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Two Different Types Predator with Two Preys

M. Krishika[1*], S. Vijaya[1]

[1]Department of Mathematics, Annamalai University, India.

Abstract. This study proposes a mathematical model involving two prey species and two predator types, where only the first prey species benefits from a refuge. The specialist predator is subjected to non-constant harvesting, reflecting fluctuating exploitation pressure. Species interactions follow a Holling type II functional response. The analysis shows that the model is bounded and identifies the equilibrium points along with their stability. Numerical results are illustrated.

Key words and Phrases: Generalist, specialist, Lotka-Volterra, refuge, harvesting.

1. INTRODUCTION

Prey-predator interactions play a central role in ecological systems and form an important area of research in mathematical ecology. Various mathematical models have been developed to understand the dynamics of interacting species and the factors affecting their coexistence and stability [1, 2].

Models involving multiple predator species have received considerable attention because they help explain competition and resource sharing mechanisms among predators. Tansky [3] investigated the effects of predator switching, while Hsu and Hubbell [4] analyzed competition between predators sharing common prey resources. Kirlinger [**?**] established conditions for the permanence of species in a two-predator two-prey system. Subsequently, Bhattacharyya and Mukhopadhyay [5] incorporated spatial migration and predator switching into prey-predator dynamics and Hadziabdic et al. [6] further examined the behavior of Lotka-Volterra systems with two predators and one prey.

Prey refuge is another important ecological mechanism that allows prey populations to avoid predation by utilizing protected habitats. Ma et al. [7] demonstrated that prey refuges significantly influence predator-prey interactions and functional responses. Ko and Ryu [8] showed that the incorporation of a constant prey refuge can enhance the stability of predator-prey systems. Li et al. [9] further

*Corresponding author : krishika561999@gmail.com

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analyzed a predator-prey model with prey refuge and discussed its effects on the long-term dynamics and stability of populations. Recent studies have also explored the combined effects of refuge and other ecological factors on predator-prey interactions [10, 11].

Harvesting is another factor that strongly influences population dynamics. In natural and managed ecosystems, harvesting rates often vary with environmental conditions, seasonal changes and human activities. Dieci and Rebaza [12] investigated dynamical transitions in nonlinear systems, while Leard et al. [13] studied ratio-dependent predator-prey models with non-constant harvesting and demonstrated its impact on population stability and persistence.

Motivated by these studies, the present work proposes a prey-predator model consisting of two predator species, a prey refuge for the first prey population, and non-constant harvesting of the specialist predator. The objective is to investigate the dynamical behavior of the system and analyze the stability characteristics of the interacting populations.

2. MATHEMATICAL MODELING

The prey populations follow logistic growth and grow rapidly when small but slow down near their carrying capacities. They interact with predators based on prey availability. A prey refuge protects a fixed proportion $\hat{m}k_1$ of the population, leaving only $(1-\hat{m})k_1$ exposed to predation. The system includes a specialist predator k_3, which feeds on a limited prey type, and a generalist predator k_4, which consumes a wider range of prey. It also incorporates non-constant harvesting, where harvesting rates vary with time, population size, or environmental conditions. The corresponding mathematical formulation and its interaction represention are given below:

$$\begin{cases} \dfrac{dk_1}{dt} &= k_1 w_1 \left(1-\dfrac{k_1}{v_1}\right) - \dfrac{l_1(1-\hat{m})k_1k_4}{(k_1(1-\hat{m})+j)}, \\ \dfrac{dk_2}{dt} &= k_2 w_2 \left(1-\dfrac{k_2}{v_2}\right) - \dfrac{l_2k_2k_3}{(k_2+j)} - \dfrac{l_3k_2k_4}{(k_2+j)}, \\ \dfrac{dk_3}{dt} &= -c_1k_3 + \dfrac{n_1l_2k_2k_3}{(k_2+j)} - \hat{h}_1k_3, \\ \dfrac{dk_4}{dt} &= -c_2k_4 + \dfrac{n_2l_1(1-\hat{m})k_1k_4}{(k_1(1-\hat{m})+j)} + \dfrac{\tilde{n}_3\tilde{l}_3k_2k_4}{(k_2+\tilde{j})}. \end{cases} \tag{1}$$

Table 1: Biological Description of the parameters for (1)

Parameters	**Description**
k_1	Prey population with refuge.
k_2	Prey population.
k_3	Specialized predator population.
k_4	Generalist predator population.
w_1, w_2	Intrinsic growth rate.

Parameters	Description
v_1, v_2	Carrying capacity.
l_1, l_2, l_3	Rate of predation.
a	Death rate of adult.
$l_1,\ l_2$	Rate of predation.
n_1, n_2, n_3	Natality rate of predators by each individual.
j	Half saturation constant.
c_1, c_2	Death rate of predator's
$\hat{h}_1$	Non constant harvesting rate.
$\hat{m}$	Refuge proportion $(0 < \hat{m} \leq 1)$.

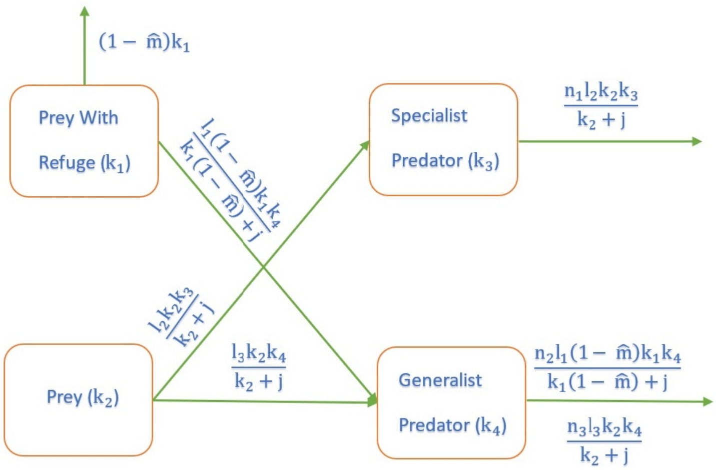

FIGURE 1. Interaction structure of the prey predator model(1)

3. POSITIVE AND BOUNDEDNESS

The positivity and boundedness properties ensure that all species populations remain non-negative and finite over time. Ecologically, this reflects that populations cannot be negative and are limited by resources and environmental constraints, keeping the model biologically realistic.

Theorem 3.1. *All solution of the system (1) initiating from R^4_+ is positive for $t \geq 0$.*

Proof. The first equation of (1) gives

$$k_1(t) = k_1(0)\exp\left(\int_0^t \left[w_1\left(1 - \frac{k_1}{v_1}\right) - \frac{l_1(1-\hat{m})k_4}{(k_1(1-\hat{m})+j)}\right] dt\right) > 0.$$

The second equation of the system (1) is

$k_2(t) = k_2(0)\text{exp}\left(\int_0^t \left[w_2\left(1 - \frac{k_2}{v_2}\right) - \frac{l_2k_3}{(k_2+j)} - \frac{l_3k_4}{(k_2+j)}\right] dt\right) > 0.$
From the third equation of the system (1), we get
$k_3(t) = k_3(0)\text{exp}\left(\int_0^t \left[-c_1 + \frac{n_1l_2k_2}{(k_2+j)} - \hat{h}_1\right] dt\right) > 0.$
The last equation of system (1) gives
$k_4(t) = k_4(0)\text{exp}\left(\int_0^t \left[-c_2 + \frac{n_2l_1(1-\hat{m})k_1}{(k_1(1-\hat{m})+j)} + \frac{n_3l_3k_2}{(k_2+j)}\right] dt\right) > 0.$
This proves the theorem. □

Theorem 3.2. *All solutions of system (1) in R_+^4 are uniformly bounded.*

Proof. Let us define a function as follows:

$$\bar{\omega} = \frac{1}{\bar{\alpha}}k_1 + \frac{1}{\bar{\beta}}k_2 + \frac{1}{\bar{\gamma}}k_3 + \frac{1}{\bar{\delta}}k_4. \tag{2}$$

Differentiate equation (1) with respect to 't', we get

$$\begin{aligned}\frac{d\bar{\omega}}{dt} &= \frac{1}{\bar{\alpha}}\frac{dk_1}{dt} + \frac{1}{\bar{\beta}}\frac{dk_2}{dt} + \frac{1}{\bar{\gamma}}\frac{dk_3}{dt} + \frac{1}{\bar{\delta}}\frac{dk_4}{dt}, \\ &= \frac{1}{\bar{\alpha}}\left[w_1 - \frac{w_1k_1}{v_1}\right] + \frac{k_2}{\bar{\beta}}\left[w_2 - \frac{w_2k_2}{v_2}\right] - \frac{k_3}{\bar{\gamma}}[c_1 + \hat{h}_1] - \frac{k_4}{\bar{\delta}}[c_2].\end{aligned}$$

For any $\xi > 0$.

$$\frac{d\bar{\omega}}{dt} + \xi\bar{\omega} \leq \frac{v_1}{4\bar{\alpha}w_1}(w_1+\xi)^2 + \frac{v_2}{4\bar{\beta}w_2}(w_2+\xi)^2.$$

we define a constant $\eta > 0$ such that $\eta = \frac{\tilde{v}_1}{4\bar{\alpha}w_1}(w_1+\xi)^2 + \frac{v_2}{4\bar{\beta}w_2}(w_2+\xi)^2$. It shows that $\frac{d\bar{\omega}}{dt} + \xi\bar{\omega} \leq \eta$. Applying theory of differential inequality, we obtain $\bar{\omega}(k_1(t), k_2(t), k_3(t), k_4(t)) \leq \frac{\eta}{\xi}\left[1 - e^{-\xi t}\right] + \bar{\omega}(k_1(0), k_2(0), k_3(0), k_4(0))e^{-\xi t}$. As $t \to \infty$, this becomes $0 \leq \bar{\omega}(k_1(t), k_2(t), k_3(t), k_4(t)) \leq \frac{\eta}{\xi}$. Hence, that concludes all solution of (1) with initial condition that initiate in R_+^4 are bounded to the region $\nabla = \{(k_1(t), k_2(t), k_3(t), k_4(t)) \in R_+^4 | \bar{\omega} = \frac{\eta}{\xi} + \epsilon \ \forall \ \epsilon\}$. □

4. EXISTENCE OF EQUILIBRIUM POINTS

Various equilibrium points of the system have been examined. The system is observed to have six possible equilibrium points. $\hat{m} \neq 1$ and $\hat{m} < 1$, all the equilibrium points satisfy the condition. These equilibrium points are now presented as follows:

(i) Trivial equilibrium $\hat{E}_1(0,0,0,0)$ represents the complete extinction of all prey and predator populations under extreme ecological stress.

(ii) Coexistence of prey without predator equilibrium $\hat{E}_2(v_1, v_2, 0, 0)$ represents the persistence of both prey species in the absence of predators.

(iii) Single prey with dual predator coexistence equilibrium $\hat{E}_3\left(\frac{c_2 j}{(1-\hat{m})(n_2 l_1 - c_2)}, 0, jw_2 - \frac{l_3}{l_2}\left[\frac{w_1 n_2 \Im}{c_2}\right], \frac{w_1 n_2 \Im}{c_2}\right)$ occurs when $n_2 l_1 > c_2$, $jw_2 > \frac{l_3 w_1 n_2 \Im}{l_2 c_2}$ and $1 > \frac{c_2 j}{v_1(1-\hat{m})(n_2 l_1 - c_2)}$ ensuring that one prey species persists and sustains both predator populations while the other prey is absent.
Where $\Im = 1 - \frac{c_2 j}{v_1(1-\hat{m})(n_2 l_1 - c_2)}$

(iv) Generalized predator absent equilibrium $\hat{E}_4\left(v_1, \mathcal{T}, \frac{w_2 j}{l_1 v_1}[v_2(\mathcal{T}+1) - \mathcal{T}], 0\right)$ exists when $n_1 l_2 > c_1 + \hat{h}_1$ and $v_2(\mathcal{T}+1) > \mathcal{T}$ under which both prey coexist with the specialist predator while the generalist predator is absent.

(v) Coexistence equilibrium point $\hat{E}_5(k_1^*, k_2^*, k_3^*, k_4^*)$ exists when all population components are positive, representing the simultaneous persistence of both prey and both predator species.

The equilibrium points are presented below:

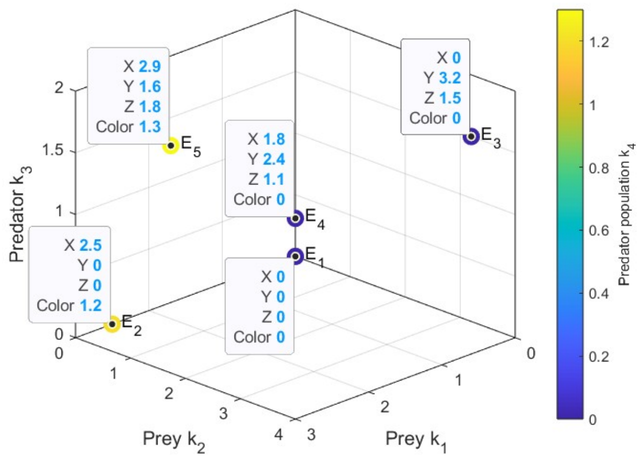

FIGURE 2. Graphical representation of the model(1) equilibria

Table 2. Key parameters and effects on ecosystem dynamics for all equilibrium points.

Equilibrium Point	**Key Parameters**	**Effect on Ecosystem Dynamics**
$\hat{E}_1$	–	All species become extinct. Recovery is only possible if prey growth and carrying capacities are sufficient.
$\hat{E}_2$	v_1, v_2	Both prey populations survive in the absence of predators.
$\hat{E}_3$	$w_1, w_2, c_2, n_2, l_1, l_2, l_3, v_1, \hat{m}$	One prey survives due to refuge and limited predation, while the other prey becomes extinct. Both predators persist with densities determined by prey growth, predator birth and death rates, and predator interactions.
$\hat{E}_4$	$v_1, v_2, w_2, l_1, l_2, c_1, n_1, j, \hat{h}_1, \mathcal{T}$	Both prey coexist with the specialist predator, while the generalist predator is absent. Population densities depend on prey growth, carrying capacities, predation rates, half-saturation effects, and harvesting.
$\hat{E}_5$	All model parameters	All prey and predator species coexist. Population densities are governed by prey growth, carrying capacities, predation rates, predator birth and death rates, prey refuge, half-saturation effects, and harvesting.

5. STABILITY ANALYSIS

The dynamics behavior of the equilibrium points can be analyzed by calculating the eigenvalue of the Jacobian matrix of system(1), given by:

$$J(k_1, k_2, k_3, k_4) = \begin{bmatrix} \gamma_{11} & 0 & 0 & \gamma_{14} \\ 0 & \gamma_{22} & \gamma_{23} & \gamma_{24} \\ 0 & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & 0 & \gamma_{44} \end{bmatrix}.$$

Where,

$$\gamma_{11} = w_1 - \frac{2w_1k_1}{v_1} - \frac{l_1(1-\hat{m})k_4}{(k_1(1-\hat{m})+j)} + \frac{l_1(1-\hat{m})k_1k_4}{(k_1(1-\hat{m})+j)^2}.$$

$$
\begin{aligned}
\bar{\gamma}_{12} &= 0,\\
\bar{\gamma}_{13} &= 0,\\
\bar{\gamma}_{14} &= -\frac{l_1(1-\hat{m})k_1}{(k_1(1-\hat{m})+j)},\\
\bar{\gamma}_{21} &= 0,\\
\bar{\gamma}_{22} &= w_2 - \frac{2w_2k_2}{v_2} - \frac{l_2k_3}{(k_2+j)} + \frac{l_2k_2k_3}{(k_2+j)^2} - \frac{l_3k_4}{(k_2+j)} + \frac{l_3k_2k_4}{(k_2+j)^2},\\
\bar{\gamma}_{23} &= -\frac{l_2k_2}{(k_2+j)},\\
\bar{\gamma}_{24} &= -\frac{l_3k_2}{(k_2+j)},\\
\bar{\gamma}_{31} &= 0,\\
\bar{\gamma}_{32} &= \frac{n_1l_2k_3}{(k_2+j)} + \frac{n_1l_2k_2k_3}{(k_2+j)^2},\\
\bar{\gamma}_{33} &= -c_1 + \frac{n_1l_2k_2}{(k_2+j)} - \hat{h}_1,\\
\bar{\gamma}_{34} &= 0,\\
\bar{\gamma}_{41} &= \frac{n_2l_1(1-\hat{m})k_4}{(k_1(1-\hat{m})+j)} - \frac{n_2l_1(1-\hat{m})k_1k_4}{(k_1(1-\hat{m})+j)^2},\\
\bar{\gamma}_{42} &= \frac{n_3l_3k_4}{(k_2+j)} - \frac{n_3l_3k_2k_4}{(k_2+j)^2},\\
\bar{\gamma}_{43} &= 0,\\
\bar{\gamma}_{44} &= -\tilde{c}_2 + \frac{\tilde{n}_2\tilde{l}_1(1-\hat{m})k_1}{(k_1(1-\hat{m})+\tilde{j})} + \frac{\tilde{n}_3\tilde{l}_3k_2}{(k_2+\tilde{j})},
\end{aligned}
$$

Theorem 5.1. *The trivial equilibrium point $\hat{E}_1(0,0,0,0)$ of model (1) is a saddle point.*

Proof. The jacobian matrix is,

$$
J(0,0,0,0) = \begin{bmatrix} \tilde{w}_1 & 0 & 0 & 0 \\ 0 & \tilde{w}_2 & 0 & 0 \\ 0 & 0 & -\tilde{c}_1 & 0 \\ 0 & 0 & 0 & -\tilde{c}_2 \end{bmatrix}.
$$

The eigenvalues are $\lambda_1 = w_1$, $\lambda_2 = w_2$, $\lambda_3 = -c_1 - \hat{h}_1$ and $\lambda_3 = -c_2$.
Thus, being a saddle point, it is unstable. □

Theorem 5.2. *Equilibrium point $\hat{E}_2\,(v_1, v_2, 0, 0)$ of model (1) is stable if the condition $c_1 > \dfrac{n_1v_2l_2}{(v_2+j)}$ and $c_2 > \dfrac{n_2l_1(1-\hat{m})v_1}{(v_1(1-\hat{m})+j)} + \dfrac{n_3l_3v_2}{(v_2+j)}$ hold.*

Proof. The jacobian matrix is,

$$J(k_1,k_2,0,0) = \begin{bmatrix} -\tilde{w}_1 & 0 & 0 & -\dfrac{l_1(1-\hat{m})v_1}{(v_1(1-\hat{m})+j)} \\ \\ 0 & -\tilde{w}_2 & -\dfrac{l_2v_2}{v_2+j} & -\dfrac{\tilde{l}_2\tilde{v}_2}{\tilde{v}_2+\tilde{j}} \\ \\ 0 & 0 & -c_1+\dfrac{n_1l_3v_2}{(v_2+j)} & 0 \\ \\ 0 & 0 & 0 & -c_2+\dfrac{n_2l_1(1-\hat{m})v_1}{(v_1(1-\hat{m})+j)}+\dfrac{n_3l_3v_2}{(v_2+j)} \end{bmatrix}.$$

The eigenvalue of $\hat{E}_2\,(v_1,v_2,0,0)$ are $\lambda_1=-w_1$, $\lambda_2=-w_2$,
$\lambda_3=-c_2+\dfrac{n_1l_2v_2}{(v_2+j)}$ and $\lambda_4=-c_2+\dfrac{n_2l_1(1-\hat{m}v_1)}{v_1(1-\hat{m})+j}+\dfrac{n_3l_3v_2}{(v_2+j)}$.
In the case where $c_1>\dfrac{n_1v_2l_2}{(v_2+j)}$ and $c_2>\dfrac{n_2l_1(1-\hat{m})v_1}{(v_1(1-\hat{m})+j)}+\dfrac{n_3l_3v_2}{(v_2+j)}$ the equilibrium point $\hat{E}_2$ is stable.

Prey populations persist while predators cannot invade. This reflects a situation where prey refuges and predator mortality prevent predator establishment, allowing prey to maintain a stable population.

□

Theorem 5.3. *Single prey with dual predator coexistence equilibrium point $\hat{E}_3$ is stable if the condition $tr(J(\hat{E}_3))<0$, $Det(J(\hat{E}_3))<0$, $DM(J(\hat{E}_3))tr(J(\hat{E}_3))<Det(J(\hat{E}_3))$ and $w_2+\dfrac{l_3w_1n_2\mathfrak{I}}{l_2c_2j}<w_2l_2+\dfrac{l_3w_1n_2\mathfrak{I}}{jc_2}$ hold.*

Proof. The jacobian matrix is,

$$J(k_1,0,k_3,k_4) = \begin{bmatrix} \chi_{11} & 0 & 0 & \chi_{14} \\ 0 & \chi_{22} & 0 & 0 \\ 0 & \chi_{32} & \chi_{33} & 0 \\ \chi_{41} & \chi_{42} & 0 & \chi_{44} \end{bmatrix}.$$

Where
$\chi_{11}=w_1-\dfrac{2c_2w_1j}{v_1(1-\hat{m})(n_2l_1)-c_2}-\dfrac{\mathfrak{I}(1-\hat{m})w_1(n_2l_1-c_2)}{c_2j}+w_1(1-\hat{m})\mathfrak{I}$,
$\chi_{12}=0$,
$\chi_{13}=0$,
$\chi_{14}=-\dfrac{c_2}{n_2}$,
$\chi_{21}=0$,

$\chi_{22} = w_2 - \dfrac{l_2^2 c_2 w_2 j - l_3 w_1 n_2 \Im}{j c_2 l_2} - \dfrac{l_3 w_1 n_2 \Im}{c_2 j}$,
$\chi_{23} = 0$,
$\chi_{24} = 0$,
$\chi_{31} = 0$,
$\chi_{32} = \dfrac{n_1 l_2^2 c_2 w_2 j - l_3 w_1 n_2 \Im}{j c_2 l_2}$,
$\chi_{33} = -c_1 - \hat{h}_1$,
$\chi_{34} = 0$,
$\chi_{41} = \dfrac{n_2^2 l_1 w_1 (1-\hat{m})(n_2 l_1 - c_2)\Im}{c_2(c_2 j + (1-\hat{m})(n_2 l_1 - c_2))} - \dfrac{n_2 l_1 w_1 (1-\hat{m})(n_2 l_1 - c_2)\Im}{(j((c_2 + (1-\hat{m})(n_2 l_1 - c_2))))^2}$,
$\chi_{42} = \dfrac{n_3 n_2 l_3 w_1 \Im}{c_2 j}$,
$\chi_{43} = 0$,
$\chi_{44} = -c_2 \hat{m}$.
The characteristic equation is
$(\chi_{22} - \lambda)(\lambda^3 -$ tr(J($\hat{E}_3$))$\lambda^2 +$ DM(J($\hat{E}_3$))$\lambda -$ Det(J($\hat{E}_3$))) $= 0$.

Where,
tr(J($\hat{E}_3$)) $= \chi_{11} + \chi_{22} + \chi_{33}$.

DM(J($\hat{E}_3$)) $= 3\chi_{11}\chi_{33}\chi_{44} - \chi_{14}\chi_{41}\chi_{44}$.

Det(J($\hat{E}_3$)) $= \chi_{11}\chi_{33}\chi_{44} + \chi_{14}\chi_{41}\chi_{33}$.

$\lambda_1 = \tilde{w}_2 - \dfrac{\tilde{l}_2^2 \tilde{c}_2 \tilde{w}_2 \tilde{j} - \tilde{l}_3 \tilde{w}_1 \tilde{n}_2 \tilde{\Im}}{\tilde{j} \tilde{c}_2 \tilde{l}_2} - \dfrac{\tilde{l}_3 \tilde{w}_1 \tilde{n}_2 \tilde{\Im}}{\tilde{c}_2 \tilde{j}}$.

Here,

$$J(\hat{E}_3) = \begin{bmatrix} \bar{\chi}_{11} & 0 & \bar{\chi}_{14} \\ 0 & \bar{\chi}_{33} & 0 \\ \bar{\chi}_{41} & 0 & \bar{\chi}_{44} \end{bmatrix}.$$

In the case where tr(J($\hat{E}_3$)) < 0, Det(J($\hat{E}_3$)) < 0,
DM(J($\hat{E}_3$))tr(J($\hat{E}_3$)) $<$ Det(J($\hat{E}_3$)) and $\tilde{w}_2 + \dfrac{\tilde{l}_3 \tilde{w}_1 \tilde{n}_2 \tilde{\Im}}{\tilde{l}_2 \tilde{c}_2 \tilde{j}} < \tilde{w}_2 \tilde{l}_2 + \dfrac{\tilde{l}_3 \tilde{w}_1 \tilde{n}_2 \tilde{\Im}}{\tilde{j} \tilde{c}_2}$ is stable.

Under these conditions, the single prey species and both predators coexist stably. The inequalities ensure that predator growth is balanced by prey availability and mortality, preventing any species from overexploiting the prey. Ecologically, this reflects a stable prey predator system where interactions are regulated and no population collapses occur.

$\square$

Theorem 5.4. *Equilibrium point $\hat{E}_4$ is stable if the condition tr($J(\hat{E}_4)$) < 0, Det($J(\hat{E}_4)$) < 0, DM($J(\hat{E}_4)$)tr($J(\hat{E}_4)$) $<$ Det($J(\hat{E}_4)$),*
$c_2 > \dfrac{n_2 l_1(1-\hat{m})v_1}{v_1(1-\hat{m})+j} + \dfrac{n_3 l_3 \mathcal{T}}{(\mathcal{T}+j)}$ *hold.*

Proof. The jacobian matrix is,

$$J(k_1, k_2, k_3, 0) = \begin{bmatrix} \zeta_{11} & 0 & 0 & \zeta_{14} \\ 0 & \zeta_{22} & \zeta_{23} & \zeta_{24} \\ 0 & \zeta_{32} & \zeta_{33} & \zeta_{34} \\ 0 & 0 & 0 & \zeta_{44} \end{bmatrix}.$$

Where
$\zeta_{11} = -w_1$,
$\zeta_{12} = 0$,
$\zeta_{13} = 0$,
$\zeta_{14} = -\dfrac{l_1(1-\hat{m})v_1}{(v_1(1-\hat{m})+j)}$,
$\zeta_{21} = 0$,
$\zeta_{22} = w_2 - \dfrac{2w_2\mathcal{T}}{v_2} - \dfrac{l_2 w_2 j[v_2(\mathcal{T}+1)-\mathcal{T}]}{l_1 v_1(\mathcal{T}+j)} + \dfrac{l_2\mathcal{T} w_2 j(v_2(\mathcal{T}+1)-\mathcal{T})}{l_1 v_1(\mathcal{T}+j)^2}$,
$\zeta_{23} = -\dfrac{l_2\mathcal{T}}{(\mathcal{T}+j)}$,
$\zeta_{24} = -\dfrac{l_3\mathcal{T}}{(\mathcal{T}+j)}$,
$\zeta_{31} = 0$,
$\zeta_{32} = \dfrac{n_1 l_2 w_2 j(v_2(\mathcal{T}+1)-\mathcal{T})}{l_1 v_1(\mathcal{T}+j)} - \dfrac{n_1 l_2\mathcal{T} w_2 j(v_2(\mathcal{T}+1)-\mathcal{T})}{l_1 v_1(\mathcal{T}+j)^2}$,
$\zeta_{33} = -c_1 + \dfrac{n_1 l_2\mathcal{T}}{(\mathcal{T}+j)} - \hat{h}_1$,
$\zeta_{34} = 0$,
$\zeta_{41} = 0$,
$\zeta_{42} = 0$,
$\zeta_{43} = 0$,
$\zeta_{44} = -c_2 + \dfrac{n_2 l_1(1-\hat{m})v_1}{(v_1(1-\hat{m})+j)} + \dfrac{n_3 l_3\mathcal{T}}{\mathcal{T}+j}$.
The characteristic equation is

$(\zeta_{44} - \lambda)(\lambda^3$ - tr(J($\hat{E}_4$))λ^4+ DM(J($\hat{E}_4$))$\lambda-$Det(J($\hat{E}_4$))) $= 0$.

Where
tr(J($\hat{E}_4$)) $= \zeta_{11} + \zeta_{22} + \zeta_{33}$,

DM(J($\hat{E}_4$)) $= 3\zeta_{11}\zeta_{22}\zeta_{33} - \zeta_{11}\zeta_{23}\zeta_{32}$,

Det($J(\hat{E}_4)$) $= \zeta_{11}\zeta_{22}\zeta_{33} - \zeta_{11}\zeta_{23}\zeta_{32}$

$\lambda_1 = -c_2 + \dfrac{n_2 l_1(1-\hat{m})v_1}{(v_1(1-\hat{m})+j)} + \dfrac{n_3 l_3 \mathcal{T}}{\mathcal{T}+j}$.

Here

$$J(\hat{E}_4) = \begin{bmatrix} \zeta_{11} & 0 & 0 \\ 0 & \zeta_{22} & \zeta_{23} \\ 0 & \zeta_{23} & \zeta_{33} \end{bmatrix}.$$

In the case where tr(J($\hat{E}_4$)) < 0, Det(J($\hat{E}_4$)) < 0, DM(J($\hat{E}_4$))tr(J($\hat{E}_4$)) $<$ Det(J($\hat{E}_4$)), $c_2 > \dfrac{n_2 l_1(1-\hat{m})v_1}{v_1(1-\hat{m})+j} + \dfrac{n_3 l_3 \mathcal{T}}{(\mathcal{T}+j)}$ is stable. This point represents sustainable coexistence where all species persist and interactions among them prevent extreme population fluctuations. □

Theorem 5.5. *Coexistence Equilibrium point $\hat{E}_5(k_1^*, k_2^*, k_3^*, k_1^*)$ of model(1) stable if the condition tr($\hat{E}_5$) < 0, [DM of order(2)($\hat{E}_5$)][tr($\hat{E}_5$)] - [DM of order(3)($\hat{E}_5$)] $>$ 0 and ([DM of order(3)($\hat{E}_5$)][(tr($\hat{E}_5$))(DM of order(2)($\hat{E}_5$)) - (DM of order(3)($\hat{E}_5$))] - Det(($\hat{E}_5$))tr($\hat{E}_5$)^2) > 0.*

Proof.

$$J(k_1, k_2, k_3, k_4) = \begin{bmatrix} \vee_{11} & 0 & 0 & \vee_{14} \\ 0 & \vee_{22} & \vee_{23} & \vee_{24} \\ 0 & \vee 32 & \vee_{33} & 0 \\ \vee_{41} & \vee_{42} & 0 & \vee_{44} \end{bmatrix}.$$

Where

$\vee_{11} = w_1 - \dfrac{2w_1 k_1^*}{v_1} - \dfrac{l_1(1-\hat{m})k_4^*}{(k_1^*(1-\hat{m})+j)} + \dfrac{l_1(1-\hat{m})k_1^* k_4^*}{(k_1^*(1-\hat{m})+j)^2}$,

$\vee_{12} = 0$,

$\vee_{13} = 0$,

$\vee_{14} = -\dfrac{l_1(1-\hat{m})k_4^*}{(k_1^*(1-\hat{m})+j)}$,

$\vee_{21} = 0$,

$\vee_{22} = w_2 - \dfrac{2w_2 k_2^*}{v_2} - \dfrac{l_2 k_3^*}{(k_2^*+j)} + \dfrac{l_2 k_2^* k_3^*}{(k_2^*+j)^2} - \dfrac{l_3 k_4^*}{(k_2^*+j)} + \dfrac{l_3 k_2^* k_4^*}{(k_2^*+j)^2}$,

$\vee_{23} = -\dfrac{l_2 k_2^*}{(k_2^*+j)}$,

$\vee_{24} = -\dfrac{l_3 k_2^*}{(k_2^*+j)}$,

$\vee_{31} = 0$,

$\vee_{32} = \dfrac{n_1 l_2 k_3^*}{(k_2^* + j)} + \dfrac{n_1 l_2 k_2^* k_3^*}{(k_2^* + j)^2},$
$\vee_{33} = -c_1 + \dfrac{n_1 l_2 k_2^*}{(k_2^* + j)} - \hat{h}_1,$
$\vee_{34} = 0,$
$\vee_{41} = \dfrac{n_2 l_1 (1-\hat{m}) k_4^*}{(k_1^*(1-\hat{m}) + j)} - \dfrac{n_2 l_1 (1-\hat{m}) k_1^* k_4^*}{(k_1^*(1-\hat{m}) + j)^2},$
$\vee_{42} = \dfrac{n_3 l_3 k_4^*}{(k_2^* + j)} = \dfrac{n_3 l_3 k_2^* k_4^*}{(k_2^* + j)^2},$
$\vee_{43} = 0,$
$\vee_{44} = -c_2 + \dfrac{n_2 l_1 (1-\hat{m}) k_1^*}{(k_1^*(1-\hat{m}) + j)} + \dfrac{n_3 l_3 k_2^*}{(k_2^* + j)}.$
The characteristic equation is
$\lambda^4 - tr(\hat{E}_5)\lambda^3 + DM\ of\ order(2)(\hat{E}_5) - DM\ of\ order(3)(\hat{E}_5) + Det(\hat{E}_5) = 0.$
Where,

$$\begin{aligned}
tr(J(\hat{E}_5)) &= \vee_{11} + \vee_{22} + \vee_{33} + \vee_{44}. \\
DM\ of\ order(2)(J(\hat{E}_6)) &= \vee_{11}(\vee_{22} + \vee_{33} + \vee_{44}) + \vee_{22}(\vee_{33} + \vee_{44}) + \vee_{33}\vee_{44} \\
&\quad - (\vee_{14}\,\vee_{41} + \vee_{24}\,\vee_{42} + \vee_{23}\vee_{32}). \\
DM\ of\ order(3)(J(\hat{E}_5)) &= 4\,\vee_{11}\,\vee_{22}\,\vee_{33}\,\vee_{44} - (\vee_{11}\,\vee_{33}\,\vee_{24}\,\vee_{42} + \vee_{11}\,\vee_{23}\,\vee_{32}\,\vee_{44}). \\
DM\ of\ order(3)(J(\hat{E}_5)) &= 4\,\vee_{11}\,\vee_{22}\,\vee_{33}\,\vee_{44} - (\vee_{11}\,\vee_{33}\,\vee_{24}\,\vee_{42} + \vee_{11}\,\vee_{23}\,\vee_{32}\,\vee_{44}). \\
Det(J(\hat{E}_5)) &= \vee_{11}[2\,\vee_{22}\,\vee_{33}\,\vee_{44}\ -2(\vee_{23}\,\vee_{32}\,\vee_{44} + \vee_{24}\,\vee_{33}\,\vee_{42})].
\end{aligned}$$

In the case where tr$(\hat{E}_5) < 0$, $[DMoforder(2)(\hat{E}_5)][tr(\hat{E}_5)] - [DMoforder(3)(\hat{E}_5)] > 0$ and $([DM\ of\ order(3)(\hat{E}_5)][(tr(\hat{E}_5))(DM\ of\ order(2)(\hat{E}_5)) - (DM\ of\ order(3)(\hat{E}_5))] - Det((\hat{E}_5))(tr(\hat{E}_5))^2) > 0$ is stable.

This equilibrium represents a fully stable coexistence of all species. Predator and prey populations are regulated by their interactions, growth and mortality (or harvesting), so no species over exploits another. Small disturbances in populations decay over time, maintaining the balance of the ecosystem. □

Table 3: Stability comparison of all equilibrium points.

Equilibrium Point	**Relative Stability**
$\hat{E}_1$	This equilibrium is the least stable because any small introduction of populations can move the system away from this state.
$\hat{E}_2$	This equilibrium is more stable than $\hat{E}_1$ because prey populations can persist without predators, but it can be affected if predators invade.

Equilibrium Point	**Relative Stability**
$\hat{E}_3$	One prey survives due to refuge and limited predation, while the other prey becomes extinct. Both predators persist with densities shaped by prey growth, predator birth and death rates, and predator interactions.
$\hat{E}_4$	This equilibrium is less stable than $\hat{E}_3$ because it involves two prey and one predator, requiring a precise balance between species interactions.
$\hat{E}_5$	This equilibrium has the most restrictive stability because all species coexist, requiring careful balance among growth, predation, and harvesting, and small disturbances can easily destabilize the system.

6. NUMERICAL SIMULATION

The proposed model is not based on real ecological data or a specific case study therefore all parameter values used in the numerical simulations are purely hypothetical and chosen for illustrative purposes. The values are selected to be positive and within reasonable ranges to ensure biologically feasible solutions and to satisfy the analytical conditions derived in this section. Different sets of parameters are employed in Figures 1, 2, and 3 to illustrate distinct dynamical behaviors of the system and to demonstrate the qualitative validity of the theoretical results. All simulations are performed using MATLAB R2024b. The values are,

Table 4: Parameter Values

Parameters	**Set 1**	**Set 2**	**Set 3**
$k_1(0)$	4.56	4.56	4.56
$k_2(0)$	7.91	7.91	7.91
$k_3(0)$	6.23	6.23	6.23
$k_4(0)$	2.034	2.034	2.034
$\tilde{w}_1$	14.67	14.67	14.67
$\tilde{w}_2$	16.75	16.75	16.75
$\tilde{v}_1$	40	40	40
$\tilde{v}_2$	45	45	45
$\tilde{l}_1$	2.74	2.74	2.74
$\tilde{l}_2$	1.91	1.91	1.91
$\tilde{l}_3$	1.05	1.05	1.05
$\tilde{n}_1$	1.51	1.51	1.51
$\tilde{n}_2$	1.95	1.95	1.95
$\tilde{n}_3$	1.33	1.33	1.33
$\tilde{j}$	2.82	2.82	2.82
$\tilde{c}_1$	1.167	1.167	1.167

Parameters	Set 1	Set 2	Set 3
$\tilde{c}_2$	1.321	1.321	1.321
$\hat{h}_1$	0.989	10.56	-
$\hat{m}$	0.47	0.47	0.47

Set 1 represents the case where the harvesting rate is less than the specialist predator population. Set 2 denotes values where the harvesting rate is higher than the population. Set 3 indicates no harvesting occurs in the specialist predator population.

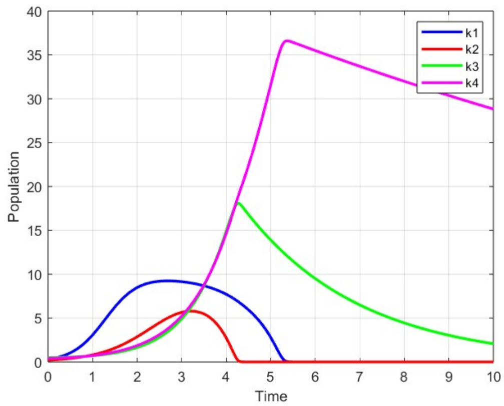

FIGURE 3. Specialist predator population>harvesting rate

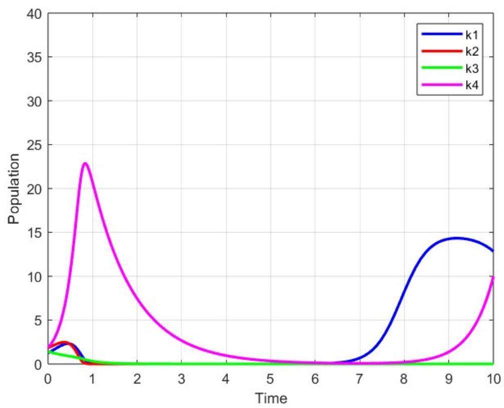

FIGURE 4. Specialist predator population < harvesting rate

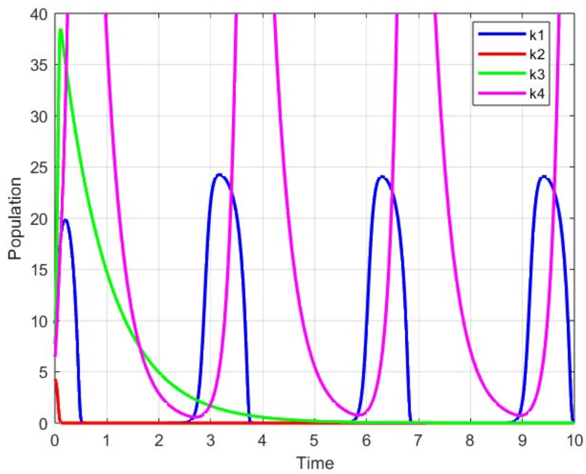

FIGURE 5. Without harvesting rate

7. CONCLUSION

This study analyzes a mathematical model with two prey species and two distinct predator types, in which the first prey species k_1 is protected by a refuge. The dynamics of the specialist predator are influenced by a non constant harvesting strategy. Figure 1 shows the interaction structure, while Figure 2 presents the equilibrium points of the system. When the harvesting rate is lower than the specialist predator population, the predator population increases even as the prey k_2 declines(Figure 3). Conversely, a higher harvesting rate reduces predator density(Figure 4). In the absence of harvesting, the specialist predator population rises while the prey population decreases(Figure 5). Non constant harvesting allows the rate to vary with population density. Simulation results align with stability theory, as population trajectories approach the predicted equilibrium points and confirming their local stability. This adaptive harvesting reduces pressure at low population levels and facilitates recovery. However, the model assumes fixed parameter values and does not consider spatial heterogeneity, which may limit its applicability. Future research could extend the model by incorporating empirical data and spatial dynamics with fear effects and anti predator behavior to enhance ecological realism and predictive power.

Data Availability Statement No data were used in this study. The results presented are based solely on mathematical assumptions and theoretical analysis of the proposed model.

Declarations. The authors declare no conflict of interest.

Author Contributions. The author solely conceived the study, performed the mathematical analysis, and wrote the manuscript.

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