4q+2-2r, 4q+1+r, 8q+3-r), \ r = 1, 2, \ldots, 2q; \\ (2q+3-2r, 9q+1+r, 11q+4-r), \ r = 1, 2, \ldots, q; \\ (4q+1-2r, 8q+3+r, 12q+4-r), \ r = 1, 2, \ldots, q-2; \\ (4q+1, 6q+2, 10q+3), (2q+3, 10q+2, 12q+5), (1, 11q+4, 11q+5). \end{cases}$$

We assign edge labels to the k-th copy of P_3 using the pair $(k, x_k + n)$, for each $k = 1, 2, \ldots, n$. It is evident that the maximum label among all the P_3 components is $11q + 4$.

To construct the labels for the C_4 components, we define the set $Y = \{y_k + n | y_k + n < 11q + 4\}$, which in this case simplifies to

$$Y = \{8q + 3 - r \mid r = 0, 1, 2, \ldots, 2q\} \cup \{11q + 4 - r \mid r = 1, 2, \ldots, q+1\}.$$

This set has cardinality $|Y| = 3q + 2$. Since each C_4 contains 4 edges and we must assign a distinct label to each edge, the number of elements in the set used for labeling must be a multiple of 4. To satisfy this divisibility condition, we augment Y depending on the residue class of q modulo 4 as follows:

(i) If $q \equiv 0 \pmod 4$, let $Y_1 = Y \cup \{11q + 5, 11q + 6\}$;
(ii) If $q \equiv 1 \pmod 4$, let $Y_2 = Y \cup \{11q + 5, 11q + 6, 11q + 7\}$;
(iii) If $q \equiv 2 \pmod 4$, let $Y_3 = Y$;
(iv) If $q \equiv 3 \pmod 4$, let $Y_4 = Y \cup \{11q + 5\}$.

We then arrange the elements of the relevant set Y_i in increasing order and denote them by $y_1 < y_2 < \cdots < y_{\lceil \frac{3n+5}{4} \rceil}$. The edge labels for j-th C_4 are assigned as $(y_{4j+1}, y_{4j+2}, y_{4j+4}, y_{4j+3})$, for $j = 0, 1, \ldots, \lceil \frac{3n+5}{16} \rceil - 1$. Since all edge labels are distinct and each vertex sum is unique, the resulting graph G is antimagic. □

Theorem 3.7. *The graph $G = nP_3 \cup \frac{n+2}{4} C_4$ is antimagic for every integer $n = 4q + 2 \geq 10$.*

Proof. We use the extended Skolem sequence for the integer $n \geq 10$, defined by the triple $(k, x_k + n, y_k + n)$ for $k = 1, 2, \ldots, n$:

$$\begin{cases} (4q + 3 - 2r, 4q + 2 + r, 8q + 5 - r), \ r = 1, 2, \ldots, 2q; \\ (4q - 2r, 8q + 6 + r, 12q + 6 - r), \ r = 1, 2, \ldots, q - 1; \\ (2q - 2r, 9q + 5 + r, 11q + 5 - r), \ r = 1, 2, \ldots, q - 2; \\ (4q + 2, 6q + 3, 10q + 5), (4q, 6q + 4, 10q + 4), (2q, 8q + 6, 10q + 6), \\ (1, 11q + 5, 11q + 6), (2, 12q + 6, 12q + 8). \end{cases}$$

To label the edges of each P_3 component, we associate to the k-th path the pair $(k, x_k + n)$ for all $k = 1, 2, \ldots, n$. The largest value among these labels is clearly $12q + 6$, which will serve as an upper bound for the labeling of the paths.

For the labeling of the C_4 components, we introduce the set $Y = \{y_k + n | y_k + n < 12q + 6\}$, which yields $Y = \{8q + 5 - r \mid r = 0, 1, 2, \ldots, 2q\} \cup \{11q + 5 - r \mid r = 1, 2, \ldots, q + 1\} \cup \{12q + 6 - r \mid r = 1, 2, \ldots, q\}$ for the values considered. This set contains $4q + 2$ distinct integers. To ensure that the total number of labels used for the cycles is a multiple of four, we extend Y by appending two additional integers greater than $12q + 6$, namely $12q + 7$ and $12q + 8$, and define the augmented set $Y' = Y \cup \{12q + 7, 12q + 8\}$, which has size $4(q + 1)$. We now arrange the elements of Y' in increasing order, denoted as $y_1 < y_2 < \cdots < y_{4(q+1)}$. For each $j = 0, 1, \ldots, q + 1$, the edge labels of j-th C_4 are assigned in the specific pattern $(y_{4j+1}, y_{4j+2}, y_{4j+4}, y_{4j+3})$, ensuring a varied distribution of values across each 4-cycle. Given that all edge labels used across P_3 and C_4 components are distinct, and each vertex of the graph receives a unique induced sum, we conclude that the resulting graph G is antimagic. □

Theorem 3.8. *For all integers $n = 4q - 1 \geq 7$, the graph $G = nP_3 \cup \lceil \frac{3(n+1)}{16} \rceil C_4$ is antimagic.*

Proof. To construct the extended Skolem sequence for any integer $n \geq 9$, we use the triple $(k, x_k + n, y_k + n)$ for $k = 1, 2, \ldots, n$:

$$\begin{cases} (4q-2r, 4q-1+r, 8q-1-r), \ r = 1, 2, \ldots, 2q-1; \\ (4q-1-2r, 8q+r, 12q-1-r), \ r = 1, 2, \ldots, q-2; \\ (2q-1-2r, 9q-1+r, 11q-2-r), \ r = 1, 2, \ldots, q-2; \\ (4q-1, 6q-1, 10q-2), (2q+1, 9q-1, 11q), \\ (2q-1, 8q, 10q-1), (1, 11q-2, 11q-1). \end{cases}$$

To label the edges of each P_3 component, we associate to the k-th path the label pair $(k, x_k + n)$ for every $k = 1, 2, \ldots, n$. Among all such labels, the maximum value assigned is $11q - 2$.

We define the set $Y = \{y_k + n | y_k + n < 11q - 2\}$ to construct the labels for the C_4 components. This yields $Y = \{8q - 1 - r \mid r = 0, 1, 2, \ldots, 2q - 1\} \cup \{11q - 2 - r \mid r = 1, 2, \ldots, q\}$, a set containing exactly $3q$ elements. Since each 4-cycle requires four distinct edge labels, the set used for labeling must be divisible by 4 in size. To achieve this when $3q \not\equiv 0 \pmod 4$, we extend the set Y by appending additional integers greater than $11q-2$ according to the value of q modulo 4. Specifically, when $q \equiv 0 \pmod 4$, the set Y already contains a multiple of 4 elements and needs no adjustment. If $q \equiv 1 \pmod 4$, we define $Y_2 = Y \cup \{11q-1\}$; if $q \equiv 2 \pmod 4$, we take $Y_3 = Y \cup \{11q-1, 11q\}$; and for $q \equiv 3 \pmod 4$, we let $Y_4 = Y \cup \{11q-1, 11q, 11q+1\}$. In each case, the augmented set has cardinality divisible by 4. Once the appropriate set Y_i is determined based on the residue of q, we arrange its elements in increasing order as $y_1 < y_2 < \cdots < y_{\lceil \frac{3(n+1)}{4} \rceil}$. The edge labels for the j-th C_4 are then assigned using the pattern $(y_{4j+1}, y_{4j+2}, y_{4j+4}, y_{4j+3})$ for each $j = 0, 1, \ldots, \lceil \frac{3(n+1)}{16} \rceil - 1$. As all edge labels across the P_3 and C_4 components are distinct, and each vertex sum is unique, it follows that the resulting graph G is antimagic. $\square$

Theorem 3.9. *Let $n = 4q \geq 8$ be a positive integer. Then the graph $G = nP_3 \cup \lfloor \frac{n}{8} + 1 \rfloor C_4$ is antimagic.*

Proof. Consider the extended Skolem sequence for integers $n = 4q \geq 8$, represented by the triple $(k, x_k + n, y_k + n)$ for each $k = 1, 2, \ldots, n$:

$$\begin{cases} (4q+1-2r, 4q+r, 8q+1-r), \ r = 1, 2, \ldots, q-1; \\ (2q+1-2r, 5q-1+r, 7q-r), \ r = 1, 2, \ldots, q-1; \\ (4q-2r, 8q+1+r, 12q+1-r), \ r = 1, 2, \ldots, q-1; \\ (2q-2r, 9q+1+r, 11q+1-r), \ r = 1, 2, \ldots, q-1; \\ (1, 6q-1, 6q), (2q+1, 7q, 9q+1), (4q, 7q+1, 11q+1), (2q, 10q+1, 12q+1). \end{cases}$$

We assign edge labels to each P_3 component as follows. For all $k = 1, 2, \ldots, n$, the k-th copy of P_3 is labeled with the pair $(k, x_k + n)$. It follows that the maximum value used among the labels of the P_3 components is $10q + 1$. We now define a set $Y = \{y_k + n | y_k + n < 10q + 1\}$ to be used in labeling the C_4 components. So, $Y = \{6q\} \cup \{7q - r,\ 8q + 1 - r \mid r = 1, 2, \ldots, q - 1\} \cup \{8q + 1, 9q + 1\}$. It is straightforward to verify that $|Y| = 2q + 1$. Furthermore, the union of all labels assigned to the P_3 components and the set Y, covers the entire set $\{j \in \mathbb{N} \mid 1 \leq j \leq 10q + 1\}$.

To ensure that the number of labels used for the C_4 components is divisible by 4, we extend the set Y depending on the parity of q. If q is even, we define $Y_1 = Y \cup \{10q+2, 10q+3, 10q+4\}$, so that $|Y_1|$ is divisible by 4. If q is odd, we define $Y_2 = Y \cup \{10q+2\}$, again ensuring that $|Y_2|$ is divisible by 4.

We begin by arranging the elements of Y_1 if q is even, or Y_2 if q is odd, in increasing order, denoted as $y_1 < y_2 < \cdots < y_{4\lfloor \frac{n}{8}+1 \rfloor}$. The edge labels for the sequence of 4-cycles C_4 are then assigned according to the following rule: the first 4-cycle is labeled with the quadruple (y_1, y_3, y_4, y_7), while the second is labeled with (y_2, y_5, y_6, y_8). For each subsequent integer k ranging from 2 to $\lfloor \frac{n}{8} \rfloor$, the k-th 4-cycle is labeled with the quadruple $(y_{4k+1}, y_{4k+2}, y_{4k+4}, y_{4k+3})$. Since the induced vertex sums are all distinct, the graph G is antimagic. □

From Theorems 3.6, 3.7, 3.8 and 3.9, we have the following result.

Theorem 3.10. *The graph $G = nP_3 \cup \lceil \frac{n+2}{4} \rceil C_4$ is antimagic for every integer $n \geq 7$.*

Proposition 3.11. *Let $m, n \in \mathbb{N}$ with $n \geq 7$ and suppose that $m \geq \lceil \frac{n+2}{4} \rceil$. Then the graph $G = nP_3 \cup mC_4$ is antimagic labeling.*

Proof. The result is clear from Theorem 3.10 and Proposition 3.3. □

Proposition 3.12. *$G = B_{n,n-2} \cup sC_4$ is antimagic for positive integers $s \geq 1$ and $n \geq 5$.*

Proof. We label the graph $B_{n,n-2}$ according to the method described in the proof of Theorem 2.8, using the integers $1, 2, \ldots, n-1$. The edges of the s disjoint 4-cycles C_4 are then labeled with the integers $n, n+1, \ldots, n+4s-1$. Specifically, for each $k \in \{1, 2, \ldots, s\}$, the edges of the k-th C_4 are assigned the labels $(n+4k-4,\ n+4k-3,\ n+4k-1,\ n+4k-2)$. Note that $\phi(v_3) = 6$, where v_3 is the unique vertex of degree 3 in $B_{n,n-2}$. Since 6 is strictly less than all vertex sums in the C_4 components, all vertex sums in the graph $G = B_{n,n-2} \cup sC_4$ are distinct. Therefore, G is antimagic. □

Proposition 3.13. *Let $G = B(n-4, 2, 2) \cup sC_4$ for positive integers $s \geq 1$ and $n \geq 6$. Then G is antimagic.*

Proof. In Section 2, we established that $B(n-4, 2, 2)$ is antimagic for $n \geq 6$. The graph $B(n-4, 2, 2)$ contains exactly two vertices of degree 3, each incident to three edges. We assign the integers $1, 2, 3, 4, 5, 6$ to these six edges, ensuring that each receives a distinct label. Consequently, the vertex sums at the two degree-3 vertices are strictly less than $2n$. For each $k \in \{1, 2, \ldots, s\}$, we label the edges of the k-th 4-cycle C_4 with the values $(n+4k-4, n+4k-3, n+4k-1, n+4k-2)$. All of these labels are all greater than those used in $D(n, 2, 2)$, and they ensure distinct vertex sums throughout the entire graph. Therefore, the union $G = B(n-4, 2, 2) \cup sC_4$ is antimagic. □

Proposition 3.14. *$G = S_{n-1} \cup tC_4$ is antimagic for integers $3 \leq n \leq 7$ and $t \geq 1$.*

Proof. We begin by labeling the star S_{n-1} with the integers $1, 2, \dots, n-1$. The vertex of degree $n-1$ in S_n has a vertex sum $p = \dfrac{n(n-1)}{2}$. Next, we assign labels to the edges of each C_4 in tC_4 using the set $(n+4i-4,\ n+4i-3,\ n+4i-1,\ n+4i-2)$ for $i = 1, 2, \dots, t$. The smallest possible vertex sum in the C_4 components is $q = n + (n+1) = 2n+1$. For $3 \le n \le 5$, we have $p \le q$, and for $n = 6$ and $n = 7$, the value p does not coincide with any vertex sum arising from the C_4 components. Therefore, all vertex sums in the graph $G = S_{n-1} \cup tC_4$ are distinct, implying that G is antimagic. □

Proposition 3.15. *Let $n \ge 8$ and $t \ge 1$ be integers. $G = S_{n-1} \cup tC_4$ is antimagic if $n > \dfrac{5+\sqrt{1+64t}}{2}$.*

Proof. We begin by labeling the edges of the star S_{n-1} with integers $1, 2, \dots, n-1$. For each $i = 1, 2, \dots, t$, we assign edge labels to the i-th 4-cycle C_4 using the values $(n+4i-4,\ n+4i-3,\ n+4i-1,\ n+4i-2)$. This labeling ensures that the largest vertex sum in each C_4 is $2n+8i-3$. Hence, the maximum vertex sum across all t copies of C_4, denoted tC_4, is $2n+8t-3$. Meanwhile, the maximum vertex sum in the star S_{n-1} is $p = \dfrac{n(n-1)}{2}$. To ensure that the entire graph $G = S_{n-1} \cup tC_4$ is antimagic, meaning that all vertex sums are distinct, it must satisfy the inequality $\dfrac{n(n-1)}{2} > 2n+8t-3$. Solving this yields the condition $n > \dfrac{5+\sqrt{1+64t}}{2}$. □

4. CONCLUDING REMARKS

In this paper, we investigated antimagic edge labelings for disjoint unions of sparse graphs, focusing on unions of paths, star– and broom–type trees, and on the role of adjoining 4-cycles. Our results combine sharp obstructions with scalable constructive labelings.

We first established a definitive limitation for linear forests: the disjoint union tP_3 is not antimagic for every $t \ge 2$. Beyond this exceptional family, we gave explicit labeling schemes showing that broad classes of forests are antimagic, including unions of stars and several families of broom forests (single and double brooms, as well as joined-star variants), by partitioning the label set into ranges and controlling induced vertex sums.

We then extended the framework to graphs containing C_4-components. We showed that the unions of 4-cycles admit antimagic labelings and that adding C_4-blocks can preserve and produce antimagicness in disjoint unions. In particular, the difficult case of many P_3-components becomes tractable once sufficiently many C_4's are adjoined: using extended Skolem sequences we derived sufficient conditions for graphs of the form $nP_3 \cup sC_4$, including an explicit bound on s in terms of n. Finally, we demonstrated that the same cycle-block strategy interacts effectively with broom and star components, yielding further antimagic families with clear parameter ranges.

Data Availability Statement
No new data were created or analyzed in this study.

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